Lecture Notes 5

Transient Flow of Ground Water

Introduction

When we derived the ground-water flow equation in Lecture 1, we focused on Darcy's Law and glossed over the concept of specific storage. However, to understand the mechanism of specific storage, it is necessary to revise our earlier derivation of the ground-water flow equation. In particular, we consider the deformation of the porous medium, and the constitutive properties of the fluid and the porous medium.

Conservation Equation in Deforming Coordinated

Recall that the principal of fluid mass conservation is a key ingredient in the derivation of the ground-water flow equation. As before, we consider an elemental control volume as shown in Figure 5-1 (this is same as Figure 1-5 of Lecture Notes 1). In the previous analysis, the control volume is assumed fixed in space. In the current analysis, we consider the control volume to represent a fixed mass of porous medium. In other words, the control volume will deform as fluid moves into and out of the volume.

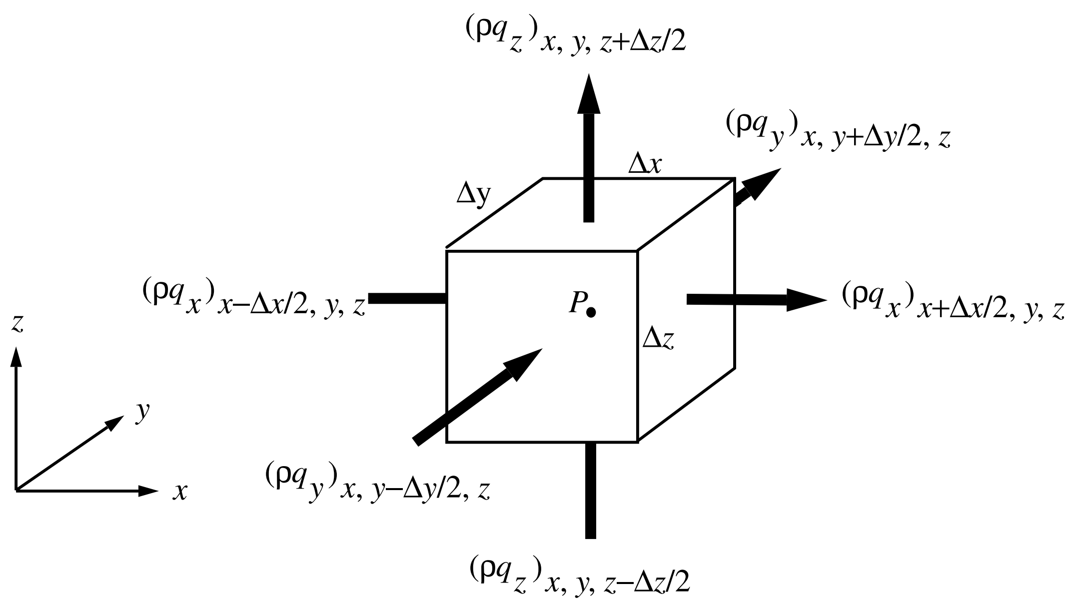

Figure 5-1. Control volume of dimension Δx, Δy, Δz, and centered at point P: (x, y, z). Arrows indicate flows into and out of the control volume. This is same as Figure 1-5 of Lecture Notes 1.

As before, the rate at which fluid mass enters the control volume is

$$(\rho q_x)_{x-\Delta x/2,\,y,\,z}\Delta y\Delta z + (\rho q_y)_{x,\,y-\Delta y/2,\,z}\Delta x\Delta z + (\rho q_z)_{x,\,y,\,z-\Delta z/2}\Delta x\Delta y$$

and the rate at which fluid mass leaves the control volume is

$$(\rho q_x)_{x+\Delta x/2,\,y,\,z}\Delta y\Delta z + (\rho q_y)_{x,\,y+\Delta y/2,\,z}\Delta x\Delta z + (\rho q_z)_{x,\,y,\,z+\Delta z/2}\Delta x\Delta y$$

For a deforming control volume, however, the rate at which fluid mass accumulates in the elemental control volume is written as

$$\frac{\partial(\rho n\Delta x\Delta y\Delta z)}{\partial t}$$

Note that it is necessary to keep the terms Δx, Δy, and Δz inside the time derivative, because the dimension of the control volume may change with time.

Invoking the principle of conservation of fluid mass, we have (analogous to Eq. 1-7)

$$\frac{(\rho q_x)_{x-\Delta x/2,\,y,\,z} - (\rho q_x)_{x+\Delta x/2,\,y,\,z}}{\Delta x} + \frac{(\rho q_y)_{x,\,y-\Delta y/2,\,z} - (\rho q_y)_{x,\,y+\Delta y/2,\,z}}{\Delta y} +$$

$$\frac{(\rho q_z)_{x,\,y,\,z-\Delta z/2} - (\rho q_z)_{x,\,y,\,z+\Delta z/2}}{\Delta z} = \frac{1}{\Delta x\Delta y\Delta z}\frac{\partial(\rho n\Delta x\Delta y\Delta z)}{\partial t} \tag{5-1}$$

To further develop eq. (5-1), we have to consider the behaviors of the fluid and the porous medium.

Constitutive Laws of Fluid and Porous Medium

A constitutive law or equation is a description of the mechanical behavior of an idealized material, e.g., an equation that describes the manner in which a linearly elastic solid deforms when it is subjected to certain stresses. For the analysis of fluid flow in a porous medium, we need to describe the compressibility of the fluid and the porous medium. The fluid compressibility, denoted by β, describes the volume change of a fluid when it is subjected to a pressure change, and is defined as

$$\beta = -\frac{1}{V_w}\frac{dV_w}{dp} \tag{5-2}$$

where V_w is the initial volume of the fluid, and dV_w is the change in volume when the fluid is subjected to a pressure change of dp. In general, the fluid compressibility is dependent on pressure and temperature. However, we will assume that it is constant. This assumption is reasonable if the flow system is under constant temperature, and the pressure change is not great. If M is the mass of the fluid of initial volume V_w, and since the mass remains constant when the fluid is compressed, we can write eq. (5-2) as

$$\beta = -\frac{M}{V_w}\frac{d(V_w/M)}{dp} \tag{5-3}$$

Recognizing that the fluid density is $\rho = M/V_w$, we can write eq. (5-3) as

$$\beta = \frac{1}{\rho}\frac{d\rho}{dp} \tag{5-4}$$

This equation for the fluid compressibility will be used below in the development of the ground-water flow equation.

Next, we need to consider the compressibility of the porous medium. Here, we accept the standard assumption that the medium deforms only in the vertical direction. (We consider three-dimensional deformation later in the class.) For now, the quantity of interest is the vertical compressibility of the medium, denoted by α_v. To define this quantity, we first have to review the concept of effective stress.

If a normal stress σ is applied to a saturated porous medium, a portion of this stress will be borne by the pressure p of the fluid in the pore space, and the remainder of the stress is borne by the skeletal matrix of the porous medium. The portion of the stress borne by the porous medium is called the effective stress, σ'. Thus we can write

$$\sigma = p + \sigma' \tag{5-5}$$

In particular, we are interested in normal stress in the vertical direction. If σ_z is the vertical normal stress, and σ_z' is the vertical effective stress, then we have

$$\sigma_z = p + \sigma_z' \tag{5-6}$$

The vertical compressibility of the porous medium is defined as

$$\alpha_v = -\frac{1}{V_T}\frac{dV_T}{d\sigma_z'} \tag{5-7}$$

where V_T is the initial (bulk) volume of the porous medium, and dV_T is the change in volume when the medium is subjected to a change in vertical effective stress of $d\sigma_z'$. Note that the compression occurs under a condition of no horizontal deformation.

The vertical compressibility can be measured in the laboratory by a loading cell in a manner described by Freeze and Cherry (1979, p. 55). As shown on Fig. 5-2, the cell is fitted with pistons at both ends. As the force is applied on the pistons, the medium is compressed, but water in the medium is allowed to drain out through vents to an external pool held at constant pressure.

Since the fluid pressure in the porous medium is kept constant, the applied stress is entirely borne by the effective stress. In other words, the experimental arrangement serves to measure the change in volume of the medium due to a change in effective stress.

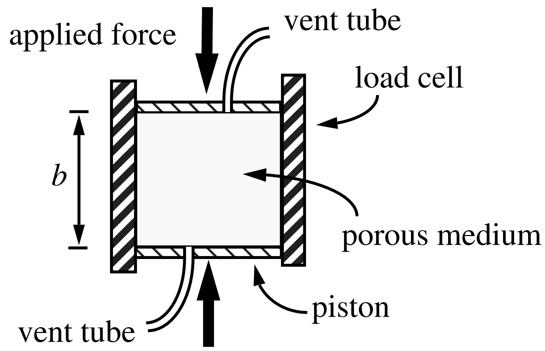

Figure 5-2. Laboratory loading cell used to determine the vertical compressibility of a porous medium

Eq. (5-7) can be written in two different ways. First, consider the assumption of no horizontal deformation. If we take a cylindrical sample of porous medium and compress it without allowing horizontal deformation, then the change in volume per unit volume of the sample is the same as the change in sample height per unit sample height. Thus, we can write (5-7) as

$$\alpha_v = -\frac{1}{b}\frac{db}{d\sigma_z'} \tag{5-8}$$

where b is the initial sample height and db is the change in sample height.

To develop another expression for α_v, we note that the compressibility of the porous medium is due to the result of two mechanisms: a reduction of the volume of the grains of the medium, and the rearrangement of the grains to a more closely packed configuration. For granular media, the second mechanism is the dominant mechanism of volume change. If we write

$$V_T = V_s + V_w \tag{5-9}$$

where V_s is the volume of grains and V_w is the volume of fluid in the pore space, then

$$dV_T = dV_s + dV_w = dV_w \tag{5-10}$$

as $dV_s = 0$. Substituting (5-10) into (5-7) leads to

$$\alpha_v = -\frac{1}{V_T}\frac{dV_w}{d\sigma_z'} \tag{5-11}$$

Noting that V_w/V_T is the porosity n, we can rewrite eq. (5-11) can be rewritten as (try it)

$$\alpha_v = -\frac{1}{1-n}\frac{dn}{d\sigma_z{}'} \qquad (5\text{-}12)$$

Now we introduce a second major assumption, namely, that the total vertical stress remains constant. This assumption is reasonable when the aquifer is not subjected to loading. Under this assumption, eq. (5-6) can be written as

$$d\sigma_z = d\sigma_z{}' + dp = 0 \qquad (5\text{-}13)$$

in which case we have

$$d\sigma_z{}' = -dp \qquad (5\text{-}14)$$

and eq. (5-8) and (5-12) become

$$\alpha_v = \frac{1}{b}\frac{db}{dp} \qquad (5\text{-}15)$$

and

$$\alpha_v = \frac{1}{1-n}\frac{dn}{dp} \qquad (5\text{-}16)$$

We will use eqs. (5-15) and (5-16) in the derivation of the equation of ground-water flow.

Revised Derivation of the Equation of Ground-Water Flow

We now have all the ingredients rederiving the equation of ground-water flow. We begin by considering the term on the right hand side of the conservation equation (5-1). Assuming no horizontal deformation (that is, Δx and Δy stay constant), we have

$$\frac{1}{\Delta x\Delta y\Delta z}\frac{\partial(\rho n\Delta x\Delta y\Delta z)}{\partial t} = \frac{1}{\Delta z}\frac{\partial(\rho n\Delta z)}{\partial t} \qquad (5\text{-}17)$$

Applying the chain rule, we have

$$\frac{1}{\Delta z}\frac{\partial}{\partial t}(\rho n\Delta z) = \frac{1}{\Delta z}\frac{d}{dp}(\rho n\Delta z)\frac{\partial p}{\partial t} = \left(n\frac{d\rho}{dp} + \rho\frac{dn}{dp} + \frac{\rho n}{\Delta z}\frac{d\Delta z}{dp}\right)\frac{\partial p}{\partial t} \qquad (5\text{-}18)$$

If we substitute eqs. (5-4), (5-16), and (5-15) into the respective terms in the parentheses on the right hand side, and identifying Δz with the sample height b, we obtain

$$\frac{1}{\Delta z}\frac{\partial}{\partial t}(\rho n\Delta z) = \rho(n\beta + \alpha_v)\frac{\partial p}{\partial t} \qquad (5\text{-}19)$$

Next, noting that $p = \rho g(h - z)$, we can write (5-18) as

$$\frac{1}{\Delta z}\frac{\partial}{\partial t}(\rho n \Delta z) = \rho^2 g\,(n\beta + \alpha_v)\frac{\partial h}{\partial t} \tag{5-20}$$

Substituting eq. (5-20) back into (5-1) and letting Δx, Δy, and Δz approach zero, we obtain

$$-\frac{\partial \rho q_x}{\partial x} - \frac{\partial \rho q_y}{\partial y} - \frac{\partial \rho q_z}{\partial z} = \rho^2 g\,(n\beta + \alpha_v)\frac{\partial h}{\partial t} \tag{5-21}$$

For the left hand side of (5-21), we note that

$$\frac{\partial \rho q_x}{\partial x} = \rho\frac{\partial q_x}{\partial x} + q_x\frac{\partial \rho}{\partial x} \tag{5-22}$$

As before, we assume that the fluid is only slightly compressible, so the second term on the right hand side of eq. (5-22) can be neglected. Thus eq. (5-21) becomes

$$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} = \rho g\,(n\beta + \alpha_v)\frac{\partial h}{\partial t} \tag{5-23}$$

If we substitute Darcy's law, expressed as

$$\begin{aligned} q_x &= -K\frac{\partial h}{\partial x} \\ q_y &= -K\frac{\partial h}{\partial y} \\ q_z &= -K\frac{\partial h}{\partial z} \end{aligned} \tag{5-24}$$

into eq. (5-23) and define the specific storage as

$$S_s = \rho g\,(n\beta + \alpha_v) \tag{5-25}$$

we obtain the equation of ground-water flow

$$\frac{\partial}{\partial x}\left(K\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial h}{\partial y}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right) = S_s\frac{\partial h}{\partial t} \tag{5-26}$$

References

Freeze, R. A., and Cherry, J. A., 1979, *Groundwater*, Prentice-Hall, Englewood Cliffs, NJ, 604 p.