Constitutive Relation

The descriptions of stress and strain individually are independent of the material.

The connection between stress and strain is a description of the particular material being studied. This connection is called the constitutive relation or the stress-strain relation.

An elastic material is one where the stress is a unique function of the strain and vice versa. An elastic solid has the following properties:

- The displacements and strains are independent of the history of loading
- When we remove the applied loads, the body returns to a unique relaxed state.
Example of an **Elastic** Medium: there is a unique relation between stress and strain.

**Inelastic** medium: non-unique relation that may depend on rate or loading history.
Linear Elastic Material: Hookean Solid

A special case is when the stresses and strains are linearly related:

\[ \sigma_{ij} = \sum_{kl} c_{ijkl} \varepsilon_{kl} \quad \varepsilon_{ij} = \sum_{kl} S_{ijkl} \sigma_{kl} \]

stiffnesses (moduli) \hspace{1cm} \text{compliances}

Since there are 9 stress components and 9 strain components, there are 81 elastic constants. However, because of the symmetry of stress and strain, plus some other thermodynamic conditions there are at most 21 independent constants.

The elastic stiffness \( c_{ijkl} \) and compliances \( S_{ijkl} \) are exactly equivalent. They are tensor inverses of each other.
Isotropic Linear Elastic Solid

For an isotropic material, only two unique constants are needed.

\[ c_{ijrs} = \lambda \delta_{ij}\delta_{rs} + \mu (\delta_{ir}\delta_{js} + \delta_{is}\delta_{jr}) \]

in simpler form:

\[ \sigma_{ij} = \lambda \delta_{ij}\varepsilon_{kk} + 2\mu \varepsilon_{ij} \]

where \( \lambda \) and \( \mu \) are the Lame constants. \( \mu \) is also known as the shear modulus.
Linear Isotropic Elasticity: the Bulk Modulus

Hooke’s Law

\[ \sigma_{ij} = \lambda \varepsilon_{\alpha \alpha} \delta_{ij} + 2\mu \varepsilon_{ij} \]

or

\[
\begin{align*}
\sigma_{xx} &= \lambda \varepsilon_{\alpha \alpha} + 2\mu \varepsilon_{xx} \\
\sigma_{yy} &= \lambda \varepsilon_{\alpha \alpha} + 2\mu \varepsilon_{yy} \\
\sigma_{zz} &= \lambda \varepsilon_{\alpha \alpha} + 2\mu \varepsilon_{zz} \\
\sigma_{xy} &= 2\mu \varepsilon_{xy} \\
\sigma_{yz} &= 2\mu \varepsilon_{yz} \\
\sigma_{xz} &= 2\mu \varepsilon_{xz}
\end{align*}
\]

By adding the first three equations:

\[ \sigma_{\alpha \alpha} = (3\lambda + 2\mu) \varepsilon_{\alpha \alpha} \]

\[ \sigma_0 = \frac{1}{3} \sigma_{\alpha \alpha} = \left( \frac{3\lambda + 2\mu}{3} \right) \varepsilon_{\alpha \alpha} \]

For convenience we define

\[ \sigma_0 = K \varepsilon_{\alpha \alpha} \]

\[ K = \left( \frac{3\lambda + 2\mu}{3} \right) \]

Here, \( K \) is the bulk modulus and gives the ratio of hydrostatic stress to volumetric strain.
Elasticity

Linear Isotropic Elasticity: the Shear Modulus

\[
\begin{align*}
\sigma_{xx} &= \lambda \varepsilon_{\alpha \alpha} + 2\mu \varepsilon_{xx} \\
\sigma_{yy} &= \lambda \varepsilon_{\alpha \alpha} + 2\mu \varepsilon_{yy} \\
\sigma_{zz} &= \lambda \varepsilon_{\alpha \alpha} + 2\mu \varepsilon_{zz} \\
\sigma_{xy} &= 2\mu \varepsilon_{xy} \\
\sigma_{yz} &= 2\mu \varepsilon_{yz} \\
\sigma_{xz} &= 2\mu \varepsilon_{xz}
\end{align*}
\]

In a simple shear experiment, only a single shear strain \( \varepsilon_{xy} \) is non-zero.

Then Hooke’s law states:

\[
\sigma_{xy} = 2\mu \varepsilon_{xy}
\]

\( \mu \) is the shear modulus, which gives the ratio of shear stress to shear strain.
Linear Isotropic Elasticity: the Young’s Modulus

\[
\begin{align*}
\sigma_{xx} &= \lambda \varepsilon_{\alpha\alpha} + 2\mu \varepsilon_{xx} \\
\sigma_{yy} &= \lambda \varepsilon_{\alpha\alpha} + 2\mu \varepsilon_{yy} \\
\sigma_{zz} &= \lambda \varepsilon_{\alpha\alpha} + 2\mu \varepsilon_{zz} \\
\sigma_{xy} &= 2\mu \varepsilon_{xy} \\
\sigma_{yz} &= 2\mu \varepsilon_{yz} \\
\sigma_{xz} &= 2\mu \varepsilon_{xz}
\end{align*}
\]

In a uniaxial stress experiment, only the axial normal stress \( \sigma_{xx} \) is non-zero.

Then from Hooke’s law we can derive:

\[
\sigma_{xx} = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \varepsilon_{xx}
\]

But it is convenient to define

\[
\sigma_{xx} = E \varepsilon_{xx}
\]

\[
E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu}
\]

This is Young’s modulus, which gives the ratio of axial stress to axial strain in a uniaxial stress state.
Linear Isotropic Elasticity: the Poisson’s Ratio

Similarly, in the uniaxial stress state we can derive the ratio of lateral strain to axial strain:

$$\varepsilon_{yy} = -\frac{\lambda}{2(\lambda + \mu)}\varepsilon_{xx}$$

And it is convenient to define Poisson’s ratio:

$$\nu = \frac{\lambda}{2(\lambda + \mu)}$$

$$\varepsilon_{yy} = -\nu\varepsilon_{xx}$$

Ideally $\nu$ must lie within the limits $-1 \leq \nu \leq \frac{1}{2}$

Practically it is almost always between $0 \leq \nu \leq \frac{1}{2}$

$\nu \rightarrow \frac{1}{2}$ when $\mu \rightarrow 0$ or $K \rightarrow \infty$

We can interpret this as a fluid state or as an incompressible state.
Uniaxial Strain

Consider the case when displacements occur only in the x direction:

This can occur in the laboratory by putting rigid lateral constraints, or in a plane P-wave.

Then we can derive:

$$\sigma_{xx} = (\lambda + 2\mu)\varepsilon_{xx}$$

So we can define a new modulus $M$ for this experiment that we will call the uniaxial strain modulus, or the “P-wave” modulus.

$$M = \left(K + \frac{4}{3}\mu\right) = (\lambda + 2\mu)$$

This gives the ratio of axial stress to axial strain in a uniaxial strain situation.
## Examples of Elastic Moduli

<table>
<thead>
<tr>
<th>Mineral</th>
<th>Density</th>
<th>Young's Modulus</th>
<th>Bulk Modulus</th>
<th>Shear Modulus</th>
<th>Vp</th>
<th>Vs</th>
<th>Poisson's Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quartz</td>
<td>2.6500</td>
<td>95.756</td>
<td>36.600</td>
<td>45.000</td>
<td>6.0376</td>
<td>4.1208</td>
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<tr>
<td>Calcite</td>
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<td>76.800</td>
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<td>6.6395</td>
<td>3.4363</td>
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<tr>
<td>Dolomite</td>
<td>2.8700</td>
<td>116.57</td>
<td>94.900</td>
<td>45.000</td>
<td>7.3465</td>
<td>3.9597</td>
<td>0.29527</td>
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<tr>
<td>Clay (kaolinite)</td>
<td>1.5800</td>
<td>3.2034</td>
<td>1.5000</td>
<td>1.4000</td>
<td>1.4597</td>
<td>0.94132</td>
<td>0.14407</td>
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<td>Muscovite</td>
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<td>6.4563</td>
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<td>Feldspar (Albite)</td>
<td>2.6300</td>
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<td>25.600</td>
<td>6.4594</td>
<td>3.1199</td>
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<td>Halite</td>
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<td>37.242</td>
<td>24.800</td>
<td>14.900</td>
<td>4.5474</td>
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<td>Anhydrite</td>
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<td>29.100</td>
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<td>Pyrite</td>
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<td>147.40</td>
<td>132.50</td>
<td>8.1076</td>
<td>5.1842</td>
<td>0.15417</td>
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<td>Siderite</td>
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<td>1.5000</td>
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<td>0.50000</td>
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<td>0.0000</td>
<td>1.0200</td>
<td>0.0000</td>
<td>1.1292</td>
<td>0.0000</td>
<td>0.50000</td>
</tr>
</tbody>
</table>

**densities in g/cm³**

**moduli in GPa**

**velocities in km/s**

C.1
## Elasticity

### Relationships Among Elastic Constants in an Isotropic Material

<table>
<thead>
<tr>
<th></th>
<th>K</th>
<th>E</th>
<th>λ</th>
<th>ν</th>
<th>M</th>
<th>μ</th>
</tr>
</thead>
<tbody>
<tr>
<td>λ+2μ/3</td>
<td>$\mu \frac{3\lambda+2\mu}{\lambda+\mu}$</td>
<td>-</td>
<td>$\frac{\lambda}{2(\lambda+\mu)}$</td>
<td>$\lambda$</td>
<td>2μ</td>
<td>-</td>
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<tr>
<td>-</td>
<td>$9K \frac{K-\lambda}{3K-\lambda}$</td>
<td>-</td>
<td>$\frac{\lambda}{3K-\lambda}$</td>
<td>$3K-2\lambda$</td>
<td>$3(K-\lambda)/2$</td>
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<td>-</td>
<td>$\frac{9K\mu}{3K+\mu}$</td>
<td>$K-2\mu/3$</td>
<td>$\frac{3K-2\mu}{2(3K+\mu)}$</td>
<td>$K+4\mu/3$</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>$\frac{E\mu}{3(3\mu-E)}$</td>
<td>-</td>
<td>$\mu \frac{E-2\mu}{(3\mu-E)}$</td>
<td>$E/(2\mu)$-1</td>
<td>$\frac{4\mu-E}{3\mu-E}$</td>
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<tr>
<td>-</td>
<td>-</td>
<td>$\frac{3K}{9K-E}$</td>
<td>$\frac{3K}{6K}$</td>
<td>$\frac{3K}{9K-E}$</td>
<td>$\frac{3KE}{9K-E}$</td>
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</tr>
<tr>
<td>$\frac{\lambda}{3v}$</td>
<td>$\lambda \frac{(1+v)(1-2v)}{v}$</td>
<td>-</td>
<td>-</td>
<td>$\frac{\lambda}{v}$</td>
<td>$\frac{\lambda}{1-2v}$</td>
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</tr>
<tr>
<td>$\frac{2(1+v)}{3(1-2v)}$</td>
<td>$2\mu(1+v)$</td>
<td>$\mu \frac{2v}{1-2v}$</td>
<td>-</td>
<td>$\mu \frac{2-2v}{1-2v}$</td>
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<tr>
<td>-</td>
<td>$3K(1-2v)$</td>
<td>$3K \frac{v}{1+v}$</td>
<td>-</td>
<td>$3K \frac{1-v}{1+v}$</td>
<td>$3K \frac{1-2v}{2+2v}$</td>
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</tr>
<tr>
<td>$\frac{E}{3(1-2v)}$</td>
<td>-</td>
<td>$E \frac{v}{(1+v)(1-2v)}$</td>
<td>-</td>
<td>$E \frac{(1-v)}{(1+v)(1-2v)}$</td>
<td>$E \frac{2+2v}{2+2v}$</td>
<td></td>
</tr>
</tbody>
</table>