Lecture Notes 2

Solution of Ground-Water Flow Problems

Mathematical Statement of a Ground-Water Flow Problem

A complete statement of a ground-water flow problem requires specifying: (a) the extent of the flow domain, (b) the governing equation, (c) spatial distribution of properties, for example, hydraulic conductivity and specific storage, (d) boundary conditions, and (e) initial condition (for transient problems). In general, specifying the flow domain might be a trivial or a major question in formulating the ground-water flow problem. The governing equation is the ground-water flow equation, expressed as

$$\frac{\partial}{\partial x}\left(K\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right) = S_s\frac{\partial h}{\partial t} \tag{2-1}$$

The boundary conditions specify the flow conditions at the boundary of the domain. As noted in the previous notes, the two common boundary conditions are (a) specified head and (b) specified flow. The initial condition defines the spatial distribution of hydraulic head everywhere in the flow domain at the initial time.

Example Problem

To illustrate the formulation of a ground-water flow problem, consider the hypothetical case of a closed, fluid reservoir as illustrated in vertical section by Figure 2-1. The reservoir is rectangular in shape, has a uniform hydraulic conductivity K and specific storage S_s, and is bounded on all sides by impervious material. A vertical fault divides the reservoir into two compartments, of width L_1 on the left, and L_2 on the right. The fault gouge is sufficiently low in hydraulic conductivity that it acts as a flow barrier. Thus, the left compartment is hydraulically isolated from the right compartment. Initially, the hydraulic head is h_1 in the left compartment and h_2 in the right compartment.

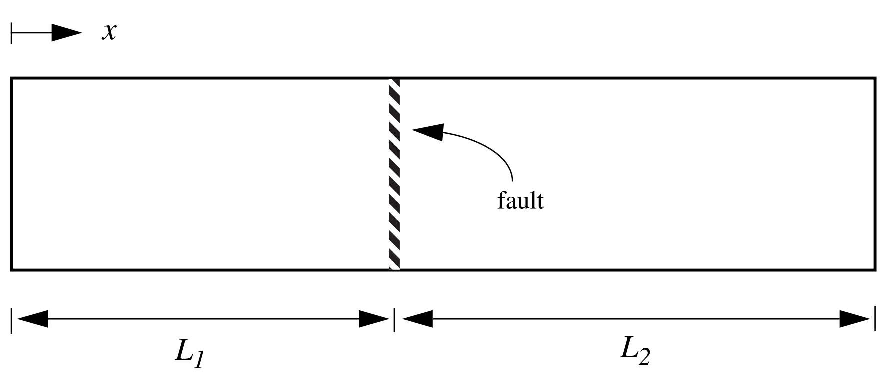

Figure 2-1. Vertical-sectional view of a rectangular reservoir bounded on all sides by impervious boundaries. A vertical fault divides the reservoir into two compartments.

Assume that at time $t = 0$, the fault is ruptured by an earthquake, so that the two compartments are now hydraulically connected. The earthquake would deform the reservoir, causing changes in hydraulic head--however, we neglect this mechanism for now. The question we want to answer is: What happens to the hydraulic-head distribution in the reservoir after the fault ruptures?

The mathematical statement of the above problem can be expressed as follows. When the fault ruptures, we expect fluid to flow form the compartment with higher head to the compartment with lower head. In other words, flow would occur in the x direction (see Figure 2-1). If we set the origin of the x axis at the left hand boundary, then the flow domain is

$$0 < x < L_1 + L_2$$

For one-dimensional flow in a homogeneous reservoir, the governing equation is

$$\frac{\partial^2 h}{\partial x^2} = \frac{S_s}{K}\frac{\partial h}{\partial t} \tag{2-2}$$

The boundary conditions are

$$\frac{\partial h}{\partial x} = 0 \quad \text{at} \quad x = 0 \tag{2-3}$$

and

$$\frac{\partial h}{\partial x} = 0 \quad \text{at} \quad x = L_1 + L_2 \tag{2.4}$$

Finally, the initial condition is

$$h(x, 0) = h_1 \quad \text{for} \quad 0 \le x \le L_1 \tag{2-5}$$

$$h(x, 0) = h_2 \quad \text{for} \quad L_1 < x \le L_1 + L_2 \tag{2-6}$$

Eqs. (2-2) to (2-6) constitute the mathematical statement of the fluid flow problem for this example.

Solution Methods

There are many methods to solve flow problems. In general they can be classified into three categories: analytical methods, semi-analytical methods, and numerical methods. Analytical methods encompass all the classical techniques such as separation of variables, Laplace transform, and Fourier transform. For steady-state problems ($\partial h/\partial t = 0$) in a two-dimensional domain, complex analysis can also be effectively employed. Analytical methods yield "closed form" or "analytical" solutions, that is, mathematical expressions that can be written on paper. An advantage of analytical solutions is that they can be examined to understand the behavior of the flow system. However, the applicability of analytical methods are typically limited to homogeneous flow domain with relatively simple geometry.

Semi-analytical methods are ones in which the mathematical problem can be partially solve by an analytical approach, and the rest is solved numerically (using a computer). A popular technique employs numerical Laplace inversion. In this method, the problem is analytically solved in the Laplace domain, but the inversion to the real domain is accomplished by a numerical algorithm. Compared to (purely) analytical methods, semi-analytical methods are applicable to a broader range of problems, but these problems must still be relatively simple.

In a numerical method, the flow domain is often divided into many cells. The differential equation as well as the boundary and initial conditions are approximated by discrete equations. Solving these discrete equations (on a computer) yields the head in discrete locations (nodes) and time. Popular numerical methods include finite difference and finite element methods. Less commonly used are boundary methods, which are based on dividing the boundaries of the flow domain into segments or cells. Among the three solutions methods mentioned above, numerical methods have the broadest applicability. Many complex problems can only be solved numerically. However, a numerical solution should be considered an approximate solution of the original mathematical problem. When using numerical methods, one should always be aware of possible errors introduced by discretization.

Equivalence between fluid flow and heat flow problems

Because fluid-flow problem is identical in form to head-flow problem (both are of the diffusive type), we can often find the solution to fluid flow problems from the heat flow literature. The largest collection of analytical solutions is in the book by Carslaw and Jaeger (1959). The equivalence between fluid flow problems and heat flow problems is summarized by Table 2-1 below.

Fluid Flow		Heat Flow	
quantity	variable	variable	quantity
hydraulic head	h	v	temperature
hydraulic conductivity	K	K	thermal conductivity
specific storage	S_s	ρc	density times specific heat
hydraulic diffusivity	K/S_s	$\kappa = K/\rho c$	thermal diffusivity

Table 2-1. Equivalence between fluid flow and heat flow quantities.

Solution to Example Problem

The solution to the example problem discussed above can be found on p. 101 of Carslaw and Jaeger (1959). The equivalent head flow problem is the linear flow of heat in a solid of length L in which the initial temperature is $f(x)$ and both ends of the solid ($x = 0$ and $x = L$) are thermally insulated. The solution to the head flow problem is

$$v = \frac{1}{L}\int_o^L f(x')\,dx' + \frac{2}{L}\sum_{n=1}^{\infty} e^{-\kappa n^2\pi^2 t/L^2}\cos\frac{n\pi x}{L}\int_0^L f(x')\cos\frac{n\pi x'}{L}dx' \tag{2-7}$$

where *x'* is the dummy variable of integration. To find the solution to the fluid flow equation, we replace L in the heat flow problem by $L_1 + L_2$ in the fluid flow problem. In addition, we replace $f(x)$ in the heat flow problem by $h(x,0)$ as defined by (2.5) and (2.6). The first term on the right hand side of (2-7) now becomes:

$$\frac{1}{L}\int_o^L f(x')\,dx' = \frac{1}{L_1 + L_2}\left(\int_0^{L_1} h_1 dx' + \int_{L_1}^{L_1+L_2} h_2 dx'\right) = \frac{h_1 L_1 + h_2 L_2}{L_1 + L_2} \tag{2-8}$$

The integral inside the summation on the right hand side of (2-7) becomes:

$$\int_0^L f(x')\cos\frac{n\pi x'}{L}dx' = \int_0^{L_1} h_1\cos\frac{n\pi x'}{L_1+L_2}dx' + \int_{L_1}^{L_1+L_2} h_2\cos\frac{n\pi x'}{L_1+L_2}dx'$$

$$= \frac{(h_1 - h_2)\,(L_1 + L_2)}{n\pi}\sin\frac{n\pi L_1}{L_1+L_2} \tag{2-9}$$

Substituting eqs. (2.8) and (2-9) into eq. (2.7) and making the variable changes according to Table 2-1 yield

$$h(x,t) = \frac{h_1 L_1 + h_2 L_2}{L_1 + L_2} + \frac{2\,(h_1 - h_2)}{\pi}\sum_{n=1}^{\infty}\frac{e^{-n^2\pi^2 Kt/S_s(L_1+L_2)^2}}{n}\cos\frac{n\pi x}{L_1+L_2}\sin\frac{n\pi L_1}{L_1+L_2} \tag{2.10}$$

One advantage of a closed form analytical solution is that it allows us to examine the behavior of the flow system. There are several of interesting features to this solution. The first term on the right hand side is the steady state part of the solution. It gives the head in the reservoir when t is very large. The second term on the right hand side is the transient part of the problem. Because t appears in the argument of the exponential function, the second term tends to zero as t becomes large. Furthermore, note that the second term goes to zero at a faster rate if K/S_s (hydraulic diffusivity) is larger. Thus, the hydraulic diffusivity is the quantity that controls the rate of hydraulic head propagation.

Another advantage of a closed form analytical solution is that it can be expressed in dimensionless form, which is simpler than the dimensional form. To express (2.10) in dimensionless form, we define the dimensionless distance as

$$x_D = \frac{x}{L_1 + L_2} \tag{2-11}$$

the dimensionless time as

$$t_D = \frac{Kt}{(L_1 + L_2)^2 S_s} \tag{2-12}$$

and the dimensionless head as

$$h_D = \frac{h - h_2}{h_1 - h_2} \tag{2-13}$$

In addition, we define

$$L_D = \frac{L_1}{L_1 + L_2} \tag{2-14}$$

Eq. (2-10) can now be written as

$$h_D(x_D, t_D) = L_D + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{e^{-n^2 \pi^2 t_D}}{n} \cos(n\pi x_D) \sin(n\pi L_D) \tag{2-15}$$

Fig. 2-2 shows a plot of h_D versus x_D for various values of t_D (indicated on plot) and for $L_D = 0.5$.

References

Carslaw, H. S., and Jaeger, J. C., 1959, Conduction of Head in Solids, Oxford University Press, 510 p.

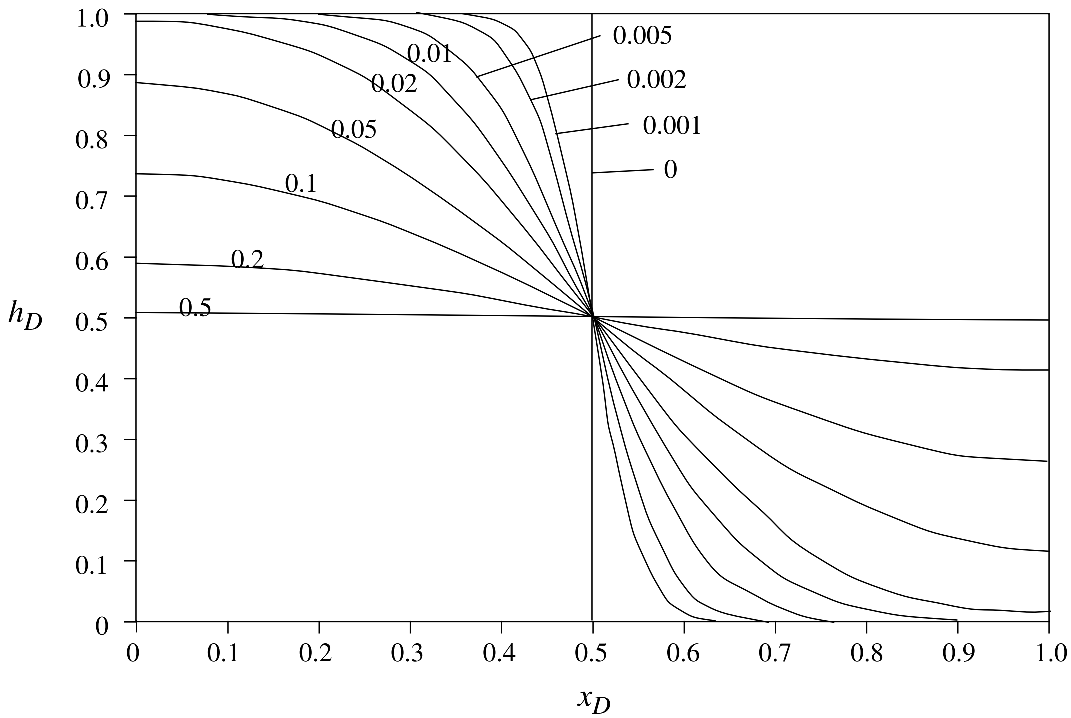

Fig. 2-2. Plot of h_D versus x_D for various values of t_D and for $L_D = 0.5$.