Equations of Equilibrium

Applying Newton’s law to balance the forces on the cube in the y-direction gives:

\[ \frac{\partial}{\partial x} \sigma_{xy} + \frac{\partial}{\partial y} \sigma_{yy} + \frac{\partial}{\partial z} \sigma_{zy} = \rho a_y \]

\[ \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \sigma_{ij} = \rho a_j \]
Equation of Motion For an Elastic Solid

Recall from before that Newton’s law gives:

\[ \frac{\partial}{\partial x} \sigma_{xy} + \frac{\partial}{\partial y} \sigma_{yy} + \frac{\partial}{\partial z} \sigma_{zy} = \rho a_y \]

Combining with Hooke’s law:

\[ \sigma_{ij} = \lambda \delta_{ij} \epsilon_{kk} + 2\mu \epsilon_{ij} \]

where

\[ \epsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

gives the equation of motion in terms of displacements:

\[ \rho \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu) \frac{\partial}{\partial x_i} \nabla \cdot \vec{U} + \mu \nabla^2 U_i \]
Seismic Waves

Consider the example of uniaxial stress

\[
\begin{pmatrix}
\sigma_{xx} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The equation of motion:

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \rho \frac{\partial^2 u_x}{\partial t^2}
\]

Simplifies to:

\[
\frac{\partial \sigma_{xx}}{\partial x} = \rho \frac{\partial^2 u_x}{\partial t^2}
\]

But for uniaxial stress we know that:

\[
\sigma_{xx} = E \epsilon_{xx} = E \frac{\partial u_x}{\partial x}
\]

Therefore, substituting in gives:

\[
E \frac{\partial^2 u_x}{\partial x^2} = \rho \frac{\partial^2 u_x}{\partial t^2}
\]

or

\[
V_E^2 \frac{\partial^2 u_x}{\partial x^2} = \frac{\partial^2 u_x}{\partial t^2}
\]

\[
V_E = \sqrt{\frac{E}{\rho}}
\]
Consider the example of uniaxial strain

\[
\begin{pmatrix}
\varepsilon_{xx} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

The equation of motion:

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \rho \frac{\partial^2 u_x}{\partial t^2}
\]

Simplifies to:

\[
\left( \frac{\lambda + 2\mu}{\rho} \right) \frac{\partial \varepsilon_{xx}}{\partial x} = \frac{\partial^2 u_x}{\partial t^2}
\]

\[
\left( \frac{\lambda + 2\mu}{\rho} \right) \frac{\partial^2 u_x}{\partial x^2} = \frac{\partial^2 u_x}{\partial t^2}
\]

\[
V_P^2 \frac{\partial^2 u_x}{\partial x^2} = \frac{\partial^2 u_x}{\partial t^2}
\]

\[
V_P = \sqrt{\frac{\lambda + 2\mu}{\rho}} = \sqrt{\frac{K + \frac{4}{3}\mu}{\rho}}
\]

P-Wave Velocity
Consider the example of a shear wave.

Assume propagation in the y direction, particle motion (displacement) in the x direction.

If the only nonzero stress is

\[ \sigma_{xy} = 2\mu \varepsilon_{xy} \]

The equation of motion:

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \rho \frac{\partial^2 u_x}{\partial t^2}
\]

Simplifies to:

\[
2\mu \frac{\partial \varepsilon_{xy}}{\partial y} = \rho \frac{\partial^2 u_x}{\partial t^2}
\]

\[
\left( \frac{\mu}{\rho} \right) \frac{\partial^2 u_x}{\partial y^2} = \frac{\partial^2 u_x}{\partial t^2}
\]

\[
V_s^2 \frac{\partial^2 u_x}{\partial y^2} = \frac{\partial^2 u_x}{\partial t^2}
\]

\[
V_s = \sqrt{\frac{\mu}{\rho}}
\]

S-Wave Velocity