Lecture Notes 1

Review of Ground-Water Flow

Hydraulic Head

In the study of ground-water flow, hydraulic head is a quantity of primary interest. If we know the spatial and temporal distribution of hydraulic head in the flow system, we can infer the movement of ground water. If, in addition, we know the hydraulic conductivity and the porosity of the flow system, we can compute flow rates and velocities.

For the purpose of this class, hydraulic head is defined as follows: In a flow system where fluid density is constant, the hydraulic head at a point P is the elevation (above a given datum) of the water level in a piezometer that is opened or screened at point P. This concept is illustrated in Figure 1-1. A piezometer is a cased well with a short screen, so that the water level in the well represents the hydraulic head in the immediate vicinity of the screen. The datum is chosen arbitrarily (for example, sea level).

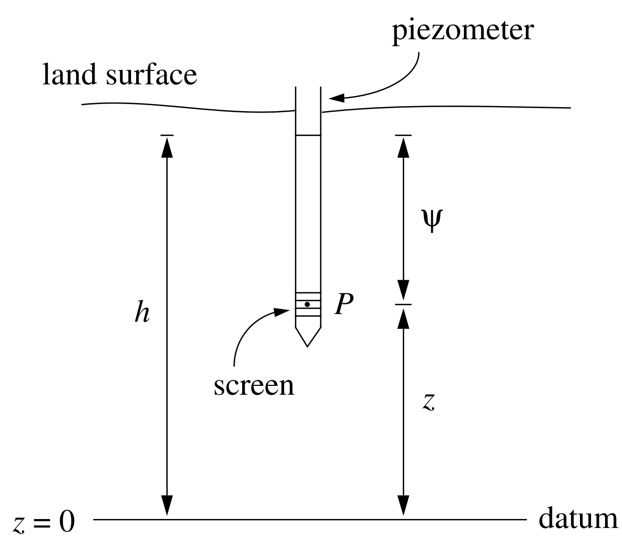

Figure 1-1. A piezometer with the screen installed at point P. The hydraulic head at point P is the height h of the water level in the piezometer. The elevation head is z, and the pressure head is ψ.

Elevation and Pressure Head

The hydraulic head at a point P is composed of two components: the elevation head, z, and the pressure head, ψ. The elevation head is the elevation of the point P above the datum. The pressure head is the height of the water column above P in the piezometer. For a constant density fluid, the pressure head is

$$\psi = \frac{p}{\rho g} \tag{1-1}$$

where p is pressure, ρ is fluid density, and g is gravitational acceleration. Thus, the hydraulic head can be expressed as

$$h = \psi + z = \frac{p}{\rho g} + z \qquad (1\text{-}2)$$

The physical meaning of the hydraulic head is discussed in an classical treatise by Hubbard (1940). Hubbard shows that the quantity gh can be interpreted as the mechanical energy per unit mass of fluid. Therefore, the hydraulic head h is the mechanical energy per unit weight of fluid. A summary of Hubbert's analysis is presented by Freeze and Cherry (1979, p. 18-22).

In the above discussion of hydraulic head, it is assumed that the fluid density is constant. If the fluid density is a function of pressure only, everything discussed above is still valid, except that the pressure head is now defined as

$$\psi = \int_{p_o}^{p} \frac{dp}{\rho g} \qquad (1\text{-}3)$$

where p_o is a reference pressure. However, if the density is a function of quantities other the pressure, for example, temperature or solute concentration, then it is not possible to define a hydraulic head. Instead, one would have to work in terms of pressure.

Hydraulic-Head Distribution Under Steady State

We use three examples to illustrate the hydraulic-head distribution in ground-water flow systems under steady state. In Example A, the flow system is stagnant, so the hydraulic head is uniform throughout the flow region. Figure 1-2 shows three piezometers with their screens installed at three different depths. As the hydraulic head is uniform, the water levels in all three piezometers are at identical elevations. A plot of hydraulic head (on the horizontal axis) versus depth (on a downward pointing vertical axis) gives a vertical straight line. A plot of pressure head versus depth yields a straight line of unit slope. Such a hydraulic-head or pressure-head distribution with depth is known as a hydrostatic profile.

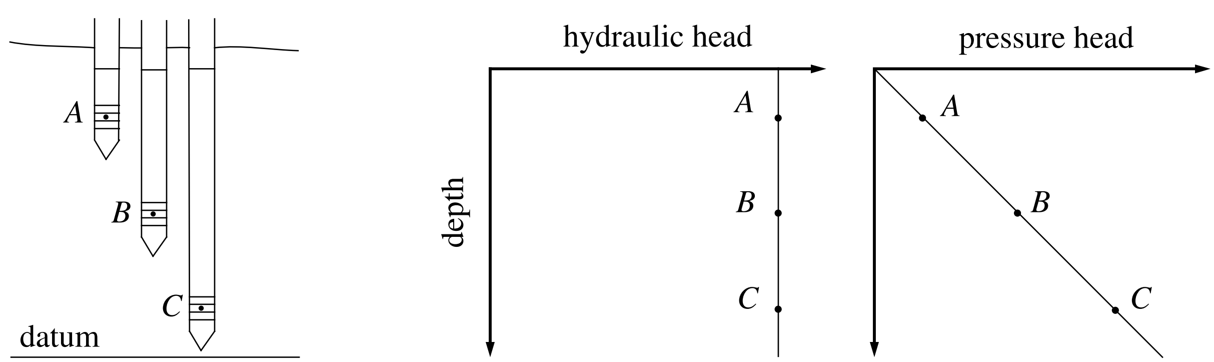

Figure 1-2. Hydraulic-head distribution in a stagnant flow system (Example A). Left: Three piezometers with screens installed at different depths. Center: Plot of hydraulic head versus depth. Right: Plot of pressure head versus depth.

Example B is a flow system in a recharge area, where ground water flows vertically downward. In such a flow system, the hydraulic head decreases with depth. Consequently, the water level in a piezometer screened at a deeper point will be lower than the water level in a piezometer screened at a shallower point (Figure 1-3). A plot of hydraulic head versus depth gives a line that deviates to the left of the hydrostatic profile. A plot of pressure head versus depth gives a line that deviates below the hydrostatic profile.

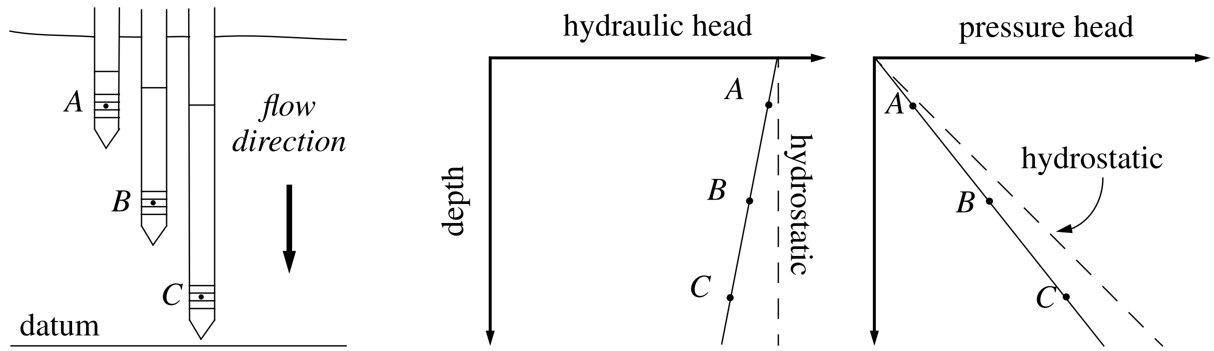

Figure 1-3.Hydraulic-head distribution in a flow system with steady downward flow (Example B). Left: Three piezometers with screens installed at different depths. Center: Plot of hydraulic head versus depth. Right: Plot of pressure head versus depth. Dotted line is hydrostatic profile. Hydraulic conductivity is assumed uniform.

Example C is a flow system in a discharge area, where the ground water flows vertically upward. In such a flow system, the hydraulic head increases with depth (Figure 1-4). In fact, in many discharge areas, the hydraulic head at depth can be higher than land surface. Here, a plot of hydraulic head versus depth gives a line that deviates to the right of the hydrostatic profile. The plot of pressure head versus depth gives a line that deviates above the hydrostatic profile.

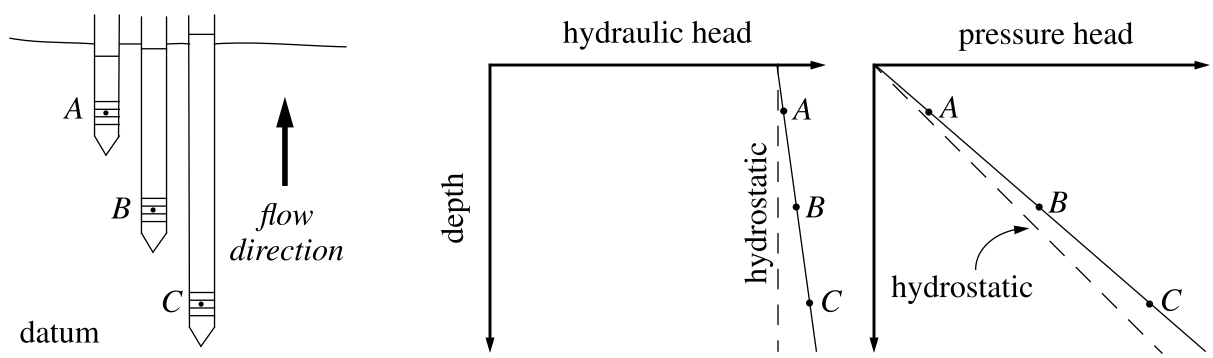

Figure 1-4.Hydraulic head distribution in a flow system with steady upward flow (Example C). Left: Three piezometers with screens installed at different depths. Center: Plot of hydraulic head versus depth. Right: Plot of pressure head versus depth. Dotted line is hydrostatic profile. Hydraulic conductivity is assumed uniform.

In many natural flow systems, the deviation of hydraulic head from the hydrostatic profile is relatively small (from less than 1 m to tens of m). These flow systems are known as normally pressured systems, that is, the hydraulic head everywhere in the flow system close to hydrostatic. By contrast, there exist flow systems where the hydraulic heads are 100's of m higher or lower then hydrostatic. These are known as anomalously pressured systems. When the hydraulic head is significantly greater then hydrostatic, the flow system is called overpressured. When the hydraulic heads are significantly smaller than hydrostatic, the flow system is called underpressured.

Darcy's Law

Darcy's Law is an empirically determined equation that describes the flow of fluid through porous media. To define Darcy's law, we first have to define two quantities: specific discharge and hydraulic gradient. Specific discharge, $\boldsymbol{q}$, is a vector whose direction is the flow direction, and whose magnitude is the volumetric flow rate per unit area normal to flow. The hydraulic gradient, ∇h, is a vector whose direction points along the maximum increase in hydraulic head, and whose magnitude is the rate of change of hydraulic head in this direction. (∇ is the gradient operator.) Darcy's law states that for an isotropic porous medium, the specific discharge is proportional to the negative of the hydraulic gradient. This can be stated in the form of a vector equation as

$$\boldsymbol{q} = -K\,\nabla h \qquad (1\text{-}4)$$

where K, the proportionality constant, is known as the hydraulic conductivity. With respect to an x-y-z cartesian coordinate system, eq. (1-4) can be written as

$$q_x = -K\frac{\partial h}{\partial x} \qquad (1\text{-}5a)$$

$$q_y = -K\frac{\partial h}{\partial y} \qquad (1\text{-}5b)$$

$$q_z = -K\frac{\partial h}{\partial z} \qquad (1\text{-}5c)$$

where q_x, q_y, and q_z are the x-, y-, and z-components of the specific discharge $\boldsymbol{q}$, and $\partial h/\partial x$, $\partial h/\partial y$, and $\partial h/\partial z$ are the x-, y-, and z-components of the hydraulic gradient ∇h.

The hydraulic conductivity, K, is a property of both the fluid and the medium through which the fluid flows. It is related to the permeability, k, by

$$K = \frac{\rho g k}{\mu} \qquad (1\text{-}6)$$

where μ is the dynamic viscosity of the fluid. The permeability (also known as intrinsic permeability) is a property of solely the medium. If the porous medium is anisotropic with respect to hydraulic conductivity, then K is a tensor quantity, and eqs. (1-4) and (1-5) are more complicated.

Law of Fluid Mass Conservation

The law of fluid mass conservation can be stated as follows: In a flow system, fluid mass is neither created nor destroyed. To translate the above statement into a mathematical expression, it is convenient to consider a "control volume" within a flow region. The principle of conservation of fluid mass can be now stated as follows: the rate at which fluid mass enters the control volume minus the rate at which fluid mass leaves the elemental control volume is equal to the rate at which fluid mass is accumulating in the elemental control volume. Figure 1-5 illustrates a control volume that is a cube centered at a point *P*: (x, y, z). The sides of the cube are Δx, Δy, and Δz.

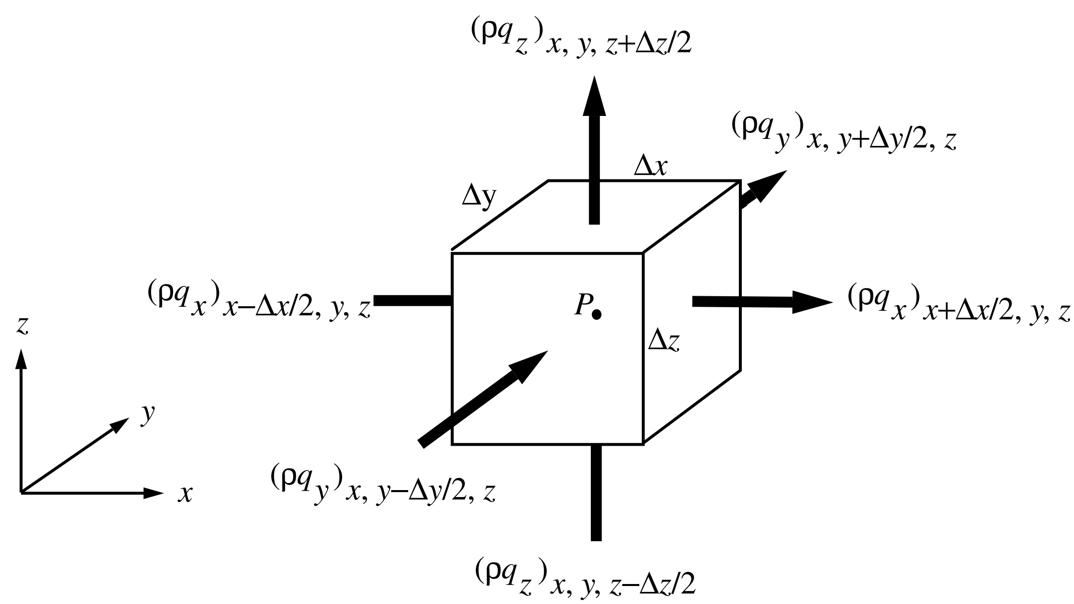

Figure 1-5.Control volume of dimension Δx, Δy, Δz, and centered at point *P*: (x, y, z). Arrows indicate flows into and out of the control volume.

The rate at which fluid mass enters the control volume in Figure 1-5 is

$$(\rho q_x)_{x-\Delta x/2,\, y,\, z}\Delta y\Delta z + (\rho q_y)_{x,\, y-\Delta y/2,\, z}\Delta x\Delta z + (\rho q_z)_{x,\, y,\, z-\Delta z/2}\Delta x\Delta y$$

The rate at which fluid mass leaves the control volume is

$$(\rho q_x)_{x+\Delta x/2,\, y,\, z}\Delta y\Delta z + (\rho q_y)_{x,\, y+\Delta y/2,\, z}\Delta x\Delta z + (\rho q_z)_{x,\, y,\, z+\Delta z/2}\Delta x\Delta y$$

The rate at which fluid mass accumulates in the elemental control volume is

$$\frac{\partial(\rho n)}{\partial t}\Delta x\Delta y\Delta z$$

where n is the porosity of the porous medium. Putting the above expressions into the principle of conservation of fluid mass, dividing through by $\Delta x \Delta y \Delta z$, and rearranging, yields

$$\frac{(\rho q_x)_{x-\Delta x/2,\, y,\, z} - (\rho q_x)_{x+\Delta x/2,\, y,\, z}}{\Delta x} + \frac{(\rho q_y)_{x,\, y-\Delta y/2,\, z} - (\rho q_y)_{x,\, y+\Delta y/2,\, z}}{\Delta y} +$$

$$\frac{(\rho q_z)_{x,\, y,\, z-\Delta z/2} - (\rho q_z)_{x,\, y,\, z+\Delta z/2}}{\Delta z} = \frac{\partial(\rho n)}{\partial t} \tag{1-7}$$

By letting Δx, Δy, and Δz approach zero, we obtain

$$-\frac{\partial \rho q_x}{\partial x} - \frac{\partial \rho q_y}{\partial y} - \frac{\partial \rho q_z}{\partial z} = \frac{\partial(\rho n)}{\partial t} \tag{1-8}$$

Eq. (1-8) is known as the equation of fluid mass conservation. It is also known as the continuity equation for flow through porous medium.

For a slightly compressible fluid such as water (as compared to a gas, which is highly compressible), it can be assumed that the density ρ does not significantly vary in space. This allows ρ to be taken outside the spatial derivative terms on the left hand side of (1-8). As a consequence, (1-8) can be written as

$$-\frac{\partial q_x}{\partial x} - \frac{\partial q_y}{\partial y} - \frac{\partial q_z}{\partial z} = \frac{1}{\rho}\frac{\partial(\rho n)}{\partial t} \tag{1-9}$$

Eq. (1-9) is call the conservation or continuity equation for a slightly compressible fluid.

Ground-Water Storage

The term $(1/\rho)[\partial(\rho n)\partial t]$ in (1-9) is known as the storage term. The two mechanisms for ground-water storage are fluid compressibility and matrix compressibility. A full analysis of ground-water storage can be quite complicated because it involves deformation of the control volume as water is stored or released. A full treatment of this topic will be postponed to a latter part of this course.

For the present, we simply note that ground-water storage described by a parameter known as specific storage, defined here as

$$S_s = \frac{1}{\rho}\frac{\partial(\rho n)}{\partial h} \tag{1-10}$$

It should be noted that (1-10) is valid only when (a) the total stress (overburden) remains constant, and (b) the porous medium deforms only in the vertical directions.

Equation of Ground-Water Flow

The equation of ground-water flow is obtained by combining Darcy's Law (1-5), the conservation equation (1-9), and the definition of specific storage (1-10). By substituting (1-5) into (1-9), we obtain

$$\frac{\partial}{\partial x}\left(K\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right) = \frac{1}{\rho}\frac{\partial(\rho n)}{\partial t} \tag{1-11}$$

Next, substituting (1-10) into (1-12) yields

$$\frac{\partial}{\partial x}\left(K\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K\frac{\partial h}{\partial z}\right) = S_s\frac{\partial h}{\partial t} \tag{1-12}$$

Eq. (1-13) is known as the equation of ground-water flow (in a porous medium that is isotropic with respect to hydraulic conductivity.) If the hydraulic conductivity is anisotropic, but the principal directions of anisotropy are aligned with the *x-y-z* axis system, then the equation of ground-water flow takes the form

$$\frac{\partial}{\partial x}\left(K_{xx}\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_{yy}\frac{\partial h}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_{zz}\frac{\partial h}{\partial z}\right) = S_s\frac{\partial h}{\partial t} \tag{1-13}$$

where K_{xx}, K_{yy}, and K_{zz} are the principal hydraulic conductivities in the x, y, and z directions.

Boundary Conditions

To solve the steady-state ground-water flow equation (1-12) or (1-13), it is necessary to specify the boundary conditions along all boundaries of the flow system. (For transient flow, it is also necessary to specify the initial condition). Of the various types of boundary conditions, the two most common are: specified head and specified flow. A specified-head boundary is expressed as

$$h = \text{specified function} \quad \text{along the boundary.} \tag{1-14}$$

A specified-flow boundary is expressed as

$$q_N = -K\frac{\partial h}{\partial N} = \text{specified function} \quad \text{along the boundary} \tag{1-15}$$

where $\boldsymbol{N}$ is a unit vector normal to the boundary (Figure 1-6). In particular, if there is no flow across the boundary (the boundary lies along an impervious barrier), the boundary condition is

$$\frac{\partial h}{\partial N} = 0 \quad \text{along the boundary.} \tag{1-16}$$

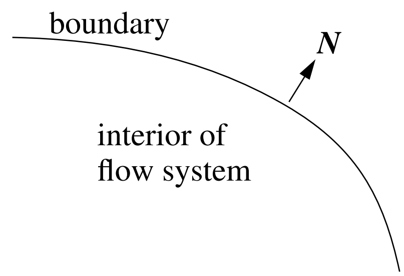

Figure 1-6. Boundary of flow system, showing the unit vector ***N*** normal to the boundary.

Initial Condition

For transient problems, the initial condition must also be specified. The initial condition is the specification of hydraulic head at all points in the flow domain at the initial time (usually $t = 0$). Initial condition is not needed for steady state (time independent) problems.

References

Freeze, R. A., and Cherry, J. A., 1979, *Groundwater*, Prentice-Hall, Englewood Cliffs, NJ, 604 p.

Hubbert. M. K., 1940, The theory of groundwater motion, *Journal of Geology*, v. 48, p. 785-944.