Lecture Notes 8

Buoyancy-Driven Flow

Introduction

Buoyancy forces due to temperature changes can be a driving mechanism for fluid flow. Because a geothermal gradient occurs in the subsurface, fluids at depth are hotter than fluids near the surface. The hotter fluids are less dense and lighter than the cooler fluids and tend to rise. Conversely, the cooler fluids are denser and heavier than the hotter fluids and tend to sink. Under certain conditions, buoyancy-driven flow can occur in the absence of topographic driving force. Such buoyancy-driven flow is known as thermal convection.

Darcy's Law in Terms of Pressure and Permeability

When changes in the fluid density are important, it is necessary to express Darcy's Law in terms of pressure instead of hydraulic head. For constant density, the relation between hydraulic head h and pressure p is

$$h = \frac{p}{\rho_f g} + z \tag{8-1}$$

where ρ_f is the fluid density, g is acceleration of gravity, and z is the elevation (above a given datum) of the point at which hydraulic head is measured. Substituting eq. (8-1) into Darcy's Law gives

$$q = -K\nabla\left(\frac{p}{\rho_f g} + z\right) \tag{8-2}$$

where K is hydraulic conductivity and is given by

$$K = \frac{\rho_f g k}{\mu} \tag{8-3}$$

with k being the permeability of the medium, and μ being the dynamic viscosity of the fluid. Because viscosity is a function of temperature, it is better to explicitly recognize it rather than lump it into the hydraulic conductivity. Thus, substituting (8-3) into (8-2) gives

$$q = -\frac{\rho_f g k}{\mu}\nabla\left(\frac{p}{\rho_f g} + z\right) \tag{8-4}$$

or equivalently

$$q = -\frac{\rho_f k}{\mu}\nabla\left(\frac{p}{\rho_f} + gz\right) \tag{8-5}$$

For a slightly compressible fluid, it is generally acceptable to assume that the spatial variation in fluid density is small so that (8-5) can be written as

$$q = -\frac{k}{\mu}\nabla (p + \rho_f g z) \tag{8-6}$$

Eq. (8-6) is the common form of Darcy's Law expressed in terms of pressure and permeability. The vector components of this equation are

$$q_x = -\frac{k}{\mu}\frac{\partial p}{\partial x} \tag{8-7a}$$

$$q_y = -\frac{k}{\mu}\frac{\partial p}{\partial y} \tag{8-7b}$$

$$q_z = -\frac{k}{\mu}\left(\frac{\partial p}{\partial z} + \rho_f g\right) \tag{8-7c}$$

Thermal Expansion of Fluid

The form of Darcy's Law in terms of permeability and pressure specifically recognize the role of fluid density. In particular, (8-7c) shows that fluid density changes will introduce a driving force in the vertical (z) direction. This is the buoyancy force. The relation between fluid density and temperature is described by the volumetric coefficient of thermal expansion of water, which is defined as the change in density per unit change in temperature, normalized by the original density:

$$\alpha_f = -\frac{1}{\rho_{fo}}\left(\frac{\rho_f - \rho_{fo}}{T - T_o}\right) \tag{8-8}$$

where ρ_{fo} is the original (or reference) density at the original (or reference) temperature of T_o. The negative sign signifies that density decreases as temperature increases. Eq. (8-8) is often written as

$$(\rho_f - \rho_{f_o}) = -\alpha_f \rho_{f_o} (T - T_o) \tag{8-8a}$$

Fluid and Heat Conservation

The conservation equations for fluid mass and heat have already been discussed in previous lectures. For fluid flow (saturated, single-phase), the conservation equation is

$$-\nabla \cdot (\rho_f q_x) = \frac{\partial(\rho_f n)}{\partial t} \tag{8-9}$$

or equivalently

$$-\frac{\partial(\rho_f q_x)}{\partial x}-\frac{\partial(\rho_f q_y)}{\partial y}-\frac{\partial(\rho_f q_z)}{\partial z} = \frac{\partial(\rho_f n)}{\partial t} \tag{8-10}$$

where n is porosity. For heat flow, the conservation equation is

$$\nabla \cdot (\kappa_{fs} \nabla T) - \nabla \cdot (\rho_f c_f q_x) = \frac{\partial(\rho_{fs} c_{fs} T)}{\partial t} \tag{8-11}$$

or equivalently

$$\frac{\partial}{\partial x}\left(\kappa_{fs}\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa_{fs}\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(\kappa_{fs}\frac{\partial T}{\partial z}\right)-\frac{\partial(\rho_f c_f q_x T)}{\partial x}-\frac{\partial(\rho_f c_f q_y T)}{\partial y}-\frac{\partial(\rho_f c_f q_z T)}{\partial z} = \frac{\partial(\rho_{fs} c_{fs} T)}{\partial t} \tag{8-12}$$

where κ is thermal conductivity, c is heat capacity, and the subscripts f and s indicate fluid and solid respectively.

Mathematical Statement of the Coupled Heat-Fluid Flow Problem

In a general sense, the mathematical statement of the coupled heat-flow problem consists of Darcy's Law (8-6) or (8-7), the relation for thermal expansion of water (8-8), the conservation equation for fluid flow (8-9) or (8-10), and the conservation equation for heat flow (8-11) or (8-12). If we neglect the variation in fluid viscosity with temperature, there are 6 equations in 6 unknowns (q_x, q_y, q_z, ρ_f, p, and T). If we want to consider the change of viscosity with temperature, then the viscosity is an additional unknown, and the additional equation would be the viscosity-temperature relationship.

To reduce the number of equations and unknowns, we can substitute the specific discharge vector q as given by eq. (8-6) into the conservation equations (8-9) and (8-11), or equivalently, substitute (8-7) into (8-10) and (8-12). Assuming that viscosity is constant, this reduces the problem to 3 equations in 3 unknowns (ρ_f, p, and T). However, the complexity of the system is not reduced, and solving these coupled set of equations generally require numerical methods.

The Boussinesq Approximation

An approximate treatment of the coupled equations of heat and fluid flow is to assume a constant fluid density everywhere except for the buoyancy term in (8-7c) and (8-8). This is known as the Boussinesq approximation. In the following discussion, we further assume that the medium is homogeneous (uniform permeability and porosity).

For the above assumption, the fluid flow conservation equation (8-10) becomes,

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = 0 \tag{8-13}$$

The heat flow conservation equation (8-12) is simplified to

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) - \frac{\rho_f c_f}{\kappa_{fs}}\left[q_x \frac{\partial T}{\partial x} + q_y \frac{\partial T}{\partial y} + q_z \frac{\partial T}{\partial z}\right] = \frac{\rho_{fs} c_{fs}}{\kappa_{fs}} \frac{\partial T}{\partial t} \tag{8-14}$$

Eq. (8-13) and (8-14) are identical to the conservation equations for constant density case. However, there is still a buoyancy effect in (8-7c). We shall see how this comes into play in the next section.

Linearized Stability Analysis

The analysis of the onset of thermal convection is a triumph in the application of classical mathematical techniques. We follow a brief treatment here. Consider a porous layer of thickness b as show in Figure 8-1. The top boundary is impervious to flow and is kept at a temperature T_{top}. The bottom boundary is also impervious to flow and is kept at the temperature T_{bot} (with T_{bot} > T_{top}). We assume that the layer is laterally extensive so we ignore the boundary conditions on the sides.

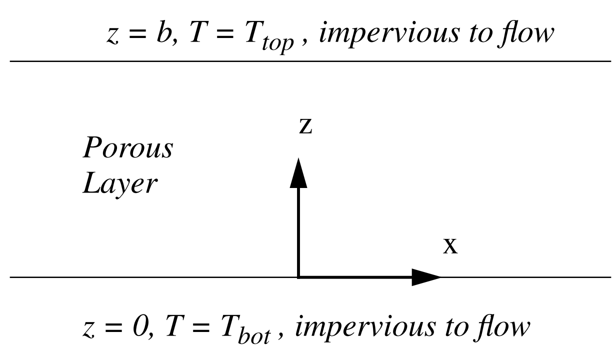

Figure 8-1. A porous layer heated from below.

If no thermal convection occurs, then q_x and q_z are both zero, and the temperature distribution is

$$T = T_{bot} - \left(\frac{T_{bot} - T_{top}}{b}\right) z \tag{8-15}$$

One can imagine that as T_{bot} increases, there will come a point when thermal convection will develop. At the point where convection begins, the temperature distribution can be written as

$$T = T_{bot} - \left(\frac{T_{bot} - T_{top}}{b}\right)z + T' \qquad (8\text{-}16)$$

where *T'* is the temperature perturbation from the non-convective (stable) case. Similarly, there will be a small perturbation in specific discharge components, and we express them as

$$q_x = 0 + q_x' = q_x' \qquad (8\text{-}17)$$

$$q_z = 0 + q_z' = q_z' \qquad (8\text{-}18)$$

The primed quantities (T', q_x', and q_z') are known as perturbation quantities. Substituting (8-17) and (8-18) into (8-7a), (8-7c) and (8-13) gives (for 2-D cases)

$$q_x' = -\frac{k}{\mu}\frac{\partial p}{\partial x} \qquad (8\text{-}19)$$

$$q_z' = -\frac{k}{\mu}\left(\frac{\partial p}{\partial z} + \rho_f g\right) \qquad (8\text{-}20)$$

$$\frac{\partial q_x'}{\partial x} + \frac{\partial q_z'}{\partial z} = 0 \qquad (8\text{-}21)$$

Substituting (8-16) through (8-18) into (8-14) yields, for the two-dimensional case

$$\left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial z^2}\right) - \frac{\rho_f c_f}{\kappa_{fs}}\left[q_x'\frac{\partial T'}{\partial x} + q_z'\frac{\partial T'}{\partial z} - q_z'\left(\frac{T_{bot} - T_{top}}{b}\right)\right] = \frac{\rho_{fs} c_{fs}}{\kappa_{fs}}\frac{\partial T'}{\partial t} \qquad (8\text{-}22)$$

Next, the standard treatment in a perturbation analysis is to neglect products of perturbation quantities. Thus, neglecting $q_x'\,\partial T'/\partial x$ and $q_z'\,\partial T'/\partial z$ leads to

$$\left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial z^2}\right) + \frac{\rho_f c_f}{\kappa_{fs}} q_z'\left(\frac{T_{bot} - T_{top}}{b}\right) = \frac{\rho_{fs} c_{fs}}{\kappa_{fs}}\frac{\partial T'}{\partial t} \qquad (8\text{-}23)$$

Also, we assume that at the onset of convection, the term $\partial T'/\partial t$ is small so we neglect it on the right hand side. This yields

$$\left(\frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial z^2}\right) + \frac{\rho_f c_f}{\kappa_{fs}} q_z'\left(\frac{T_{bot} - T_{top}}{b}\right) = 0 \qquad (8\text{-}24)$$

Now, it is possible to eliminate q_z' by a series of differentiations and substitutions. First we take the second partial derivative of (8-24) with respect to *x*. This gives

$$\left(\frac{\partial^4 T'}{\partial x^4} + \frac{\partial^4 T'}{\partial x^2 \partial z^2}\right) + \frac{\rho_f c_f}{\kappa_{fs}}\left(\frac{T_{bot} - T_{top}}{b}\right)\frac{\partial^2 q_z'}{\partial x^2} = 0 \qquad (8\text{-}25)$$

Next, we take the second partial derivative of (8-24) with respect to z. This gives

$$\left(\frac{\partial^4 T'}{\partial x^2 \partial z^2} + \frac{\partial^4 T'}{\partial z^4}\right) + \frac{\rho_f c_f}{\kappa_{fs}}\left(\frac{T_{bot} - T_{top}}{b}\right)\frac{\partial^2 q_z'}{\partial z^2} = 0 \tag{8-26}$$

Adding eq. (8-25) and (8-26) gives

$$\left(\frac{\partial^4 T'}{\partial x^4} + 2\frac{\partial^4 T'}{\partial x^2 \partial z^2} + \frac{\partial^4 T'}{\partial z^4}\right) + \frac{\rho_f c_f}{\kappa_{fs}}\left(\frac{T_{bot} - T_{top}}{b}\right)\left(\frac{\partial^2 q_z'}{\partial x^2} + \frac{\partial^2 q_z'}{\partial z^2}\right) = 0 \tag{8-27}$$

Now we work on Darcy's Law. Taking the partial derivative of (8-19) with respect to z yields

$$\frac{\partial q_x'}{\partial z} = -\frac{k}{\mu}\frac{\partial^2 p}{\partial x \partial z} \tag{8-28}$$

Taking the partial derivative of (8-20) with respect to x yields

$$\frac{\partial q_z'}{\partial x} = -\frac{k}{\mu}\left(\frac{\partial^2 p}{\partial z \partial x} + g\frac{\partial \rho_f}{\partial x}\right) \tag{8-29}$$

Subtracting (8-29) from (8-28) yields

$$\frac{\partial q_x'}{\partial z} - \frac{\partial q_z'}{\partial x} = \frac{gk}{\mu}\frac{\partial \rho_f}{\partial x} \tag{8-30}$$

Substituting (8-8a) into (8-30) gives

$$\frac{\partial q_x'}{\partial z} - \frac{\partial q_z'}{\partial x} = -\frac{gk\alpha_f \rho_{fo}}{\mu}\frac{\partial T'}{\partial x} \tag{8-31}$$

Now we take the partial derivative of (8-31) with respect to x. This yields

$$\frac{\partial^2 q_x'}{\partial x \partial z} - \frac{\partial^2 q_z'}{\partial x^2} = -\frac{gk\alpha_f \rho_{fo}}{\mu}\frac{\partial T'}{\partial x^2} \tag{8-32}$$

Next we take the partial derivative of (8-21) with respect to z. This yields

$$\frac{\partial^2 q_x'}{\partial x \partial z} + \frac{\partial^2 q_z'}{\partial z^2} = 0 \tag{8-33}$$

By subtracting (8-32) from (8-33), we obtain

$$\frac{\partial^2 q_z'}{\partial x^2} + \frac{\partial^2 q_z'}{\partial z^2} = \frac{gk\alpha_f \rho_{fo}}{\mu}\frac{\partial T'}{\partial x^2} \tag{8-34}$$

The final step is to substitute (8-34) into (8-27).

$$\left(\frac{\partial^4 T'}{\partial x^4} + 2\frac{\partial^4 T'}{\partial x^2 \partial z^2} + \frac{\partial^4 T'}{\partial z^4}\right) + \frac{\rho_f^2 g k c_f \alpha_f (T_{bot} - T_{top})}{\mu \kappa_{fs} b} \frac{\partial T'}{\partial x^2} = 0 \tag{8-35}$$

We now have a single equation in terms of the temperature perturbation. We can normalize the x and z coordinates by defining

$$x_D = \frac{x}{b} \tag{8-36}$$

$$z_D = \frac{z}{b} \tag{8-37}$$

so that (8-35) is of the form

$$\left(\frac{\partial^4 T'}{\partial x_D^4} + 2\frac{\partial^4 T'}{\partial x_D^2 \partial z_D^2} + \frac{\partial^4 T'}{\partial z_D^4}\right) + \frac{\rho_f^2 g k c_f \alpha_f b (T_{bot} - T_{top})}{\mu \kappa_{fs}} \frac{\partial T'}{\partial x_D^2} = 0 \tag{8-38}$$

The dimensionless grouping in the equation is known as the Rayleigh number:

$$\mathrm{Ra} = \frac{\rho_f^2 g k c_f \alpha_f b (T_{bot} - T_{top})}{\mu \kappa_{fs}} \tag{8-39}$$

and is a measure of the tendency of the fluid in the porous medium to develop thermal convection. From additional theoretical analysis, it is shown that when $\mathrm{Ra} > 4\pi^2$, thermal convection will likely occur.