A Finite Difference Method for Irregular Geometries: Application to Dynamic Rupture on Rough Faults

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Abstract

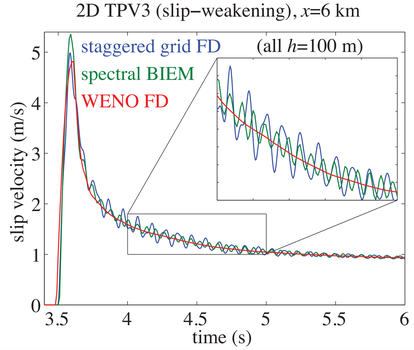
We have developed a finite difference method to solve dynamic rupture problems in irregular geometries. Our objective is to connect properties of high frequency radiation produced during slip on rough faults to statistical measures of fault roughness. To handle irregular geometries, we transform the governing equations from a non-Cartesian coordinate system that conforms to the irregular boundaries of the physical domain to a Cartesian coordinate system in a rectangular computational domain, and solve the equations in the computational domain. To accurately capture the high frequency wavefield, we use a method that produces far smaller numerical oscillations than those plaguing conventional finite difference/element methods. The governing equations (momentum conservation and Hooke's law) are written as a system of first-order equations for velocity and stress, which are defined at a common set of grid points and time steps (i.e., there is no staggering in space or time). Time stepping is done using an explicit third-order Runge-Kutta method. The equations are hyperbolic and the fields can be decomposed into a set of waves (with associated wave speeds) propagating in each coordinate direction. Spatial derivatives are computed with fifth-order WENO (weighted essentially non-oscillatory) finite differences in the upwind direction associated with each wave [Jiang and Shu, J. Comp. Phys., 126(1), 202-228, 1996]. Rather than using data from a single stencil (i.e., set of grid points) to calculate the derivative, a weighted combination of data from several candidate stencils is used. The weights are assigned based on solution smoothness within each stencil, and stencils in which the solution exhibits excessive variations are given minimal weight. Consequently, numerical oscillations are suppressed, even in the vicinity of the rupture front and at wavefronts. Boundary conditions are implemented by again appealing to the hyperbolic nature of the governing equations. At each point on a boundary (or fault), the solution is decomposed into a set of waves propagating into and out of the boundary. The amplitudes of incoming waves are preserved, while those of outgoing waves are modified to satisfy the boundary conditions. On the fault, this amounts to solving the friction law together with an equation expressing shear stress as the sum of a load, the radiation damping response, and the stress change carried by the incoming waves.

Motivation and Examples

Reduce Numerical Oscillations, Retain High Order Accuracy

Oscillations plague dynamic rupture simulations due to steep gradients in fields around rupture fronts and wavefronts. Researchers in computational fluid dynamics routinely deal with shocks and have developed high order methods that work with even discontinuous solutions without introducing oscillations. We use one such method, the WENO (weighted essentially nonoscillatory) method by Jiang and Shu [1996].

2D TPV3 (slip−weakening), x=6 km



Target Problem: Connect Fault Roughness to High Frequency Ground Motion

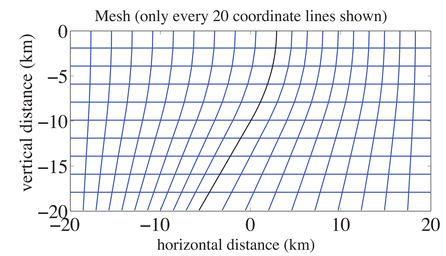
Stochastic rupture models (with heterogeneous stresses, for example) are widely used to produce a less coherent rupture process that generates high frequency ground motion. However, those models are for planar faults, and one often thinks of heterogeneous stress or strength on a flat fault as a proxy for unmodeled geometrical heterogeneities. With proper tools, we can investigate this problem more rigorously. Since high frequency radiation is of interest, numerical methods must get this part of the spectrum right.

Use Finite Difference Method Even For Irregular Geometries

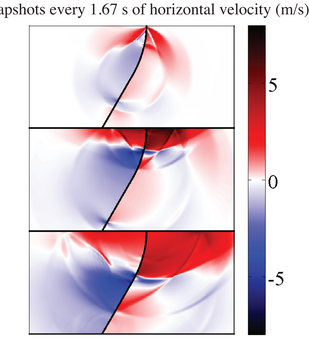
Irregular geometries are typically handled using finite element methods, which suffer from numerical oscillations and often require special treatment of nonphysical hourglass modes. But finite difference methods (like WENO) can also be used via a global mapping between irregular physical domain and Cartesian (rectangular) computational domain. Governing equations are transformed and solved in computational domain.

Example: supershear rupture on listric fault breaking free surface (2D in-plane geometry), rate-and-state friction with flash heating, absorbing boundaries except for free surface (top)

Mesh (only every 20 coordinate lines shown)



Snapshots every 1.67 s of horizontal velocity (m/s)



Numerical Method

Governing Equations

momentum conservation and Hooke's law as (hyperbolic) system of first order PDEs (antiplane case with constant material properties for simplicity)

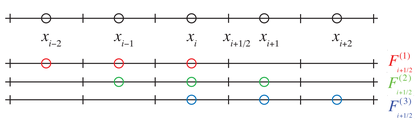
$$\frac{\partial}{\partial t}\begin{bmatrix} v_z \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} + \begin{bmatrix} 0 & -c^2/\mu & 0 \\ -\mu & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} v_z \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} + \begin{bmatrix} 0 & 0 & -c^2/\mu \\ 0 & 0 & 0 \\ -\mu & 0 & 0 \end{bmatrix}\frac{\partial}{\partial y}\begin{bmatrix} v_z \\ \sigma_{xz} \\ \sigma_{yz} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or simply $\frac{\partial q}{\partial t} + A\frac{\partial q}{\partial x} + B\frac{\partial q}{\partial y} = 0$ or $\frac{\partial q}{\partial t} + \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} = 0$

WENO Reconstruction and Conservative Differencing

Strategy: Approximate spatial derivatives using nearby stencils (set of grid points) in which solution is smoothest. Avoid differentiation across discontinuities (causes oscillations!). Consider 1D case (2D and 3D treated dimension by dimension):

1. discretize in space, defining point values as $q_i(t) = q(x_i,t)$ on mesh $\{x_i\}=\{i\Delta x\}$
2. write conservative difference formula $\frac{\partial q}{\partial t} + \frac{\partial F}{\partial x} = 0 \rightarrow \frac{\partial q_i}{\partial t} + \frac{F_{i+1/2} - F_{i-1/2}}{\Delta x} = 0$
3. evaluate $F_i = F(q_i)$ (at grid points)
4. reconstruct (approximate) $F_{i+1/2}$ and $F_{i-1/2}$ using values of $\{F_i\}$ as follows (for $F_{i+1/2}$)



a. form three alternative reconstructions, using three different stencils

b. evaluate "smoothness indicator" for each stencil and assign weight ω to associated reconstruction (small weights for stencils crossing discontinuities or steep gradients in q, large weights if solution is smooth inside stencil)

c. form weighted linear combination of alternative reconstructions

$$F_{i+1/2} = \omega_1 F^{(1)}_{i+1/2} + \omega_2 F^{(2)}_{i+1/2} + \omega_3 F^{(3)}_{i+1/2}$$

Wave Decomposition and Upwinding

Notice that stencils are biased to left, which is appropriate for waves travelling from left (upwind) to right (downwind). Similar treatment for waves travelling from right to left. But we have waves propagating in both directions, so we must decompose fields into these waves and identify associated wave speeds. Again consider 1D case.

Diagonalize A: $\frac{\partial q}{\partial t} + A\frac{\partial q}{\partial x} = 0 \quad \rightarrow \quad \frac{\partial (Rq)}{\partial t} + \left(RAR^{-1}\right)\frac{\partial (Rq)}{\partial x} = 0$

Eigenvalues are wave speeds, eigenvectors are transported by waves:

$$RAR^{-1} = \begin{bmatrix} +c & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} \sigma_{xz} - (\mu/c)v_z & \text{propagates to right at speed } +c \\ \sigma_{xz} + (\mu/c)v_z & \text{propagates to left at speed } -c \end{matrix}$$

(formally, these are Riemann invariants of hyperbolic system)

μ/c is shear impedence, and these are stress changes carried by plane S waves.

Split A into left- and right-going waves as $A=A^+ + A^-$, where

$$A^+ = R^{-1}\begin{bmatrix} +c & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}R \quad \text{and} \quad A^- = R^{-1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & -c & 0 \\ 0 & 0 & 0 \end{bmatrix}R$$

Now split $F=Aq$ into left- and right-going waves as $F=F^+ + F^-$, and perform upwind-biased reconstruction of F^+ (biased to left) and F^- (biased to right).

Boundary Conditions

We use similar wave decomposition when applying boundary conditions. Use one-sided reconstructions near boundaries (and faults). Preserve Riemann invariants (as calculated by finite difference solution of PDEs) propagating from medium into boundary, set amplitudes of outward propagating waves to satisfy boundary conditions. For example, consider 1D case with fault at x=0.

1. update all fields with finite difference solution of PDEs
2. preserve value of Riemann invariants propagating into fault

$$\sigma^-_{xz} - (\mu/c)v^-_z = \sigma^{-,FD}_{xz} - (\mu/c)v^{-,FD}_z \qquad \sigma^+_{xz} = \sigma^+_{xz} = \tau - \tau_0,\ V = v^+_z - v^-_z$$

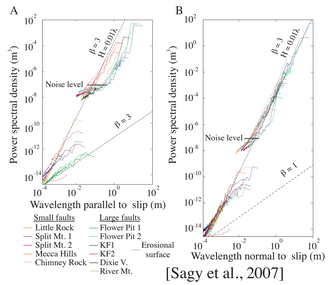
$$\sigma^+_{xz} + (\mu/c)v^+_z = \sigma^{+,FD}_{xz} + (\mu/c)v^{+,FD}_z \qquad \phi = \frac{1}{2}\left[\left(\sigma^{+,FD}_{xz} + \sigma^{-,FD}_{xz}\right) + (\mu/c)\left(v^{+,FD}_z - v^{-,FD}_z\right)\right]$$

$$\Rightarrow \tau = \tau_0 - (\mu/2c)V + \phi$$

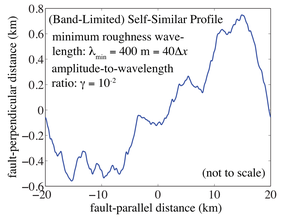
3. solve together with friction law: $\tau = f(V)\sigma$

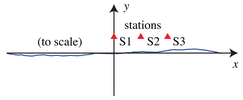
Dynamic Ruptures on Rough Faults

Faults are rough at all scales [Power and Tullis, 1991; Sagy et al., 2007] in a self-similar manner: amplitude-to-wavelength ratio of roughness, γ, is independent of scale of observation. Faults are rougher in direction perpendicular to slip ($\gamma=10^{-2}$) than parallel to slip ($\gamma=10^{-3}$). Antiplane ruptures experience more roughness, but in-plane slip on even slightly rough faults alters normal (and shear) stress. How does roughness affect rupture propagation? What are characteristics of high frequency ground motion and how are they related to γ, amount of slip, etc.?

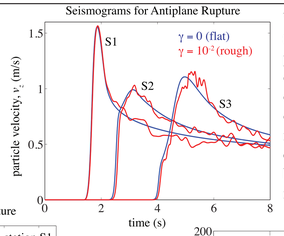


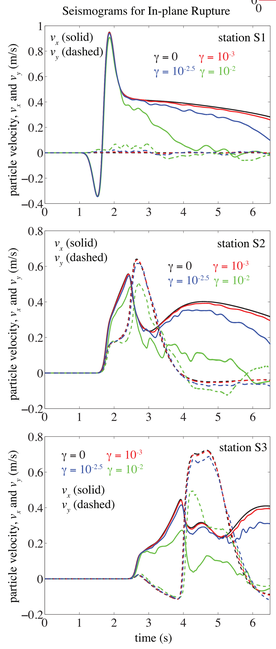
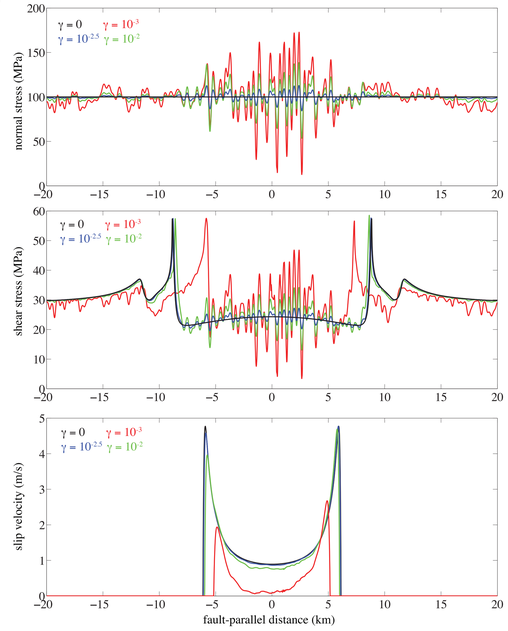
[Sagy et al., 2007]





Seismograms for Antiplane Rupture



We modeled antiplane and in-plane ruptures on rough faults (having band-limited self-similar profile). Antiplane rupture process was minimally altered by roughness, but high frequencies are evident in seismograms. In-plane rupture process was far more sensitive to roughness due to large changes in normal and shear stress. Changes will be even more extreme when shorter wavelength roughness is modeled. Caution: While these figures show only minor perturbations to rupture process at $\gamma=10^{-3}$ for in-plane case, this may change when using smaller cut-off wavelength!

Seismograms for In-plane Rupture