High-Order Treatment of Fault Boundary Conditions Using Summation-By-Parts Finite Difference Methods

Jeremy E. Kozdon (jkozdon@stanford.edu)(1), Eric M. Dunham(1), Jan Nordström(2)

(1) Department of Geophysics, Stanford University; (2)Department of Information Technology, Uppsala University

Introduction

High-order numerical methods are ideally suited for earthquake problems since they require fewer grid points to achieve the same solution accuracy as low-order methods. Though it is relatively straightforward to apply high-order methods in the interior of the domain, it can be challenging to maintain stability and accuracy near boundaries (e.g., free surface) and internal interfaces (e.g., faults and layer interfaces). This is particularly problematic for earthquake models since numerical errors near faults degrade the global accuracy of the solution, including ground motion predictions. Despite several efforts to develop high-order fault boundary treatment, no codes have demonstrated greater than 2nd-order accuracy for dynamic rupture problems, even on rate-and-state friction problems with smooth solutions. Here we demonstrate a method which achieves global higher-order accuracy.

Governing Equations and Energy Dissipation

Momentum conservation and Hooke's law as a (symmetrized) first order hyperbolic system

$$\begin{aligned}\rho\frac{\partial v}{\partial t} &= \frac{\partial \sigma}{\partial y}\\ \frac{\partial \sigma}{\partial t} &= G\frac{\partial v}{\partial y}\end{aligned} \implies \frac{\partial q}{\partial t} = \begin{pmatrix}0 & c_s\\ c_s & 0\end{pmatrix}\frac{\partial q}{\partial y} = A\frac{\partial q}{\partial y},\quad q = \begin{pmatrix}Z\, v\\ \sigma\end{pmatrix}$$

$v =$ velocity

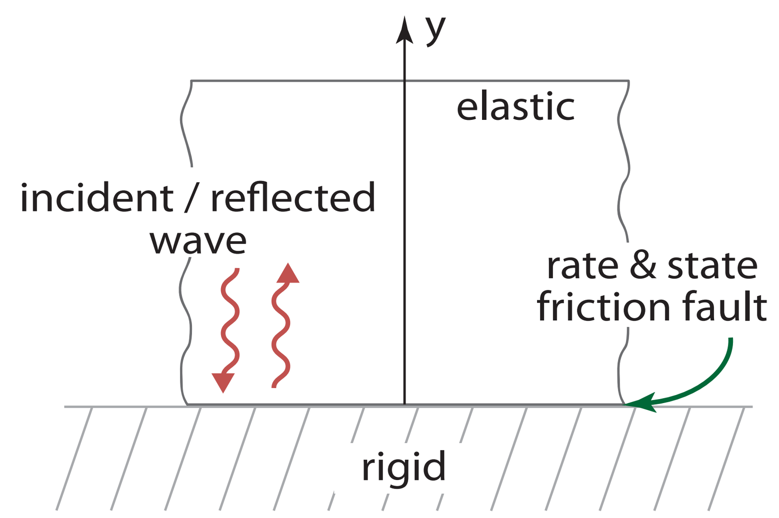
$\sigma =$ stress

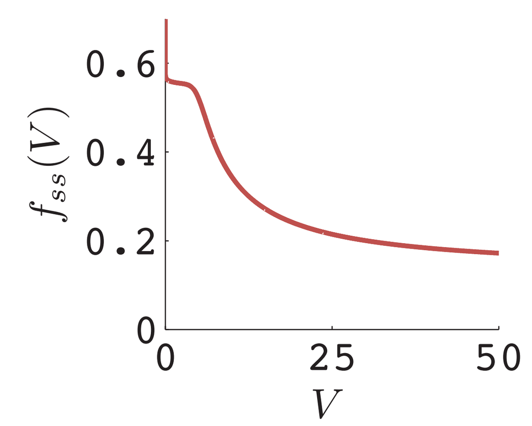
$\rho =$ density

$G =$ Shear Modulus

$c_s = \sqrt{\dfrac{G}{\rho}} =$ shear wave speed

$Z = \dfrac{G}{c_s} =$ shear impedance





$V = v(0,t)$ slip velocity

$\tau = \sigma(0,t)$ fault strength

$\tau = \tau(V,\psi)$ rate & state friction law

$$\tau(V,\psi) = \sigma_n f(V,\psi) = \sigma_n\, a \,\mathrm{arcsinh}\left(\frac{V}{2v_0}\exp\frac{\psi}{a}\right)$$

$$\frac{d\psi}{dt} = -\frac{V}{L}\left(f(V,\psi) - f_{ss}(V)\right)$$

Solution energy is defined as

$$\|q\|^2 = \frac{1}{2G}\int_0^\infty q^T q\, dt = \int_0^\infty \underbrace{\frac{\rho v^2}{2}}_{\text{kinetic energy}} dy + \int_0^\infty \underbrace{\frac{\sigma}{2G}}_{\text{elastic energy}} dy$$

then energy dissipation at fault (rate of frictional work on fault)

$$\frac{d\|q\|^2}{dt} = \frac{1}{G}\int_0^\infty q^T\frac{dq}{dt}\,dy = -\tau\, V \le 0$$

Problem can be reformulated in terms of incident (-) and reflected (+) characteristics (waves)

$$w^{\pm} = \sigma \mp Z\, v$$

then boundary condition states that reflected wave is a nonlinear function incident wave

$$w^+(0,t) = W\left(w^-(0,t)\right)$$

and energy dissipation at fault

$$\frac{d\|q\|^2}{dt} = -\frac{1}{Z}\left(w^-(0,t)^2 - W\left(w^-(0,t)\right)^2\right) \le 0$$

Summation-by-Parts Finite Difference Methods

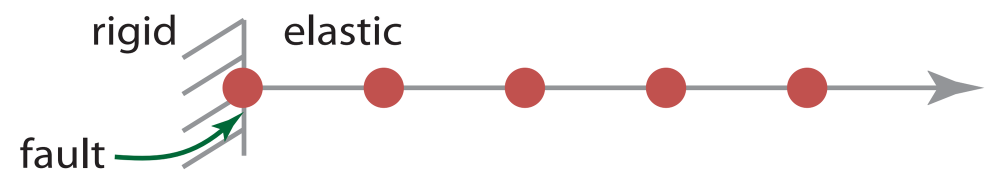
When numerically approximating governing equations it is desirable that discrete solution dissipates energy at least as fast as continuous solution. This ensures stability (and therefore accuracy) of approximation.

Summation-by-parts (SBP) operators are specially formed so such energy dissipation estimates can be obtained. A difference operator is said to be an SBP operator if (D, H, Q are matrices and H is positive definite)

$$\frac{d}{dx} \Rightarrow D = H^{-1}Q,\quad Q^T + Q = \mathrm{diag}(-1,0,\cdots,0,1)$$

A semi-discretization of governing equations can be written as (excluding the boundary terms)

$$\frac{dq}{dt} = \left(H^{-1}Q\otimes A\right)q,\quad q = \left(Z\, v_0,\ \sigma_0,\ Z\, v_1,\ \sigma_1,\ \cdots\right)^T$$



i	0	1	2	3	$\cdots$
y_i	0	Δy	$2\Delta y$	$3\Delta y$	$\cdots$
v_i	v_0	v_1	v_2	v_3	$\cdots$
σ_i	σ_0	σ_1	σ_2	σ_3	$\cdots$

where $\otimes$ stands for Kronecker product of two matrices. Note that in this formulation σ and v are collocated at grid points.

After inclusion of boundary conditions (to be discussed) semi-discretization is integrated in time along with state variable ψ using an appropriate ODE method (we use a 4th order Runge-Kutta method).

Energy is then defined as (second equality only for diagonal H)

$$\|q\|^2 = \frac{1}{2G}q^T\left(H\otimes I_2\right)q = \sum_{i=0}^\infty\left(\frac{\rho v_i^2}{2} + \frac{\sigma_i^2}{2G}\right)\underbrace{H_{ii}}_{\gamma_i\Delta y}$$

Boundary Treatment: Injection Method

Boundary conditions strongly enforced by modification of method. Concept: advance solution using discretization, then modify fault values (i=0) to satisfy rate-and-state friction law.

A single forward Euler step (Runge-Kutta iteration) would be

- Update solution using difference method

$$q := q + dt\left(H^{-1}Q\otimes A\right)q$$

- Compute new values on fault ($\hat v,\ \hat\sigma$) which satisfy boundary condition while preserving incident characteristic (nonlinear solve)

$$\begin{aligned}\hat\sigma + Z\,\hat v &= \sigma_0 + Z\, v_0\\ \hat\sigma &= \tau(\hat v,\psi)\end{aligned}$$

- Reset grid values to fault values

$$q_0 := \begin{pmatrix}Z\,\hat v\\ \hat\sigma\end{pmatrix}$$

Conceptually this method is straightforward, but energy estimation is (generally) not possible, therefore stability & accuracy not guaranteed.

Boundary Treatment: Simultaneous Approximation Term

Boundary conditions weakly enforced through penalty term

$$\begin{aligned}\frac{dq}{dt} &= \left(H^{-1}Q\otimes A\right)q + \left(H^{-1}E_0\otimes\Sigma\right)\left(e_0\otimes B(q_0)\right)\\ E_0 &= \mathrm{diag}(1,0,0,\cdots),\quad e_0 = (1,0,0,\cdots)^T\end{aligned}$$

where $B(q_0)$ is boundary term and penalty matrix Σ which will be chosen so that there is a stable energy estimate (depends on boundary term). We consider both non-characteristic and characteristic boundary condition formulation.

In order to estimate energy dissipation, and therefore Σ, we multiply scheme by $(H\otimes I_2)$ and add transpose giving

$$\begin{aligned}\frac{d\|q\|^2}{dt} &= \frac{1}{2G}q^T\left(Q+Q^T\otimes A\right)q + \frac{1}{G}q^T\left(E_0\otimes\Sigma\right)\left(e_0\otimes B(q_0)\right)\\ &= -v_0\,\sigma_0 + \frac{1}{G}q_0^T\,\Sigma\, B(q_0)\end{aligned}$$

Σ will then be chosen so that discrete solution energy decays at least as fast as continuous solution.

Non-Characteristic Boundary Condition

Non-characteristic boundary conditions & resulting penalty matrix are

$$B(q_0) = \left(\sigma_0 - \tau(v_0,\psi)\right)\begin{pmatrix}1\\1\end{pmatrix},\quad \Sigma = \begin{pmatrix}c_s & 0\\ 0 & 0\end{pmatrix}$$

then energy dissipation estimate is of same form as continous solution

$$\frac{d\|q\|^2}{dt} = -\tau(v_0,\psi)\, v_0 \le 0$$

Unfortunately this results in an arbitrarily stiff ODE, hence implicit time integration must be used with this formulation.

Characteristic Boundary Condition

Alternatively characteristic (wave-form) boundary condition give

$$B(q_0) = \left(w_0^+ - W\left(w_0^-\right)\right)\begin{pmatrix}1\\1\end{pmatrix},\quad \Sigma = \begin{pmatrix}c_s/2 & 0\\ 0 & -c_s/2\end{pmatrix}$$

energy dissipation then has form

$$\frac{d\|q\|^2}{dt} = \underbrace{-\frac{1}{Z}\left(\left(w_0^-\right)^2 - W\left(w_0^-\right)^2\right)}_{\text{continuous form}}\ \underbrace{-\frac{1}{Z}\left(w_0^+ - W\left(w_0^-\right)\right)^2}_{\text{small numerical damping}}$$

Characteristic formulation results in a well-conditioned (non-stiff) ODE!

If formula from injection method for $\hat\sigma$ and $\hat v$ is used and H is diagonal, characteristic method reduces to

$$\begin{aligned}\frac{dq_0}{dt} &= \left(d_0^T\otimes A\right)q - \left(H^{-1}\right)_{11}c_s\begin{pmatrix}Z\left(v_0 - \hat v\right)\\ \sigma_0 - \hat\sigma\end{pmatrix}\\ \frac{dq_i}{dt} &= \left(d_i^T\otimes A\right)q,\quad i = 1,2,\cdots\end{aligned}$$

which can be interpreted as relaxation of computed values $(v,\ \sigma)$ to "target" $(\hat v,\ \hat\sigma)$ values with relaxation time $T = H_{11}\,/\,c_s = \gamma_1\,\Delta y\,/\,c_s$

SBP-SAT References

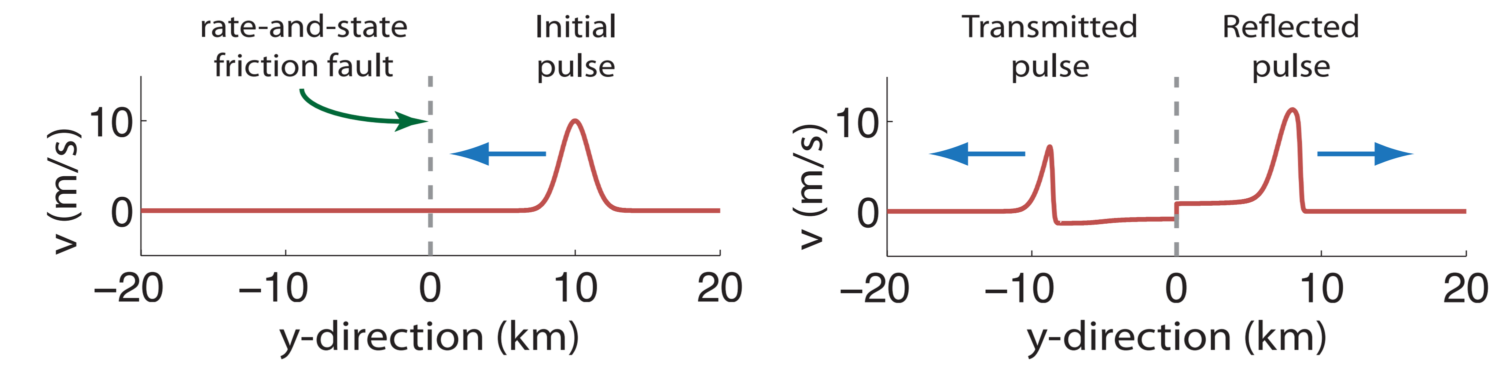
Carpenter et al., *Journal of Computational Physics*, 1999

Nordström & Gustafsson, *Journal of Scientific Computing*, 2003

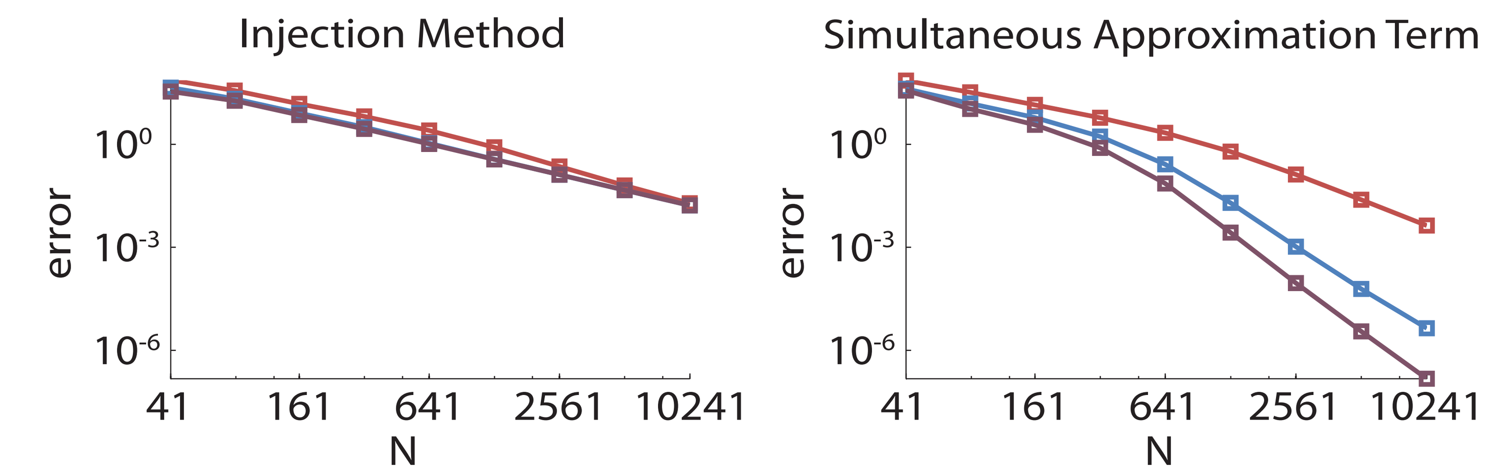
Nordström, *SIAM Journal of Scientific Computing*, 2007

Results: Two-Sided Fault with Rate-and-State Friction

Previous formulation can be generalized to a two-sided fault. Here we present results for this problem using both injection and SAT method with characteristic boundary treatment



We run above test problem and determine the convergence rate, in l_2-norm, for injection method and SAT against a semi-analytic solution. We use SBP difference operators which are 2nd, 3rd, and 4th order accurate.



Rates of convergence can be estimated using

$$p = \log\left(\frac{\|q_e - q_{\Delta y}\|_{\Delta y}}{\|q_e - q_{\Delta y/2}\|_{\Delta y/2}}\right)\Big/\log\left(\frac{\Delta y}{\Delta y/2}\right),\quad \|q\|^2_{\Delta y} = \sum_{i=1}^N \Delta y\ q_i^2$$

$q_{\Delta y}$ is numerical solution with step size Δy and q_e is semi-analytic solution

	Injection Method			Simultaneous Approximation Term		
N =	2nd Order	3rd Order	4th Order	2nd Order	3rd Order	4th Order
41	1.03	1.07	0.87	1.12	1.40	1.79
81	1.27	1.37	1.38	1.19	1.38	1.53
161	1.21	1.38	1.35	1.25	1.82	2.23
321	1.34	1.55	1.47	1.46	2.69	3.46
641	1.63	1.55	1.49	1.82	3.74	4.76
1281	1.85	1.50	1.48	2.22	4.24	4.89
2561	1.85	1.49	1.49	2.45	4.08	4.67
5121	1.71	1.49	1.49	2.50	3.81	4.58

Conclusions

SBP finite difference methods with SAT boundary treatment have been used to achieve higher-order convergence with rate & state friction laws

By using characteristic variables stiffness has been avoided

Verified for both one-sided (not shown) and two-sided fault problems

Future Work

Generalize to in-plane case

Implement in 2-D & 3-D codes

Include artificial dissipation to minimize spurious oscillations