Lecture Notes 7

Effect of Ground-Water Flow on Heat Transport

Introduction

In this lecture, we continue the discussion of heat transport in the presence of ground-water flow. We make the assumptions that (1) the hydraulic driving force (for example, due to topography) on the water is significantly stronger than the buoyancy force due to temperature change, and (2) we can neglect the change in fluid viscosity due to temperature change. These assumptions are reasonable in situations involving active ground-water flow and where changes in temperature are small. We have already seen an one-dimensional analysis based on these assumptions in the paper by Bredehoeft and Papadopulos (1965). In this lecture, we extend the discussion to two-dimensions.

Governing Equations

The flow of ground water and heat involves two governing equations, one for water and one for heat. If we assume that temperature-induced changes are negligible, then we can write the ground-water flow equation in terms of hydraulic head and hydraulic conductivity:

$$\nabla \cdot (K \nabla h) = S_s \frac{\partial h}{\partial t} \tag{7-1}$$

The heat flow equation is

$$\nabla \cdot (\kappa_{ws} \nabla T) - \nabla \cdot (q \rho_w c_w T) = \rho_{ws} c_{ws} \frac{\partial T}{\partial t} \tag{7-2}$$

where

T = temperature
κ_{ws} = thermal conductivity of the water-solid complex
ρ_{ws} = density of the water-solid complex
c_{ws} = specific heat of the water-solid complex
ρ_w = density of water
c_w = specific heat of water
q = specific discharge vector

Equations (7-1) and (7-2) are “coupled” in the sense the specific discharge vector q in (7-2) is obtained from the hydraulic head h in (7-1) through Darcy’s Law:

$$q = -K \nabla h \tag{7-3}$$

However, in the present analysis, the coupling is only "one-way" in that ground-water flow affects heat flow, but heat-flow does not affect ground-water flow. This simplification allows one to solve the ground-water/head flow problem in two steps. First, the ground-water flow equation (7-1) is solved to obtain hydraulic head h. Then, the specific discharge vector q is obtained from Darcy's Law (7-3). Finally, the heat flow equation (7-2) is solved to obtain temperature T.

Two Dimensional, Steady-State Flow in Homogeneous Medium

For two-dimensional, steady-state flow in a homogeneous medium, equations (7-1) through (7-3) take the forms:

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial z^2} = 0 \tag{7-4}$$

$$\kappa_{ws}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) - \rho_w c_w\left(q_x \frac{\partial T}{\partial x} + q_z \frac{\partial T}{\partial z}\right) = 0 \tag{7-5}$$

$$q_x = -K\frac{\partial h}{\partial x} \tag{7-6a}$$

$$q_z = -K\frac{\partial h}{\partial z} \tag{7-6b}$$

One might wonder how did q_x and q_y come outside the partial derivatives? The answer is that $\nabla \cdot qT = q \cdot (\nabla T) + T(\nabla \cdot q)$. However, under steady flow, $\nabla \cdot q = 0$ and so we are left with $\nabla \cdot qT = q \cdot (\nabla T)$.

For flow domains of simple geometry and boundary conditions, (7-4) can sometimes be solved to yield an analytical solution. However, when q_x and q_z are computed and substituted into (7-5), the resultant heat flow equation cannot usually be solved analytically. Thus, even for simple problems, numerical methods are needed. However, Domenico and Palciauskas (1973) has developed an approximate analytical solution. We consider this solution next.

Approximate Analytical Solution by Domenico and Palciauskas (1973)

Figure 7-1 illustrates the rectangular flow domain analyzed by Domenico and Palciauskas. The governing equations are (7-4) through (7-6) above. For water flow, the boundary conditions are specifies a sinusoidal varying head at the top boundary

$$h = A - B\cos\left(\frac{\pi x}{L}\right) \qquad \text{at } z = H,\ 0 < x < L \tag{7-7}$$

where B is the amplitude of the sinusoid, no water flow across the vertical sides:

$$\frac{\partial h}{\partial x} = 0 \qquad \text{at } x = 0 \text{ and } x = L,\ 0 < z < H \tag{7-8}$$

and no water flow at the bottom boundary

$$\frac{\partial h}{\partial z} = 0 \qquad \text{at } z = 0,\ 0 < x < L \tag{7-9}$$

For heat flow, the boundary conditions are specified temperature (T_{top}) at the top boundary:

$$T = T_{top} \qquad \text{at } z = H,\ 0 < x < L \tag{7-10}$$

no heat flow across the vertical sides

$$\frac{\partial T}{\partial x} = 0 \qquad \text{at } x = 0 \text{ and } x = L,\ 0 < z < H \tag{7-11}$$

and a specified uniform conductive heat flux (F) at the bottom boundary

$$-\kappa_{ws}\frac{\partial T}{\partial z} = F \tag{7-12}$$

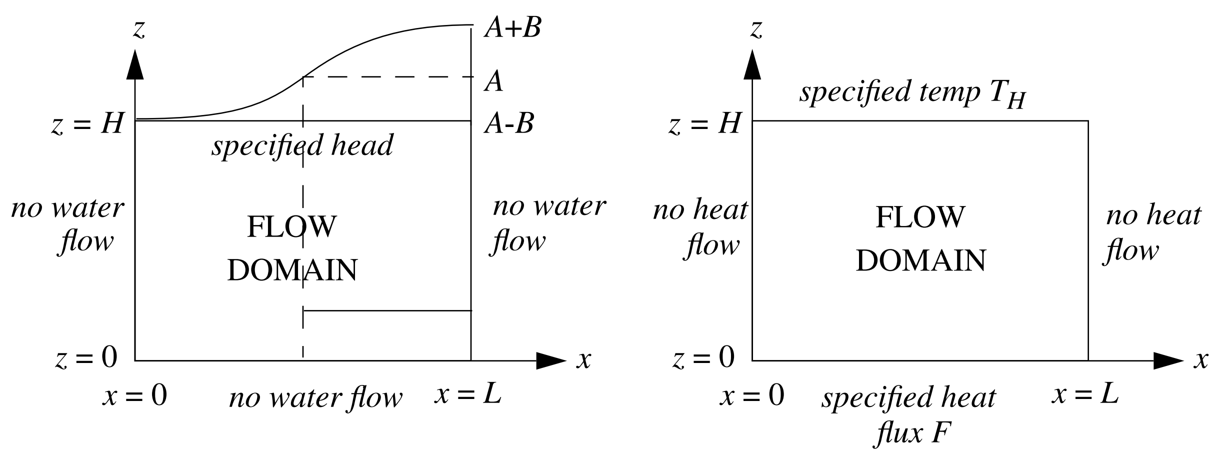

Figure 7-1. Flow domain and boundary conditions of problem analyzed by Domenico and Palciauskas (1973).

The solution for the water flow problem is simply:

$$h(x, z) = A - B\left[\frac{\cosh(\pi z/L)\cos(\pi x/L)}{\cosh(\pi H/L)}\right] \tag{7-13}$$

The components of specific discharge vector are, according to (7-6a) and (7-6b):

$$q_x = -K\frac{\partial h}{\partial x} = -\frac{\pi BK}{L}\left[\frac{\cosh(\pi z/L)\sin(\pi x/L)}{\cosh(\pi H/L)}\right] \tag{7-14}$$

$$q_z = -K\frac{\partial h}{\partial z} = \frac{\pi BK}{L}\left[\frac{\sinh(\pi z/L)\cos(\pi x/L)}{\cosh(\pi H/L)}\right] \tag{7-15}$$

To solve the heat flow equation (6-5), Domenico and Palciauskas (1973) introduced the following approximation: the components of the temperature gradient $\partial T/\partial x$ and $\partial T/\partial z$ are approximated by the corresponding quantities in the absence of ground-water flow (that is, heat flow under pure conduction). Now, in the absence of ground-water flow, the components of the temperature gradient that satisfy the heat flow boundary conditions (7-10) through (7-12) are:

$$\frac{\partial T}{\partial x} = 0 \tag{7-16}$$

$$\frac{\partial T}{\partial z} = -\frac{F}{\kappa_{ws}} \tag{7-17}$$

Therefore, the heat flow equation (7-5) is approximated as

$$\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{\rho_w c_w \pi BKF}{\kappa_{ws}^2 L}\left[\frac{\sinh(\pi z/L)\cos(\pi x/L)}{\cosh(\pi H/L)}\right] = 0 \tag{7-18}$$

The solution of (7-18) that satisfies boundary conditions (7-10) through (7-12) turns out to be

$$T(x, z) = T_{top} + \frac{F}{\kappa_{ws}}(H - z) +$$

$$\frac{\rho_w c_w BKFH}{2\kappa_{ws}^2}\left\{\frac{\cos(\pi x/L)}{\cosh(\pi H/L)}\left[\frac{(H-z)}{H}\cosh(\pi z/L) + \frac{L}{\pi H}\frac{\sinh[\pi(z-H)/L]}{\cosh(\pi H/L)}\right]\right\} \tag{7-19}$$

Note that the first two terms represents the temperature distribution in the absence of flow (that is, under pure conduction). Therefore, it is the third term that represents the effect of ground water flow on heat transport (convection). If we write eq. (7-19) as

$$T(x, z) = T_{top} + \frac{FL}{\kappa_{ws}}\left[\frac{(H-z)}{L} + \frac{\rho_w c_w BKH}{2\kappa_{ws} L} \times \{\text{term involving cos etc.}\}\right] \tag{7-20}$$

The dimensionless quantity

$$\frac{\rho_w c_w BKH}{\kappa_{ws} L} \tag{7-21}$$

has the form of a Peclet number, which can be interpreted as a measure of the relative magnitudes of convective versus conductive transport. Recall the quantities in the Peclet number are

- ρ_w = density of water
- c_w = specific heat of water
- B = one half the vertical relief of the water table across the top boundary
- K = hydraulic conductivity
- H = thickness of the flow region
- κ_{ws} = thermal conductivity of the water-solid complex
- L = horizontal length of the flow region
- c_{ws} = specific heat of the water-solid complex

Through computational experimentation, Domenico and Palciauskas (1973) found that when the Peclet number is substantially less than 1, conductive transport dominates. Conversely, when the Peclet number is substantially greater than 1, then convective transport dominates.

References

Bredehoeft, J.D., and Papadopulos, 1965, Rates of vertical groundwater movement estimated from the Earth's thermal profile, *Water Resources Research,* v. 1, p. 325-328

Domenico, P. A., and Palciauskas, V. V., 1973, Theoretical analysis of forced convective heat transfer in regional ground-water flow, *Geological Society of America Bulletin,* v. 84, p. 3803-3814,