PRESSURE TRANSIENT ANALYSIS OF RESERVOIRS WITH LINEAR OR INTERNAL CIRCULAR BOUNDARIES

By

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ABSTRACT

In this work, a practical pressure transient analysis method is presented for a drawdown test in a well near an internal circular boundary. Both no-flow and constant pressure boundaries are considered. The problem is mathematically posed and solved using Green's Function theory and the Laplace Transformation. Both the Laplace solutions and the analytical solutions are presented.

Linear boundaries are viewed as circles with infinite radii and act as a known limiting case for finite radii internal boundaries. The size of an internal circular boundary and the distance to it can be estimated using generalized type curves. Using a new method developed here, the distance to a linear boundary can be determined by semilog type curve matching without using the usual double straight line technique.

In developed systems containing compressible subregions, interference testing can provide estimates of the sizes of these subregions. However, detecting no-flow subregions with interference testing is not practical. Also presented are type curves for interference between a well flowing at a constant rate and one at constant pressure. These type curves may be applied to interpretation of pressure interference between oil and gas fields sharing a common aquifer.

The superposition method which can be used for assembling circular subregions intersected by linear faults is discussed as well.

Finally, a new generalized semilog type curve is presented that can be used for analyzing pressure transient data for both the linear and circular boundary cases.
TABLE OF CONTENTS

DEDICATION ............................................................ iii
ACKNOWLEDGMENT ..................................................... iv
ABSTRACT ............................................................... v
LIST OF FIGURES ...................................................... viii
LIST OF TABLES ....................................................... xiv

CHAPTER 1: INTRODUCTION ............................................ 1
  1.1 APPLICATIONS ................................................... 1
  1.2 PROBLEM DESCRIPTION .......................................... 3
  1.3 BACKGROUND ................................................... 4
  1.4 PROBLEM STATEMENT ........................................... 8

CHAPTER 2: LINEAR BOUNDARIES ...................................... 9
  2.1 PROBLEM STATEMENT ........................................... 9
  2.2 SOLUTION .......................................................... 11
  2.3 LOG-LOG AND SEMILOG ANALYSIS METHODS ...................... 12
  2.4 A NEW SEMILOG TYPE CURVE MATCHING METHOD ................ 16
  2.5 TYPE CURVE MATCHING EXAMPLE ................................ 19

CHAPTER 3: CONSTANT PRESSURE INTERNAL CIRCULAR BOUNDARY .... 26
  3.1 PROBLEM STATEMENT ........................................... 26
  3.2 API TRANSFORMATION .......................................... 29
  3.3 THE LAPLACE TRANSFORMATION SOLUTION ...................... 29
  3.4 THE ANALYTIC SOLUTION ....................................... 34
  3.5 NUMERICAL ESİ̇G OF THE LAPLACE TRANSFORM .................. 45
  3.6 TYPE CURVE MATCHING FOR THE PRODUCTION WELL .......... 46
      3.6.1 GENERAL DISCUSSION .................................... 46
      3.6.2 TYPE CURVE MATCHING EXAMPLE ......................... 54
  3.7 INTERFERENCE CURVE MATCHING ................................. 57
      3.7.1 GENERAL DISCUSSION .................................... 57
      3.7.2 INTERFERENCE TESTING IN AN UNKNOWN GEOMETRY ....... 69
      3.7.3 INTERFERENCE TESTING IN A KNOWN GEOMETRY ......... 69
3.8 INTERFERENCE BETWEEN OIL AND GAS FIELDS ........................................ 85
3.9 SEMICIRCULAR AND QUARTERCIRCULAR SUBREGIONS ........................... 88

CHAPTER 4: NO-FLOW INTERNAL CIRCULAR BOUNDARY .......................... 92
4.1 PROBLEM STATEMENT ........................................................................... 92
4.2 LAPLACE TRANSFORMATION .............................................................. 94
4.3 THE LAPLACE TRANSFORMATION SOLUTION ................................. 94
4.4 THE ANALYTICAL SOLUTION .............................................................. 98
4.5 NUMERICAL INVERSION OF THE LAPLACE TRANSFORMATION .......... 110
4.6 TYPE CURVE MATCHING FOR THE PRODUCTION WELL .................. 111
4.7 INTERFERENCE .................................................................................... 114

CHAPTER 5: A GENERALIZED SEMILOG TYPE CURVE .......................... 121

CHAPTER 6: CONCLUSIONS ...................................................................... 124

NOMENCLATURE ......................................................................................... 128
REFERENCES .............................................................................................. 130

APPENDICES ............................................................................................... 133

APPENDIX A: CIRCLES OF CONSTANT $r_2/r_1$ RATIO ............................ 133
APPENDIX B: DIMENSIONLESS PRESSURE VS. REDUCED DIMENSIONLESS
TIME FOR POINTS WITH A CONSTANT $r_2/r_1$ RATIO .......................... 135
APPENDIX C: SHIFTING OF THE SEMILOG CURVES ................................. 141
APPENDIX D: DERIVATION OF THE LATE TIME DIMENSIONLESS PRESSURE
FOR THE CONSTANT PRESSURE HOLE USING THE
DOUBLET MODEL ....................................................................................... 148
APPENDIX E: DIMENSIONLESS DEPARTURE TIME FROM THE LINE SOURCE
.. ................................................................................................................. 150
APPENDIX F: ASYMPTOTIC EXPANSIONS FOR MODIFIED BESSEL
FUNCTIONS .................................................................................................. 151
APPENDIX G: THE COMPUTER PROGRAM ................................................ 156
APPENDIX H: TABLES FOR PRESSURE TIME TYPE CURVES ............... 209
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>A schematic diagram of the constant pressure linear boundary system</td>
<td>10</td>
</tr>
<tr>
<td>2.2</td>
<td>Log-log type curves for the linear boundary case</td>
<td>13</td>
</tr>
<tr>
<td>2.3</td>
<td>Example of the double straight line analysis method</td>
<td>15</td>
</tr>
<tr>
<td>2.4</td>
<td>A semi-log type curve for the linear boundary case</td>
<td>17</td>
</tr>
<tr>
<td>2.5</td>
<td>A generalized semi-log type curve for the linear boundary case</td>
<td>17</td>
</tr>
<tr>
<td>2.6</td>
<td>Log-log graph of the drawdown data for an interference well</td>
<td>21</td>
</tr>
<tr>
<td>2.7</td>
<td>Log-log match for the drawdown data for an interference well</td>
<td>21</td>
</tr>
<tr>
<td>2.8</td>
<td>Semi-log graph of the drawdown data in dimensionless pressure form</td>
<td>23</td>
</tr>
<tr>
<td>2.9</td>
<td>Log-log and semi-log graphs for the drawdown data in dimensionless pressure form</td>
<td>23</td>
</tr>
<tr>
<td>2.10</td>
<td>Semi-log match of the drawdown data to the generalized type curve</td>
<td>24</td>
</tr>
<tr>
<td>3.1</td>
<td>A schematic diagram of the constant pressure hole system</td>
<td>28</td>
</tr>
<tr>
<td>3.2</td>
<td>The geometry for type curve matching, constant pressure hole</td>
<td>47</td>
</tr>
<tr>
<td>3.3</td>
<td>Log-log curves for (2c=100) and (F) from 0.1 to 0.9</td>
<td>48</td>
</tr>
<tr>
<td>3.4</td>
<td>Semi-log curves for (2c=100) and (F) from 0.1 to 0.9</td>
<td>48</td>
</tr>
<tr>
<td>3.5</td>
<td>The doublet model for the constant pressure hole at steady state</td>
<td>49</td>
</tr>
<tr>
<td>3.6</td>
<td>Semi-log curves for (2c=100, 250) and (F=0.1) to 0.9.</td>
<td>51</td>
</tr>
<tr>
<td>3.7</td>
<td>Semi-log curve for (2c=500) (F=0.5) matched with a shifted</td>
<td></td>
</tr>
</tbody>
</table>
CURVE FOR $2c=100$ $F=0.5$, CONSTANT PRESSURE HOLE ................. 51

3.8 A GENERALIZED SEMILOG TYPE CURVE FOR THE CONSTANT PRESSURE
INTERNAL CIRCULAR BOUNDARY .................................. 53

3.9 ONE PERCENT DIMENSIONLESS DEPARTURE TIME FROM THE LINE
SOURCE AS A FUNCTION OF $2c'$ .................................. 53

3.10 TYPE CURVE MATCH EXAMPLE : DATA FOR $2c=20$ AND $F=0.5$.
CONSTANT PRESSURE HOLE ........................................ 55

3.11 TYPE CURVE MATCH EXAMPLE : LOG-LOG MATCH TO THE LINE
SOURCE AND A PRELIMINARY VALUE OF $2c$ ....................... 55

3.12 TYPE CURVE MATCH EXAMPLE : SEMILOG MATCH FOR THE RELATIVE
SIZE OF THE HOLE AND THE DISTANCE TO IT ..................... 56

3.13 INTERFERENCE LOG-LOG CURVES FOR $F=0.5$, $E=1.5$ AND $\theta=0$,
45, 90, 135, 180 DEG. CONSTANT PRESSURE HOLE .............. 58

3.14 THE GEOMETRY OF THE OBSERVATION POINTS IN THE CONSTANT
PRESSURE HOLE SYSTEM .............................................. 58

3.15 INTERFERENCE LOG-LOG CURVES FOR THREE POINTS ON THE LONG
TIME CONSTANT PRESSURE CIRCLE ................................ 60

3.16 INTERFERENCE LOG-LOG CURVES FOR $E=0.9$, $F=0.1$ TO 0.8 AND
$\theta=45$ DEG. CONSTANT PRESSURE HOLE ....................... 60

3.17 INTERFERENCE NORMALIZED LOG-LOG CURVES FOR $E=0.9$, $F=0.1$
TO 0.8 AND $\theta=45$ DEG. CONSTANT PRESSURE HOLE .......... 61

3.18 INTERFERENCE LOG-LOG CURVES FOR $F=0.4$, $E=0.5$ TO 1.0 AND
$\theta=45$ DEG. CONSTANT PRESSURE HOLE ....................... 61

3.19 INTERFERENCE NORMALIZED LOG-LOG CURVES FOR $F=0.4$, $E=0.5$
TO 1.0 AND $\theta=45$ DEG. CONSTANT PRESSURE HOLE .......... 62

3.20 INTERFERENCE LOG-LOG CURVES FOR $F=0.5$, $E=0.99$ AND
$\theta=0, 45, 90, 135, 180$ DEG. CONSTANT PRESSURE HOLE .... 62

3.21 INTERFERENCE NORMALIZED LOG-LOG CURVES FOR $F=0.5$, $E=0.99$
AND $\theta=0, 45, 90, 135, 180$ DEG. CONSTANT PRESSURE HOLE 63

3.22 INTERFERENCE NORMALIZED LOG-LOG CURVES FOR $\theta=45$ DEG. AND
$E-F=0.5$. CONSTANT PRESSURE HOLE ......................... 64

3.23 INTERFERENCE NORMALIZED LOG-LOG CURVES FOR $\theta=45$ DEG. AND
$(E-F)/(1-F)=5/6$. CONSTANT PRESSURE HOLE ................ 64

3.24 INTERFERENCE LOG-LOG CURVES FOR $F=0.4$, $\theta=0, 45, 90, 135, 180$
DEG. AND $E=0.5$ TO 2.0. CONSTANT PRESSURE HOLE ........... 66

3.25 INTERFERENCE NORMALIZED SEMILOG CURVES FOR $F=0.4$, $\theta=45$.
90,135,180 DEG. AND E-0.5 TO 2.0. CONSTANT PRESSURE HOLE

3.26 INTERFERENCE LOG-LOG TYPE CURVES FOR E=1.4 AND 8-45 DEG.
CONSTANT PRESSURE HOLE

3.27 INTERFERENCE SEMILOG TYPE CURVES FOR E=1.4 AND 8=45 DEG.
CONSTANT PRESSURE HOLE

3.28 INTERFERENCE LOG-LOG TYPE CURVES FOR E=0.7 AND 8=0 DEG.
CONSTANT PRESSURE HOLE

3.29 INTERFERENCE SEMILOG TYPE CURVES FOR E=0.7 AND 8=0 DEG.
CONSTANT PRESSURE HOLE

3.30 INTERFERENCE LOG-LOG TYPE CURVES FOR E=1.4 AND 8=0 DEG.
CONSTANT PRESSURE HOLE

3.31 INTERFERENCE SEMILOG TYPE CURVES FOR E=1.4 AND 8=0 DEG.
CONSTANT PRESSURE HOLE

3.32 INTERFERENCE LOG-LOG TYPE CURVES FOR E=0.7 AND 8=45 DEG.
CONSTANT PRESSURE HOLE

3.33 INTERFERENCE SEMILOG TYPE CURVES FOR E=0.7 AND 8=45 DEG.
CONSTANT PRESSURE HOLE

3.34 INTERFERENCE LOG-LOG TYPE CURVES FOR E=1.0 AND 8=45 DEG.
CONSTANT PRESSURE HOLE

3.35 INTERFERENCE SEMILOG TYPE CURVES FOR E=1.0 AND 8=45 DEG.
CONSTANT PRESSURE HOLE

3.36 INTERFERENCE LOG-LOG TYPE CURVES FOR E=1.4 AND 8=45 DEG.
CONSTANT PRESSURE HOLE

3.37 INTERFERENCE SEMILOG TYPE CURVES FOR E=1.4 AND 8=45 DEG.
CONSTANT PRESSURE HOLE

3.38 INTERFERENCE LOG-LOG TYPE CURVES FOR E=0.7 AND 8=90 DEG.
CONSTANT PRESSURE HOLE

3.39 INTERFERENCE SEMILOG TYPE CURVES FOR E=0.7 AND 8=90 DEG.
CONSTANT PRESSURE HOLE

3.40 INTERFERENCE LOG-LOG TYPE CURVES FOR E=1.0 AND 8=90 DEG.
CONSTANT PRESSURE HOLE

3.41 INTERFERENCE SEMILOG TYPE CURVES FOR E=1.0 AND 8=90 DEG.
CONSTANT PRESSURE HOLE

3.42 INTERFERENCE LOG-LOG TYPE CURVES FOR E=1.4 AND 8=90 DEG.
CONSTANT PRESSURE HOLE

3.43 INTERFERENCE SEMILOG TYPE CURVES FOR E=1.4 AND 8=90 DEG.
CONSTANT PRESSURE HOLE ........................................ 78
3.44 INTERFERENCE LOG-LOG TYPE CURVES FOR $E=0.7$ AND $\theta=135$ DEG.
CONSTANT PRESSURE HOLE ........................................ 79
3.45 INTERFERENCE SEMILOG TYPE CURVES FOR $E=0.7$ AND $\theta=135$ DEG.
CONSTANT PRESSURE HOLE ........................................ 79
3.46 INTERFERENCE LOG-LOG TYPE CURVES FOR $E=1.0$ AND $\theta=135$ DEG.
CONSTANT PRESSURE HOLE ........................................ 80
3.47 INTERFERENCE SEMILOG TYPE CURVES FOR $E=1.0$ AND $\theta=135$ DEG.
CONSTANT PRESSURE HOLE ........................................ 80
3.48 INTERFERENCE LOG-LOG TYPE CURVES FOR $E=1.4$ AND $\theta=135$ DEG.
CONSTANT PRESSURE HOLE ........................................ 81
3.49 INTERFERENCE SEMILOG TYPE CURVES FOR $E=1.4$ AND $\theta=135$ DEG.
CONSTANT PRESSURE HOLE ........................................ 81
3.50 INTERFERENCE LOG-LOG TYPE CURVES FOR $E=0.7$ AND $\theta=180$ DEG.
CONSTANT PRESSURE HOLE ........................................ 82
3.51 INTERFERENCE SEMILOG TYPE CURVES FOR $E=0.7$ AND $\theta=180$ DEG.
CONSTANT PRESSURE HOLE ........................................ 82
3.52 INTERFERENCE LOG-LOG TYPE CURVES FOR $E=1.0$ AND $\theta=180$ DEG.
CONSTANT PRESSURE HOLE ........................................ 83
3.53 INTERFERENCE SEMILOG TYPE CURVES FOR $E=1.0$ AND $\theta=180$ DEG.
CONSTANT PRESSURE HOLE ........................................ 83
3.54 INTERFERENCE LOG-LOG TYPE CURVES FOR $E=1.4$ AND $\theta=180$ DEG.
CONSTANT PRESSURE HOLE ........................................ 84
3.55 INTERFERENCE SEMILOG TYPE CURVES FOR $E=1.4$ AND $\theta=180$ DEG.
CONSTANT PRESSURE HOLE ........................................ 84
3.56 A COMPARISON BETWEEN TWO CASES :
1 : A CONSTANT RATE WELL AND A CONSTANT PRESSURE WELL
2 : TWO CONSTANT RATE WELLS .................................. 87
3.57 SEMILOG TYPE CURVES FOR THE RATE-PRESSURE MODEL .......... 87
3.58 SUPERPOSITION FOR A CONSTANT PRESSURE SEMI-CIRCLE AND
A NO-FLOW LINEAR BOUNDARY ................................... 89
3.59 SUPERPOSITION FOR A CONSTANT PRESSURE QUARTER CIRCLE
BOUND BY NO-FLOW LINEAR BOUNDARIES ......................... 89
3.60 SUPERPOSITION SEMILOG CURVES FOR A SEMI-CIRCLE $E=0.5$
AND $0^\circ.1$. ANGLE BETWEEN THE WELL AND THE BOUNDARY $22.5$ DEG.
CONSTANT PRESSURE HOLE ...................................... 91
4.1 A SCHEMATIC DIAGRAM OF THE NO-FLOW BOUNDARY HOLE SYSTEM ........................................... 93
4.2 SEMILOG CURVES FOR $F = 0.3$ TO 0.95 AND $2c = 100$. NO-FLOW BOUNDARY HOLE ............................... 112
4.3 SEMILOG CURVES FOR $F = 0.3$ TO 0.95 AND $2c = 250$. NO-FLOW BOUNDARY HOLE ............................... 112
4.4 SEMILOG CURVE FOR $2c = 500$ AND $F = 0.5$ MATCHED WITH A SHIFTED CURVE FOR $2c = 100$ AND $F = 0.5$. NO-FLOW BOUNDARY HOLE ............. 113
4.5 A GENERALIZED SEMILOG TYPE CURVE FOR THE NO-FLOW INTERNAL CIRCULAR BOUNDARY .......................... 114
4.6 INTERFERENCE LOG-LOG CURVES FOR $E = 0.9999$, $F = 0.9$ AND $\theta = 0, 45, 90, 135, 180$ DEG. NO-FLOW BOUNDARY HOLE ...................... 115
4.7 INTERFERENCE SEMILOG CURVES FOR $E = 0.9999$, $F = 0.9$ AND $\theta = 45, 90, 135, 180$ DEG. NO-FLOW BOUNDARY HOLE ................. 115
4.8 LONG TIME LOCATION OF $p_d$ L.S. NO-FLOW BOUNDARY HOLE .......................................................... 117
4.9 INTERFERENCE LOG-LOG CURVES FOR $E = 0.7$, $F = 0.1$ TO 0.6 AND $\theta = 0$ DEG. NO-FLOW BOUNDARY HOLE ......................................................... 118
4.10 INTERFERENCE LOG-LOG CURVES FOR $E = 0.7$, $F = 0.1$ TO 0.6 AND $\theta = 45$ DEG. NO-FLOW BOUNDARY HOLE ......................................................... 118
4.11 INTERFERENCE LOG-LOG CURVES FOR $E = 0.7$, $F = 0.1$ TO 0.6 AND $\theta = 90$ DEG. NO-FLOW BOUNDARY HOLE ......................................................... 119
4.12 INTERFERENCE LOG-LOG CURVES FOR $E = 0.7$, $F = 0.1$ TO 0.6 AND $\theta = 135$ DEG. NO-FLOW BOUNDARY HOLE ......................................................... 119
4.13 INTERFERENCE LOG-LOG CURVES FOR $E = 0.7$, $F = 0.1$ TO 0.6 AND $\theta = 180$ DEG. NO-FLOW BOUNDARY HOLE ......................................................... 120
5.1 A GENERALIZED SEMILOG TYPE CURVE FOR THE INTERNAL CIRCULAR BOUNDARY CASE INCLUDING LINEAR BOUNDARIES ........................................ 122

B. 1 THE GEOMETRY FOR THE POINTS ON THE LATE TIME CONSTANT PRESSURE CIRCLE .................................. 136
B.2 CURVES FOR TWO POINTS ON THE LATE TIME CONSTANT PRESSURE CIRCLES. CONSTANT PRESSURE LINEAR BOUNDARY .............................. 139

D. 1 THE GEOMETRY OF THE DOUBLET MODEL FOR THE CONSTANT
PRESSURE HOLE AT STEADY STATE .................................................. 149

F.1 \( R(m) \) AND \( L(m) \) AS A FUNCTION OF \( m \) IN THE ASYMPTOTIC
EXPANSION OF \( K_{50}(5) \), \( K_{50}(10) \) AND \( K_{50}(20) \) .................. 153

F.2 ASYMPTOTIC EXPANSION FOR \( K_{50}(5) \) AS A FUNCTION OF THE
NUMBER OF TERMS IN THE EXPANSION. \( m \) ................................. 155

G.1 A FLOW DIAGRAM FOR THE COMPUTER PROGRAM .................... 158

G.2 A SCHEMATIC FLOW DIAGRAM OF THE NAVIGATION PROGRAM .... 159
## LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>DRAWDOWN DATA FOR A LINEAR BOUNDARY CASE TEST</td>
<td>20</td>
</tr>
<tr>
<td>3.1</td>
<td>A COMPARISON BETWEEN THE STEADY STATE DIMENSIONLESS PRESSURE FOR TWO MODELS:</td>
<td></td>
</tr>
<tr>
<td></td>
<td>1. RATE - RATE</td>
<td>88</td>
</tr>
<tr>
<td></td>
<td>2. RATE - PRESSURE</td>
<td></td>
</tr>
<tr>
<td>c.1</td>
<td>NUMERICAL DATA FOR SHIFTING A SEMILOG CURVE FOR THE CONSTANT PRESSURE LINEAR</td>
<td>146</td>
</tr>
<tr>
<td></td>
<td>BOUNDARY CASE</td>
<td></td>
</tr>
<tr>
<td>C.2</td>
<td>NUMERICAL DATA FOR SHIFTING A SEMILOG CURVE FOR THE CONSTANT PRESSURE HOLE</td>
<td>147</td>
</tr>
<tr>
<td>C.3</td>
<td>NUMERICAL DATA FOR SHIFTING A SEMILOG CURVE FOR THE NO-FLOW BOUNDARY HOLE</td>
<td>147</td>
</tr>
<tr>
<td>G.1</td>
<td>THE NAVIGATION PROGRAM DECISION TREE</td>
<td>160</td>
</tr>
<tr>
<td>H. 1</td>
<td>CONSTANT PRESSURE INTERNAL BOUNDARY</td>
<td>210</td>
</tr>
<tr>
<td>H. 2</td>
<td>NO-FLOW INTERNAL BOUNDARY</td>
<td>219</td>
</tr>
<tr>
<td>H. 3</td>
<td>CONSTANT PRESSURE LINEAR BOUNDARY</td>
<td>226</td>
</tr>
<tr>
<td>H. 4</td>
<td>NO-FLOW LINEAR BOUNDARY</td>
<td>227</td>
</tr>
<tr>
<td>H. 5</td>
<td>THE LINE SOURCE</td>
<td>228</td>
</tr>
</tbody>
</table>
CHAPTER 1: INTRODUCTION

In this chapter, we discuss the need for a pressure transient analysis method for reservoirs with internal circular boundaries and the conditions under which this method is applicable. Then, we present a general discussion of the problem followed by a background review. Finally, we conclude this chapter with a statement of the problem.

1.1 APPLICATIONS

Constant pressure internal subregions occur naturally in oil fields as gas caps and in geothermal fields as steam or noncondensable gas caps. These subregions can also be induced artificially during steam flooding, in-situ combustion, immiscible gas drive, aquifer gas storage and growth of steam or gas bubbles below the bubble point pressure. In any of these cases, testing a well completed in the liquid zone, exterior to the circular discontinuity, can provide estimates of the size of this internal gaseous subregion and the distance to it if a technique can be derived to analyze such tests.

In this research, we are concerned with drawdown testing in a well exterior to a circular boundary such as a gas cap. Ramey (1983) and Standing (1983) were concerned about the effect such a gas cap can have on pressure responses during well tests. Ramey (1983) has observed a high total system compressibility when analyzing interference tests in
such systems.

The new method presented here for pressure transient analysis is applicable whenever a two composite system resembles an infinite system containing an internal circular boundary.

This method offers a way to detect naturally occurring compressible regions when exploratory wells are completed in the liquid zone. Even a locally small gas cap would have a large impact on pressure transients in nearby wells. Using this method, we can distinguish between the total system compressibility and an effect of a compressible subregion.

We can apply this analysis method to study pressure interference between oil and gas fields sharing the same aquifer. If the oil field is far enough from the gas field and is relatively small, it can be approximated as a line source. The gas field can be relatively large, resembling a constant pressure circle, or small and resembling a constant pressure line source.

In developed systems, the method derived here permits monitoring of the location of fronts without having to shut in the injectors, which in many cases is undesirable.

The analysis assumes a horizontal slab reservoir with the compressible subregion completely penetrating the vertical thickness. However, the analysis is applicable to systems with liquid zones underlying gas bubbles in thin formations. Examples of such cases are naturally occurring gas caps or gas storage in aquifers.

Internal subregions with no-flow boundaries occur when a low mobility and compressibility fluid is injected into a reservoir containing a mobile compressible fluid. This condition may occur during reinjection of geothermal water into steam or two phase zones.
enhanced oil recovery processes, such as polymer flooding or water injection during gas fillup, may develop similar conditions. The analysis of pressure transient tests in the mobile region may yield estimates of the size of the affected zone.

It is demonstrated here that useful results from interference testing are limited to the cases of constant pressure boundaries and only in systems where the location of the bubble is known. Interference testing may lead to an estimate of the shape of the affected zone, whether it is circular or not.

The application of pressure transient analysis methods for linear boundary cases is discussed extensively in the water and petroleum literature. A new method which improves the semilog portion of the analysis is developed as a corollary to this work. The use of a single semilog type curve replaces the straight line analysis method for determining the distance to the linear boundary.

1.2 Problem Description

Composite reservoirs are flow systems composed of two or more different regions. In this work, we consider models of oil, gas or geothermal reservoirs as two region infinite slab systems. One region is continuous and homogeneous and is bounded internally by a circular subregion. A production well and interference wells are exterior to the internal circular region, which may be conveniently described as a "hole" in the exterior region.
In a fully composite system, each region has its own flow characteristics. We have taken a simplifying approach, where the interface conditions of the internal circular region are specified as boundary conditions to the exterior regions instead of allowing the pressure transients to travel through the internal region. If the mobility and compressibility of the hole are high in comparison to the other region, the hole acts like a constant pressure source. On the other hand, if the mobility and compressibility of the hole are low in comparison to the extended region, the hole acts like a no-flow internal boundary. Hence, we consider both the no-flow and the constant pressure boundary conditions.

A linear boundary is a limiting case of a circular boundary with an infinite radius. If a linear boundary were to be wrapped around the well, it would form the known case of a well located within a circular boundary. If a linear boundary were to be wrapped away from the well, it forms an internal circular region that does not include the well.

The thrust of this research is to develop a pressure transient analysis method for a drawdown constant rate test for a well near an internal circular boundary. This reservoir limit test may be analyzed to determine the distance to and the size of the circular discontinuity.

1.3 BACKGROUND

Pressure transient tests are performed in order to gain knowledge about reservoir flow properties. The early and intermediate time
pressure data can provide estimates of the flow characteristics of the local area around the well. The late time pressure data can provide information about the extent or the limits of the reservoir. This research is mainly concerned with the effects of internal reservoir limits on the pressure response of a well, but is nevertheless heavily dependent on the ability to determine the flow characteristics of the area near the well. Hence, both the intermediate and late time pressure responses are needed.

Carslaw and Jaeger (1946) and Van Everdingen and Hurst (1949) presented the solutions for a constant flow rate line source well and for a finite radius well in an infinite slab system. At intermediate times, many wells follow this behavior, known as "infinite-acting". Ramey (1970) presented an interpretation method for early time pressure data in the presence of wellbore storage and skin effect. The present study does not consider the effects of storage and skin on reservoir limit testing.

Carslaw and Jaeger (1946) and Bixel, et al. (1963) presented the most general analysis for an infinite reservoir with a linear discontinuity. They considered two regions with different flow properties and continuity of pressure and flow rate at the linear boundary.

Other authors simplified the approach to the composite system by specifying the condition at the linear boundary using the method of images. Stallman (1952) published log-log type curves for both the no-flow and the constant pressure linear boundaries. His curves are applicable for the analysis of single well tests and also for interference tests. These curves may be used to find the distance to

Davis and Larkin (1963), Standing (1964), Witherspoon, et al. (1967) and Kruseman and de Ridder (1970) extended the log-log method for determining the distance to a linear boundary. They introduced the semilog method for identifying the intersection of two semilog straight lines representing the superposition of two line source solutions.

Cinco, et al. (1976) presented a solution to the transient pressure behavior of a well near a conductive linear fracture. This is a unique paper since they considered the fracture as an internal finite linear boundary.

Composite systems with circular discontinuities have been studied extensively in the literature. However, very few authors have considered internal circular discontinuities. Carslaw and Jaeger (1946) presented the Green's function for a point source external to an infinite cylinder with specified boundary conditions. Miller (1963) and Witherspoon, et al. (1967) considered testing near gas storage bubbles to be the same as testing near a constant pressure linear boundary. Most of the work using composite systems considered concentric systems with wells in the central region. Mathematically oriented works, describing heat conduction in composite materials were presented by Jaeger (1938 and 1943). Jaeger presented the Green's source function for an instantaneous line source centered and noncentered in a two
Larkin (1963) and Temeng and Horne (1983) applied Jaeger's work (1938 and 1934) on an eccentric line source to the flow of fluids in reservoirs. Hantush and Jacob (1960) considered an eccentric well within a bounded aquifer with a leaky caprock.

Loucks and Guerrero (1961) and Bixel and Van Poole (1967) presented type curves for a well centered in a two region radial flow system. Ramey (1970) presented approximate solutions for unsteady liquid flow for a well centered in a radially concentric composite system.

Katz, et al. (1959) and Coats, et al. (1959) studied gas storage in aquifers. Although the gas bubble may ride on top of the aquifer, they treated the aquifer as a radially concentric two region system. This approach is valid when the underlying aquifer is relatively thin. Coats (1962) included vertical flow under the gas bubble in his solution to this aquifer problem.

Hurst (1960) and Mortada (1960) considered interference between oil fields sharing a common aquifer. Their common approach was to treat the oil fields as line sources, hence, avoiding the dependence of the pressure on the angle of rotation.

The present work concentrates on internal circular boundaries, yet, the same mathematical methods apply also to linear boundary configurations.
1.4 **PROBLEM STATEMENT**

The objective of this study is to provide a practical method for estimating the size of and the distance to an internal circular discontinuity.

In order to achieve this objective, we pose a mathematical description of the pressure change in a reservoir due to a constant rate line source exterior to a circular boundary with specified boundary condition. We consider both no-flow and constant pressure boundaries.

We use the mathematical solutions to generate type curves that can be used in practical type curve matching procedures for analyzing transient pressure drawdown data from well tests.

First, we describe the limiting linear boundary cases, and then we consider constant pressure and no-flow internal circular boundary cases.
CHAPTER 2: LINEAR BOUNDARIES

In this chapter, we consider a drawdown pressure transient analysis method for determining the distance between a well and a nearby constant pressure or no-flow linear boundary.

First, we pose and solve the problem using the method of images. Then, we describe the current log-log and semilog methods of analysis. Finally, we present a new semilog method followed by a type curve matching example. Some of the techniques developed in this chapter are useful in later chapters considering internal circular boundaries.

2.1 PROBLEM STATEMENT

The problem is two dimensional with one axis of symmetry along the line perpendicular to the linear boundary that includes the line source well (see Fig. 2.1).

It is assumed that the system has an infinite radial extent, constant thickness, viscosity, porosity and compressibility, and constant and isotropic permeability. It is also assumed that the pressure gradients are small, so that the gradient squared terms can be neglected and that the flow is isothermal. Gravity effects are neglected.

The pressure \( p(r, \theta, t) \) must satisfy the following equation and boundary conditions:
FIGURE 2.1: A SCHEMATIC DIAGRAM OF THE CONSTANT PRESSURE LINEAR BOUNDARY SYSTEM

\[
\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} = \frac{1}{n} \frac{\partial p}{\partial r} \quad \text{at} \quad p(\infty, \theta, t) = 0 \quad (2.1)
\]

\[
p \text{ at the boundary} = 0 \quad (2.3)
\]

\[
\lim_{r \to 0} \frac{r}{\partial r} = -\frac{q\mu}{2\pi kh} \quad (2.4)
\]

\[
p(r, \theta, 0) = 0 \quad (2.5)
\]
2.2 **SOLUTION**

The solution uses the method of images for generating linear boundaries, superposing the line source due to the infinite acting well and an opposing line source or sink due to an image well. A constant pressure linear boundary is generated by a line source well and a line sink image. A no-flow linear boundary is generated by a line source well and a line source image. The use of imaging for generating linear boundaries was discussed by Carslaw and Jaeger (1946), Ferris, et al. (1962) and Ramey, et al. (1973).

The dimensionless pressure drop for a line source near a constant pressure linear boundary takes the form:

\[ p_D = -\frac{1}{2} \left[ E_1(-x_1) - E_1(-x_2) \right] \]  \hspace{1cm}  (2.6)

For a no-flow linear boundary the solution is:

\[ p_D = -\frac{1}{2} \left[ E_1(-x_1) + E_1(-x_2) \right] \]  \hspace{1cm}  (2.7)

where:

\[ x_1 = \frac{r_{Di}^2}{4\tau_D} \]  \hspace{1cm}  (2.8)

\( E_1 \) denotes the Exponential Integral. The dimensionless terms are defined in the conventional manner.
2.3 LOG-LOG AND SEMILOG ANALYSIS METHODS

Stallman (1952) presented log-log type curves for a line source well near a linear boundary (see Fig. 2.2). The curves below the line source curve approach steady state values and represent constant pressure linear boundaries. The curves that deviate above the line source curve are for no-flow linear boundaries. The parameter of the various curves is the ratio of the distance between the pressure point and the image well to the distance between the pressure point and the production well:

\[
\frac{r_{D2}}{r_{D1}} = \frac{r_{2}}{r_{1}} = \sqrt{\frac{h}{H}} = \text{constant} \tag{2.12}
\]

Stallman's type curve (Fig. 2.2) can be used for analyzing pressure responses from production wells and interference wells.

Brigham (1979) has shown that Eq. 2.12 represents circles that are centered at:

\[
P_{D} = \frac{2\pi kh(p_i - p)}{qB\mu} \tag{2.9}
\]

\[
t_{D} = \frac{kt}{\phi\mu c\tau_{w}^{2}} \tag{2.10}
\]

\[
r_{DL} = \frac{r_{D2}}{r_{w}} \tag{2.11}
\]
Figure 2.2: Log-log type curves for the linear boundary case.

After Stallman (1952)

\[
\begin{bmatrix}
  c' \\
  \frac{1 + H}{1 - H}, 0
\end{bmatrix}
\]

(2.13)

and have radii of:

\[
2c' \frac{\sqrt{H}}{1 - H}
\]

(2.14)

The distance between the well and the linear boundary is denoted by \( c' \) (see Fig. 2.1). The derivation of Eqs. 2.13 and 2.14 is presented in Appendix A.

Also, interference points which lie on the circles having the same \( r_2/r_1 \) ratio have the same dimensionless pressure response as a function
of reduced dimensionless time \( \frac{p_D}{t_D/r_D^2} \), hence, the use of the parameter \( r_2/r_1 \) in Stallman’s type curve. This behavior is discussed in Appendix B.

Using Stallman’s type curves, we can match the pressure response to one of the curves and determine the ratio \( r_2/r_1 \). However, it is difficult to interpolate between the curves. Addressing this difficulty, Davis and Hawkins (1963) and Witherspoon, et al. (1970) extended Stallman’s log-log type curve matching analysis using a double straight line semilog method. They observed that when the pressure - time data are graphed in a semilog fashion, two straight lines develop. Figure 2.3 presents an example of these two straight lines taken from Witherspoon, et al. (1970). The first straight line is the infinite acting period of the production well and develops after a dimensionless time of 10. The second straight line has a slope which is double that of the first line and represents the sum of two line sources. This second straight line develops when both the exponential integrals can be represented by the logarithmic approximation:

\[
- E_i(-X) = - \gamma - \ln(X)
\]  

(2.15)

where \( \gamma \) is the Euler constant.

Davis and Hawkins (1963) showed that for a production well, the distance to the linear boundary can be determined from the intersection point of the two straight lines:

\[
d = \left[ 0.561 nt_1 \right]^{1/2}
\]  

(2.16)
where:

d = distance between the well and the boundary

and:

\[ \eta = \frac{k}{\phi \mu c} \]  \hspace{1cm} (2.17)

and \( t_1 \) is the time of the intersection point (see Fig. 2.3).

Witherspoon, et al. (1970) presented a method based on two intersection points. The first point is the intersection of the first straight line and the time axis. The second point is the intersection point of the two straight lines. The ratio \( r_2/r_1 \) becomes:

\[ \frac{r_2}{r_1} = \sqrt{t_1/t_0} \] \hspace{1cm} (2.18)
Davis and Hawkins (1963) and Witherspoon, et al. (1970) pointed out several limitations of the double straight line method. If either of the straight lines did not fully develop, the method cannot be used. Furthermore, if the first straight line has a small slope, a small error in the location of the straight line will have a large effect on the calculated distance between the well and the image well.

So far, we have discussed the existing log-log and semilog methods for determining the distance between a well and an image well. In the next section, we present a new semilog type curve matching method for determining this distance.

2.4 A NEW SEMILOG TYPE CURVE MATCHING METHOD

In this section, we present a new semilog method for determining the ratio $r_2/r_1$ based on type curve matching. First, we describe how the type curve was generated. Then, a procedure for using the type curve is presented. Finally, we discuss the advantages of the new semilog method.

Figure 2.4 presents the same pressure-time data of Stallman's type curve (Fig. 2.2) in a semilog scale. The curves branching off horizontally from the line source curve represent constant pressure linear boundaries. The curves branching off above the line source curve represent no-flow linear boundaries.

Except for the early time transition part, prior to a dimensionless time of 10, all these curves can be collapsed to a single curve.
have chosen arbitrarily to shift all the curves and match them to the curve where $r_2/r_1 = 100$. The shifting method is discussed in Appendix C. As a result of this shifting of all the curves in Fig. 2.4, we generated a generalized semilog type curve presented in Fig. 2.5. The dimensionless pressure and time are modified based on the derivations presented in Appendix C.

The new semilog type curve can be used for both production and interference wells. An early time infinite acting period is needed in order to use this semilog type curve. The early time log-log match to the line source curve enables us to convert pressures to dimensionless pressures. The following procedure describes the use of the generalized semilog type curve (Fig. 2.5):
1) Make a log-log graph of the pressure-time response using the same scales as the log-log line source type curve.

2) Match the early time part of the data to the line source curve and pick a match point.

3) Convert all the pressures to a dimensionless form.

4) Make a semilog graph of the dimensionless pressure-time response using the same scale as in the generalized semilog type curve.
5) Match to one of the curves (constant pressure or no-flow boundary) and pick a match point. The transition and the late time data are the most important portion of the match.

6) Using the match point and the modified pressure equation, solve for the ratio \(r_2/r_1\), which in the case of a production well is approximately twice the distance to the linear boundary.

A type curve matching example is presented in the next section.

There are several advantages in using this semilog type curve matching method in comparison to the double straight line method. The new method can be used under all the conditions when the double straight line method is applicable. The first semilog straight line corresponds to an early time match to the line source curve, hence, pressures can be converted to a dimensionless form. However, there are conditions when the new method can be used and the previous method fails. Such a condition may occur when a test is terminated early, and only the first straight line and the transition between the two lines have developed. Another condition may occur when the first straight line is not defined, but we can still have a log-log match to the line source curve prior to a dimensionless time of 10.

In both these conditions, the new method can be used to determine the distance between a production well and the linear boundary or the distance between an interference well and the image well.

Note that the time axis can remain in real time units since the
2.5 **Type Curve Matching Example**

In this section, a synthetic drawdown test is analyzed using the new generalized semilog type curve. Table 2.1 presents hypothetical drawdown data given by Witherspoon, et al. (1970). The pressures are for an observation well, 325 feet away from the pumping well.

Figure 2.6 is a log-log graph of the data. Figure 2.7 is a log-log match of the data to the line source log-log type curve. The log-log match yields an approximate value for $r_2/r_1$ and a conversion factor between $p$ and $p_D$.

The match point is:

\[ p_D = p / 90 \]  (2.19)

Next, we convert the pressures to dimensionless pressures using Eq. 2.19 and make a semilog graph of the dimensionless pressure vs. real time (Fig. 2.8). Note that the time axis need not be converted to a dimensionless form. This can simplify the procedure by graphing the semilog data on the same sheet of paper with the log-log graph (see Fig. 2.9).
<table>
<thead>
<tr>
<th>Time (min)</th>
<th>Drawdown (feet)</th>
<th>Time (min)</th>
<th>Drawdown (feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.15</td>
<td>1000</td>
<td>74.61</td>
</tr>
<tr>
<td>30</td>
<td>0.31</td>
<td>1300</td>
<td>85.33</td>
</tr>
<tr>
<td>40</td>
<td>0.56</td>
<td>1600</td>
<td>87.70</td>
</tr>
<tr>
<td>50</td>
<td>1.31</td>
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<td>101.41</td>
</tr>
<tr>
<td>60</td>
<td>1.50</td>
<td>2200</td>
<td>105.36</td>
</tr>
<tr>
<td>70</td>
<td>2.51</td>
<td>2500</td>
<td>111.17</td>
</tr>
<tr>
<td>80</td>
<td>3.50</td>
<td>3000</td>
<td>119.60</td>
</tr>
<tr>
<td>90</td>
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<td>3500</td>
<td>125.01</td>
</tr>
<tr>
<td>100</td>
<td>3.33</td>
<td>4000</td>
<td>135.12</td>
</tr>
<tr>
<td>150</td>
<td>11.87</td>
<td>5000</td>
<td>148.81</td>
</tr>
<tr>
<td>200</td>
<td>17.42</td>
<td>6000</td>
<td>168.34</td>
</tr>
<tr>
<td>250</td>
<td>22.40</td>
<td>7000</td>
<td>170.01</td>
</tr>
<tr>
<td>300</td>
<td>28.61</td>
<td>8000</td>
<td>176.12</td>
</tr>
<tr>
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<td>9000</td>
<td>184.40</td>
</tr>
<tr>
<td>400</td>
<td>36.31</td>
<td>10000</td>
<td>191.93</td>
</tr>
<tr>
<td>500</td>
<td>44.70</td>
<td>15000</td>
<td>222.41</td>
</tr>
<tr>
<td>600</td>
<td>52.13</td>
<td>20000</td>
<td>247.33</td>
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<td>700</td>
<td>60.46</td>
<td>25000</td>
<td>267.00</td>
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<tr>
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<td>65.03</td>
<td>30000</td>
<td>282.13</td>
</tr>
<tr>
<td>900</td>
<td>68.50</td>
<td>40000</td>
<td>301.24</td>
</tr>
</tbody>
</table>
FIGURE 2.6: LOG-LOG GRAPH OF THE DRAWDOWN DATA FOR AN INTERFERENCE WELL. AFTER WITHERSPOON (1970)

FIGURE 2.7: LOG-LOG MATCH FOR THE DRAWDOWN DATA FOR AN INTERFERENCE WELL. AFTER WITHERSPOON (1970)
FIGURE 2.8: SEMILOG GRAPH OF THE DRAWDOWN DATA IN DIMENSIONLESS PRESSURE FORM

FIGURE 2.9: LOG-LOG AND SEMILOG GRAPHS FOR THE DRAWDOWN DATA IN DIMENSIONLESS PRESSURE FORM
Finally, the semilog graph of the data is matched to the generalized semilog type curve in Fig. 2.10. This match concentrates on the late time data and on the transition. The early time data, that do not match to the first straight line, correspond to early time line source behavior prior to a dimensionless time of 10. This early time portion of the data can be matched to the lowermost portion of the type curve. This is an example where the first straight line has not fully developed, yet, we have a good log-log match to the line source curve. At the match point:

\[
\begin{align*}
\bar{p}_D &= -1 \\

\bar{p}_D' &= 1.6
\end{align*}
\]

Next, we solve for \(2c'\) in the equivalent system using the modified pressure equation:

\[
p^*_D = p_D + \ln(100) - \ln(2c')
\]

\[
2c' = 7.427
\]

Since, in this case, the pressure is measured at the production well, \(r_{D2} + r_{D1} = r_{Dp} + 1 \times 2c'\), therefore:

\[
\frac{r_2}{r_1} = 6.427
\]

hence:

\[
r_2 = 6.427 \quad r_1 = 2089 \text{ ft}
\]
Using the double straight line method, Witherspoon, et al. (1970) found \(r_2\) to be 2025 ft.

In summary, two type curves are used in this new method. The log-log type curve of the line source is used to convert the pressure data to a dimensionless form. The new generalized semilog type curve (Fig. 2.5) is used to determine the distance between the pressure point and the image well. If the pressure point is at the production well, this distance is twice the distance between the well and the linear boundary.

The method of shifting the semilog curves and the semilog type curve matching technique are used in establishing the analysis method for internal circular boundaries presented in the next two chapters.
CHAPTER 3 : CONSTANT PRESSURE INTERNAL CIRCULAR BOUNDARY

This chapter presents the transient pressure analysis for a constant flow rate well near a constant pressure circular boundary. The problem is mathematically stated and solved using the Laplace transformation method. Then, the practical applications of the solution are discussed.

3.1 PROBLEM STATEMENT

The problem is two dimensional with one axis of symmetry along the line between the well and the center of the hole (see Fig. 3.1). The constant pressure hole cannot be treated as a line source if it is of finite radius, hence, the pressure at a given point is a function of three parameters: distance r, angle θ and time t.

It is assumed that the system has: an infinite radial extent, constant thickness, constant and isotropic permeability, constant viscosity, porosity and compressibility. It is also assumed that the pressure gradients are small so that the gradient squared terms can be neglected and that the flow is isothermal. The well produces at a constant flow rate.

The pressure \(p(r, \theta, t)\) must satisfy the following equation and boundary conditions:
Figure 3.1: A schematic diagram of the constant pressure hole system.

\[ \frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} = -\frac{1}{\eta} \frac{\partial p}{\partial t} \quad (3.1) \]

\[ p(\infty, \theta, t) = 0 \quad \text{or} \quad p_i \quad (3.2) \]

\[ p(a, \theta, t) = 0 \quad \text{or} \quad p_i \quad (3.3) \]

\[ \lim_{R \to 0} \frac{\partial p}{\partial R} = -\frac{q\mu}{2\pi kh} \quad (3.4) \]

\[ p(r, \theta, 0) = 0 \quad \text{or} \quad p_i \quad (3.5) \]
where:

\[ R^2 = r^2 + r'^2 - 2rr' \cos \theta \]  \hspace{1cm} (3.6)

and:

\[ n = \frac{k}{\phi \mu c_t} \]  \hspace{1cm} (3.7)

Equation 3.4 is the condition at the line source well exterior to the constant pressure boundary. The derivation of Eq. 3.4 follows.

The flow around the well is assumed radial:

\[ q(R) = \frac{2\pi R k h}{\mu} \frac{\partial p}{\partial R} \]

Now, as \( R \) tends to 0, \( q(R) \) tends to \( q \), so that the rate of production out of the system is maintained constant, hence Eq. 3.4:

\[ \lim_{R \to 0} \frac{\partial p}{\partial R} = \frac{-q \mu}{2\pi kh} \]
3.2 **LAPLACE TRANSFORMATION**

We transform Eqs. 3.1 through 3.4 into Laplace space using the initial condition of Eq. 3.5. In general:

\[ p(r, \theta, t) \rightarrow \tilde{p}(r, \theta, s) \]

\[
\frac{\partial^2 \tilde{p}}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{p}}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \tilde{p}}{\partial \theta^2} - \sqrt{s/\eta} \tilde{p} = 0 \tag{3.8}
\]

\[ \tilde{p}(\infty, \theta, s) = 0 \tag{3.9} \]

\[ \tilde{p}(a, \theta, s) = 0 \tag{3.10} \]

\[ \lim_{R \to 0} \frac{\partial \tilde{p}}{\partial R} = \frac{q \mu}{2\pi \kappa h} \tag{3.11} \]

3.3 **THE LAPLACE TRANSFORMATION SOLUTION**

The solution for the homogeneous boundary conditions, Eqs. 3.8, 3.9 and 3.11, in a coordinate system centered at the well is:

\[ \tilde{p} = \frac{q \mu}{2\pi \kappa h} K_0(R \sqrt{s/\eta}) \tag{3.12} \]

By the addition theorem for Bessel Functions, Carslaw and Jaeger
we translate Eq. 3.12 to a coordinate system centered at the center of the hole:

\[
\bar{p} = \frac{qu}{2\pi skh} \sum_{n=-\infty}^{\infty} \cos(n\theta) I_n(r\sqrt{s/\eta}) K_n(r'\sqrt{s/\eta})
\]

\(\text{for } r < r'\) \hspace{1cm} (3.13)

\[
\bar{p} = \frac{qu}{2\pi skh} \sum_{n=-\infty}^{\infty} \cos(n\theta) I_n(r'\sqrt{s/\eta}) K_n(r\sqrt{s/\eta})
\]

\(\text{for } r > r'\) \hspace{1cm} (3.14)

In order to satisfy the condition of constant pressure at the internal boundary, we assume that \(\bar{p}\) takes the following form:

\[
\bar{p} = \frac{qu}{2\pi skh} \sum_{n=-\infty}^{\infty} \cos(n\theta) \left[ I_n(r\sqrt{s/\eta}) K_n(r'\sqrt{s/\eta}) + A_n K_n(r\sqrt{s/\eta}) \right]
\]

\(\text{for } r < r'\) \hspace{1cm} (3.15)

\[
\bar{p} = \frac{qu}{2\pi skh} \sum_{n=-\infty}^{\infty} \cos(n\theta) \left[ I_n(r'\sqrt{s/\eta}) K_n(r\sqrt{s/\eta}) + A_n K_n(r'\sqrt{s/\eta}) \right]
\]

\(\text{for } r > r'\) \hspace{1cm} (3.16)
where the constants $A_n$ are to be set by the boundary condition. The particular solution, $K_{n'}$, is picked in order to satisfy the condition at infinite radii. A similar method for constructing the solution to the problem of an eccentric well within a circular subregion was presented by Carslaw and Jaeger (1946).

Equations 3.15 and 3.16 can be written as:

\[
\tilde{p} = \frac{qu}{2\pi skh} \sum_{n=0}^{\infty} \epsilon_n \cos(n\theta) \left[ I_n(r\sqrt{s/\eta})K_n(r'\sqrt{s/\eta}) + A_n K_n(r\sqrt{s/\eta}) \right]
\]

for $r < r'$

\[
\tilde{p} = \frac{qu}{2\pi skh} \sum_{n=0}^{\infty} \epsilon_n \cos(n\theta) \left[ I_n(r'\sqrt{s/\eta})K_n(r\sqrt{s/\eta}) + A_n K_n(r'\sqrt{s/\eta}) \right]
\]

for $r > r'$

where:

for $n = 0$, $\epsilon_n = 1$

for $n > 0$, $\epsilon_n = 2$

The internal boundary condition determines the coefficients $A_n$:

\[
A_n = - \frac{I_n(a\sqrt{s/\eta})K_n(r\sqrt{s/\eta})}{K_n(a\sqrt{s/\eta})} \quad (3.19)
\]
Substituting Eq. 3.19 into Eqs. 3.17 and 3.18 yields:

\[ \bar{p} = \frac{q \mu}{2\pi \varepsilon_0kh} \sum_{n=0}^{\infty} \varepsilon \cos(n\theta) \left[ I_n(r'\sqrt{s/\eta})K_n(r'\sqrt{s/\eta}) - \frac{I_n(a\sqrt{s/\eta})K_n(r'\sqrt{s/\eta})}{K_n(a\sqrt{s/\eta})} K_n(r\sqrt{s/\eta}) \right] \]

for \( r < r^* \)

\[ (3.20) \]

\[ \bar{p} = \frac{q \mu}{2\pi \varepsilon_0kh} \sum_{n=0}^{\infty} \varepsilon \cos(n\theta) \left[ I_n(r'\sqrt{s/\eta})K_n(r\sqrt{s/\eta}) - \frac{I_n(a\sqrt{s/\eta})K_n(r\sqrt{s/\eta})}{K_n(a\sqrt{s/\eta})} K_n(r'\sqrt{s/\eta}) \right] \]

for \( r > r^* \)

\[ (3.21) \]

Next, we make the problem dimensionless using the standard definitions:

\[ p_D = \frac{2\pi \varepsilon_0(p_1 - p)}{\eta \mu} \]

\[ (3.22) \]

\[ \tau_D = \frac{kt}{\phi \mu c \tau_w^2} \]

\[ (3.23) \]
\[ r_D^2 = \frac{r}{r_w} \quad (3.24) \]

\[ r'_D = \frac{r'}{r_w} \quad (3.25) \]

\[ a_D = \frac{a}{r_w} \quad (3.26) \]

\[ R_D = \frac{R}{r_w} \quad (3.27) \]

Substituting Eqs. 3.22 through 3.27 into Eqs. 3.20 and 3.21 yields:

\[
\overline{p}_D = \frac{1}{s} \sum_{n=0}^{\infty} \epsilon_n \cos(n\theta) \left[ I_n \left( r_D \sqrt{s} \right) K_n \left( r'_D \sqrt{s} \right) \right. \\
- \frac{I_n \left( a_D \sqrt{s} \right) K_n \left( r'_D \sqrt{s} \right)}{K_n \left( a_D \sqrt{s} \right) K_n \left( r_D \sqrt{s} \right)} \left. K_n \left( r_D \sqrt{s} \right) \right] 
\]

for \( r_D < r'_D \) \quad (3.28)

\[
\overline{p}_D = \frac{1}{s} \sum_{n=0}^{\infty} \epsilon_n \cos(n\theta) \left[ I_n \left( r'_D \sqrt{s} \right) K_n \left( r_D \sqrt{s} \right) \right. \\
- \frac{I_n \left( a_D \sqrt{s} \right) K_n \left( r_D \sqrt{s} \right)}{K_n \left( a_D \sqrt{s} \right) K_n \left( r'_D \sqrt{s} \right)} \left. K_n \left( r'_D \sqrt{s} \right) \right] 
\]

for \( r_D > r'_D \) \quad (3.29)
Note that Eq. 3.29 is Eq. 3.28 with \( r_D \) and \( r'_D \) interchanged. The Laplace solution was inverted numerically using an algorithm developed by Stehfest (1970). A description of the algorithm is presented in Section 3.5.

3.4 THE ANALYTICAL SOLUTION

The following presents the analytical inversion of the Laplace solution into the real time solution using the method of residues.

The Laplace solution of Eq. 3.28 can be written as:

\[
\bar{p}_D = \frac{1}{s} \sum_{n=0}^{\infty} \varepsilon \cos(n\theta) \frac{1}{K_n(a_D \sqrt{s})} \left\{ I_n(r_D \sqrt{s}) K_n(r'_D \sqrt{s}) K_n(a_D \sqrt{s}) - K_n(r_D \sqrt{s}) I_n(a_D \sqrt{s}) K_n(r'_D \sqrt{s}) \right\}
\]

for \( r_D < r'_D \) (3.30)

At \( s=0 \) we have a single pole, hence, we can use the small argument approximations for the Modified Bessel Functions:

\[
K_0(z) = -(1n \frac{z}{2} + \gamma)
\]

(3.31)

\[
K_n(z) = 2^{n-1}(n-1)! \frac{z^{-n}}{n!}
\]

(3.32)
\[ I_0(z) = 1 \]  
\[ I_n(z) = 2^{-n} z^n / n! \]  

As \( s \) tends to 0, \( t \) tends to \( \infty \) and the residue at \( s=0 \) is the steady state pressure drop, \( p_{Dss} \). Substituting Eqs. 3.31 through 3.34 into Eq. 3.30 yields:

\[ \tilde{p}_{D1} = \frac{1}{s} \ln \frac{r_D}{a_D} + \sum_{n=1}^{\infty} \cos(n\theta) \left[ \frac{r_D^n}{n} \left( \frac{r_D}{r_D} \right)^n - \frac{a_D^n}{r_D r_D^n} \right] \]  

Using the following relation:

\[ \sum_{k=1}^{\infty} \frac{1}{k} p^k \cos(k\theta) = \frac{1}{2} \ln(1 - 2pcos\theta + p^2) \]  

and the Laplace inversion formula:

\[ \frac{1}{s} (b) + b \]  

Equation 3.35 inverts into the following:
This steady state solution can also be derived using superposition of line sources, with an identical result. This derivation is presented in Appendix D.

For \( r_D > r'_D \) we interchange \( r_D \) and \( r'_D \) in Eq. 3.38:

\[
\rho_{Dss} = \ln \left( \frac{r'_D}{a_D} \right) + \frac{1}{2} \ln \frac{1 - 2 \frac{a_D^2}{r_D r'_D} \cos \theta + (\frac{a_D^2}{r_D r'_D})^2}{1 - 2 \frac{r_D^2}{r'_D^2} \cos \theta + (\frac{r_D}{r'_D})^2}
\]

for \( r_D > r'_D \) \hspace{1cm} (3.39)

Factoring the term \((r'_D/r_D)^2\) out of the denominator of the last term in Eq. 3.39, and joining it to the first term, Eq. 3.39 becomes identical to Eq. 3.38. This is expected from the reciprocity principle.

When \( s \neq 0 \) we use the residues at the roots of \( K_n(a_D \sqrt{s}) \). Let \( \xi_{n/m} \) denote the \( m \)th zero of \( K_n(a_D \sqrt{s}) = K_n(\xi_{n/m}) \). \( K_n(z) \) has \( n \) zeroes in the second and third quadrants, Macdonald (1938), Watson (1948) and Abramowitz (1964). Using the method of residues we evaluate the inversion of \( \rho_{D2} \):
\[
\text{RES}(\xi_{n/m}) = \lim_{s \to \xi^2_{n/m}} \sum_{n=0}^{\infty} e \cos(n\theta) \frac{(s - \xi^2_{n/m}) e^{st_D} K_n(r_D^{1/s})}{sK_n(a_D^{1/s})}
\]

\[
= \left[ I_n(r_D^{1/s})K_n(a_D^{1/s}) - K_n(r_D^{1/s})I_n(a_D^{1/s}) \right]
\]

Rearranging Eq. 3.40:

\[
\text{RES}(\xi^2_{n/m}) = \sum_{n=0}^{\infty} e \cos(n\theta) B_n \frac{e^{\xi^2_{n/m} t_D} K_n(r_D^{1/s})}{\xi^2_{n/m}}
\]

where:

\[
B_n = \lim_{s \to \xi^2_{n/m}} \frac{(s - \xi^2_{n/m})}{K_n(a_D^{1/s})}
\]

\[
= \left[ I_n(r_D^{1/s})K_n(a_D^{1/s}) - K_n(r_D^{1/s})I_n(a_D^{1/s}) \right]
\]

Using L'Hôpital's rule, we evaluate \( B_n \):

\[
B_n = -\frac{2\xi_{n/m} K_n(\xi_{n/m} r_D) I_n(\xi_{n/m} a_D)}{a_D K_n(\xi_{n/m} a_D)}
\]

From Abramowitz (1964, p.361):
Using Eq. 3.44 and the fact that \( K_n(\xi_{n/m} a_D) = 0 \), we find that:

\[
B_n = \frac{2\xi_{n/m}}{a_D} \frac{K_n(\xi_{n/m} r_D) I_n(\xi_{n/m} a_D)}{K_n+1(\xi_{n/m} a_D)}
\]  

Substituting Eq. 3.45 into Eq. 3.41 yields:

\[
\text{RES}(\xi_{n/m}^2) = \sum_{n=0}^{\infty} \varepsilon_n \cos(n\theta) \frac{\xi_n^2}{a_D} \frac{K_n(\xi_{n/m} r_D) I_n(\xi_{n/m} a_D)}{K_n+1(\xi_{n/m} a_D)}
\]  

Now, in order to complete the inversion of \( p_{D2} \), we use the residues from Eq. 3.46:

\[
p_{D2} = 2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} \varepsilon_n \cos(n\theta) e^{\xi_{n/m} t_D} \frac{\xi_{n/m}^2}{a_D} \frac{K_n(\xi_{n/m} r_D) I_n(\xi_{n/m} a_D)}{K_n+1(\xi_{n/m} a_D)}
\]  

\[
\frac{K_n(\xi_{n/m} r_D) K_n(\xi_{n/m} r_D) I_n(\xi_{n/m} a_D)}{a_D \xi_{n/m} K_n+1(\xi_{n/m} a_D)}
\]
We can express \( p_{D2} \) in terms of Bessel Functions instead of Modified Bessel Functions. We use the following relations:

\[
I_n(z) = i^{-n} J_n(iz)
\]

\[
K_n(z) = \frac{\pi}{2} i^{n+1} [ J_n(iz) + Y_n(iz) ] = \frac{\pi}{2} i^{n+1} H_n^{(1)}(iz)
\]

The second and third quadrants for \( K_n(z) \) correspond to the third and fourth quadrants for \( H_n^{(1)}(z) \) since the argument of the Hankel Function is rotated by \( \pi/2 \).

Substituting Eqs. 3.48 and 3.49 into Eq. 3.30 yields:

\[
p_{D2} = \frac{\pi i}{2s} \sum_{n=0}^{\infty} \varepsilon \cos(n\theta) \frac{H_n^{(1)}(ir\sqrt{s})}{H_n^{(1)}(ia_Dr\sqrt{s})} [ J_n(ia_Dr\sqrt{s})H_n^{(1)}(ia_Dr\sqrt{s}) ]
\]

\[
- J_n(ia_Dr\sqrt{s})H_n^{(1)}(ia_Dr\sqrt{s})
\]

\( H_n^{(1)}(ia_Dr\sqrt{s}) \) has zeroes at \( ia_Dr\sqrt{s} = \mu_1, \mu_2, \ldots, \mu_m \) and \( p_{D2} \) has simple poles at:

\[
s = -\left( \frac{\mu_m}{a_D} \right)^2 = -\frac{\alpha^2}{a_D}
\]

or:

\[
a_m = \left( \frac{\mu_m}{a_D} \right) = i\sqrt{s}
\]
\[ a_{n/m} \] denotes the \( m \)th zero of \( H_n^{(1)}(ia_D \sqrt{s}) = H_n^{(1)}(\alpha n/m) \).

Using the method of residues we evaluate the inversion of \( \frac{1}{D_2} \):

\[
RES(-\alpha_{n/m}^2) = \pi \lim_{s \to -\alpha_{n/m}} \sum_{n=0}^{\infty} \epsilon_n \cos(n\theta) \]

\[
\cdot \frac{i(s + a_{n/m}^2) e^{t_D} H_n^{(1)}(ir_D \sqrt{s})}{s H_n^{(1)}(ia_D \sqrt{s})} \]

\[
\cdot \left[ J_n(ir_D \sqrt{s}) H_n^{(1)}(ia_D \sqrt{s}) - J_n(ia_D \sqrt{s}) H_n^{(1)}(ir_D \sqrt{s}) \right] \]

Rearranging Eq. 3.53:

\[
RES(-\alpha_{n/m}^2) = \pi \sum_{n=0}^{\infty} \epsilon_n \cos(n\theta) A_n \frac{e^{-\alpha_{n/m}^2 t_D} (1)}{H_n^{(1)}(\alpha n/m r_D')} \]

where:

\[ A = \lim_{n \to -\alpha_{n/m}} \frac{i(s + a_{n/m}^2)}{H_n^{(1)}(ia_D \sqrt{s})} \]

\[
\cdot \left[ J_n(ir_D \sqrt{s}) H_n^{(1)}(ia_D \sqrt{s}) - J_n(ia_D \sqrt{s}) H_n^{(1)}(ir_D \sqrt{s}) \right] \]
Using L'Hôpital's rule, we evaluate $A_n$:

\[
A_n = \lim_{s \to -\frac{\alpha_{n/m}}{n/m}} \frac{i}{H_n^{(1)}(ia_D \sqrt{s}) \left(\frac{i a_D}{2\sqrt{s}}\right)}
\]

\[
\cdot \{ \left[ J_n(ia_D \sqrt{s})H_n^{(1)}(ia_D \sqrt{s}) - J_n(iD_D \sqrt{s})H_n^{(1)}(iD_D \sqrt{s}) \right]
\]

\[
+ (s + \frac{\alpha_{n/m}^2}{n/m}) \frac{d}{ds} \left[ J_n(iD_D \sqrt{s})H_n^{(1)}(ia_D \sqrt{s}) \right]
\]

\[- J_n(ia_D \sqrt{s})H_n^{(1)}(iD_D \sqrt{s}) \} =
\]

\[
= \lim_{s \to -\frac{\alpha_{n/m}}{n/m}} - \frac{J_n(ia_D \sqrt{s})H_n^{(1)}(iD_D \sqrt{s})}{\frac{a_D}{2\sqrt{s}} H_n^{(1)}(ia_D \sqrt{s})}
\]

Substituting $s = -\frac{2}{\alpha_{n/m}}$ yields:

\[
A_n = - \frac{2\sqrt{s} J_n(\alpha_{n/m} a_D)H_n^{(1)}(\alpha_{n/m} a_D)}{a_D H_n^{(1)}(\alpha_{n/m} a_D)}
\]

(3.56)

From Abramowitz (1964, p. 361):
Using Eq. 3.57 and the fact that $H_n^{(1)}(\alpha a_D) = 0$, we find that:

$$A_n = \frac{2\alpha n/m}{J_n(\alpha a_D) J_n(\alpha a_D) + J_n(\alpha a_D) J_n(\alpha a_D)}$$

Substituting Eq. 3.58 into Eq. 3.54 yields:

$$\text{RES} \left( -\frac{a^2}{n/m} \right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{e^{-an/m}}{\sum_{m=0}^{\infty} \cos(n\theta)}$$

Now, in order to complete the inversion of $P_{D2}$, we use the residues from Eq. 3.59:

$$P_{D2} = -\pi \sum_{n=0}^{\infty} \frac{\sum_{m=0}^{\infty} \cos(n\theta)}{\sum_{m=0}^{\infty} \cos(n\theta)}$$

$$= \frac{H_n^{(1)}(\alpha a_D) H_n^{(1)}(\alpha a_D) J_n(\alpha a_D) J_n(\alpha a_D)}{a_D H_n a_D}$$

(3.60)
Finally, the complete real time solution is \( p_D = p_{D1} + p_{D2} \). In terms of Modified Bessel Functions, the solution is:

\[
p_D = \ln\left(\frac{r}{a_D}\right) + \frac{1}{2} \ln \frac{1 - 2\frac{a_D^2}{r_Dr'_D} \cos \delta + \left(\frac{a_D^2}{r_Dr'_D}\right)^2}{1 - 2\frac{r}{r'_D} \cos \delta + \left(\frac{r}{r'_D}\right)^2}
\]

\[
+ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \epsilon_n \cos(n\delta) e^{\xi\xi} \sum_{D}^{2} \frac{2}{\xi} \frac{K_{\xi}(\xi_{n/m} r'_D)K_{\xi}(\xi_{n/m} r_d)I_{\xi}(\xi_{n/m} a_d)}{a_D 5n/m K_{\xi+1}(\xi_{n/m} a_d)}
\]

for \( r_D < r'_D \)  \hspace{1cm} (3.61)
In terms of Bessel Functions the solution is:

\[ p_D = \ln \left( \frac{r_D}{a_D} \right) + \frac{1}{2} \ln \left( \frac{1 - \frac{a_D}{r_D} \cos \theta + \left( \frac{a_D}{r_D} \right)^2}{1 - \frac{a_{D'}}{r_{D'}} \cos \theta + \left( \frac{a_{D'}}{r_{D'}} \right)^2} \right) \]

\[ - \pi \sum_{n=0}^{\infty} \sum_{m=0}^{n} \epsilon_n \cos(n \theta) e^{-\frac{a_{n/m}}{t_D}} \]

\[ \frac{H_n^{(1)}(\alpha_{n/m} r_D) H_n^{(1)}(\alpha_{n/m} r_{D'}) J_n^{(1)}(\alpha_{n/m} a_D)}{a_D a_{n/m} H_{n+1}^{(1)}(\alpha_{n/m} a_D)} \]

for \( r_D < r_{D'} \) (3.62)

For \( r_D > r_{D'} \), we interchange \( r_D \) and \( r_{D'} \) in Eqs. .61 and .62.

The transient part of the real time solution was not used in the numerical evaluations due to complexities of computation. However, the general behavior of the pressure response can be deduced from the real time solution. The steady state part of the solution is a limiting value for the numerically evaluated pressures at late time.
3.5 **NUMERICAL INVERSION OF THE LAPLACE TRANSFORM**

Due to the complexity of the numerical evaluation of Eq. 3.62, we use the Laplace form of the solution, Eqs. 3.28 and 3.29 and invert the transform numerically. The numerical Laplace transform inverter was presented by Stehfest in 1970.

If \( \overline{p}(s) \) is the Laplace transform of \( p(t) \), then, the equations used in the algorithm are:

\[
p(t) = \frac{\ln 2}{t} \sum_{i=1}^{N} v_i \overline{p} \left( \frac{\ln 2}{t} i \right)
\]

(3.63)

where:

\[
v_i = (-1)^{\frac{N}{2} + 1} \min \left( \frac{N}{2}, \frac{N}{2} \right) \sum_{k=1}^{\min(i, N)} \frac{N}{k} (2k)! \left( \frac{N}{2} - k \right) ! k ! (k-1)! (i-k)! (2k-i)!
\]

(3.64)

\( N \) is the number of sampling points where \( \overline{p}(s) \) is evaluated for each inversion. Some of the limitations of the Stehfest algorithm are described by Stehfest (1970) and Shinohara (1980). The pressure function should be continuous, moderately varying and not rapidly oscillating. The number of sampling points, \( N \), was taken as 8. For the 64 bit arithmetic that was used, a value of \( N=16 \) is the best in terms of accuracy for the type of function presented here. With \( N=16 \) it is possible to produce an accuracy of 6 to 7 digits in most ranges. The program was tested at various \( N \)'s. With \( N=8 \) the accuracy is within 4 to

-45-
5 digits which is sufficient for practical purposes and requires significantly less computer time than with N=16.

The Laplace solutions are given by Eqs. 3.28 and 3.29. In the numerical inversion, only Eq. 3.28 was used. Although Eq. 3.28 is written for \( r_D < r'_D \), it can be used for \( r_D > r'_D \) making use of the reciprocity principle. The pressure drop at \( r_D \) due to the well at \( r'_D \) is the same as the pressure drop at \( r'_D \) due to a well at \( r_D \).

The computer program for the Stehfest algorithm is presented in Appendix G.

### 3.6 TYPE CURVE MATCHING FOR THE PRODUCTION WELL

In this section we show how to type curve match for the size of the hole and the distance to it, using the production well pressure data.

#### 3.6.1 GENERAL DISCUSSION

The pressure point representing the production well is one dimensionless radius away from the line source, at angle zero and in the direction of the hole. The pressure - time behavior depends on two factors:

a. The ratio \( F \), which is the ratio of the diameter of the hole to the distance between the center of the hole and the line...
FIGURE 3.2: THE GEOMETRY FOR TYPE CURVE MATCHING.

CONSTANT PRESSURE HOLE

source (see Fig. 3.2).

b. The actual size of the system, c (see .Fig. 3.2).

Figures 3.3 and 3.4 present the pressure-time behavior for a constant c=50 and various ratios of F, from 0.1 to 0.9. Figure 3.3 is in log-log scale and Fig. 3.4 is in semilog scale. Every curve starts off on the line source solution then undergoes a transition to approach a steady state value. The steady state values of $p_D$ can be calculated directly using Eq. 3.38 with an angle of zero.

At long time, the system will approach a steady state condition which can be represented simply by a doublet model. This is because at steady state all the equipotential lines are circles (see Appendix A). We can find the location of the image well at steady state (see Fig. 3.5). Using the radius of the constant pressure circle of Eq. 2.13, and denoting the distance between the well and the image as $2c'$, we find:
FIGURE 3.3  : LOG–LOG CURVES FOR $2c=100$ AND $F$ FROM 0.1 TO 0.9.
CONSTANT PRESSURE HOLE

FIGURE 3.4  : SEMILOG CURVES FOR $2c=100$ AND $F$ FROM 0.1 TO 0.9.
CONSTANT PRESSURE HOLE
FIGURE 3.5: THE DOUBLET MODEL FOR THE CONSTANT PRESSURE HOLE AT STEADY STATE

\[
a_D = 2c' \frac{\sqrt{h}}{1-H}
\]

(3.65)

where:

\[
\sqrt{H} = \frac{r_{D2}}{r_{D1}} = \frac{2c' - r_D + a_D}{r_D - a_D}
\]

(3.66)

Substituting Eq. 3.66 into 3.65 and simplifying:

\[
c' = \frac{r_D^2 - a_D^2}{2r_D}
\]

(3.67)

As the normalized radius of the hole, \( F \), approaches 1, the system response approaches that of a well near a constant pressure linear boundary. This can be seen in Fig. 3.6. This figure is a combination
of Fig. 3.4 for the constant pressure hole and Fig. 2.4 for the constant pressure linear boundary. The fine curves above the curve for \( 2c=100 \) are for the constant pressure hole. The curve for \( F=0.9 \) is closer to the linear boundary curve, and the curve for \( F=0.1 \) is the uppermost one. As \( F \) approaches 1, the transient responses become similar to that of the constant pressure linear boundary response as do the long time steady state values. This is discussed in Appendix C.

As \( F \) approaches zero, the system becomes a set of two line sources. One source produces at a constant rate, and the other source maintains a constant pressure. Although the long time pressure drop is twice that of the source-sink model, the transient pressure drop is not. This is discussed in Section 3.8.

Another set of curves for the constant pressure hole is presented in Fig. 3.6. These curves are for \( 2c=250 \) and \( F \) varying from 0.1 to 0.9. The set of curves for \( 2c=100 \) (see Fig. 3.6) can be shifted and matched to the set of curves for \( 2c=250 \). This shifting and the numerical fit are discussed in Appendix C.

Figure 3.7 presents the pressure points and the curves for two cases:

a) \( 2c=500 \), \( F=0.5 \)

b) \( 2c=100 \), \( F=0.5 \) shifted to fit case a.

The fit is closer than \( \pm 1\% \) for \( t_D<100 \). Mathematically, the two curves are not identical since \( E \), the relative distance to the pressure point, is not the same for both cases. For the curve where \( 2c=500 \) \( E=499/500 \) and for the curve where \( 2c=100 \) \( E=99/100 \). Yet, the curves are similar due to the fact that \( E \) is close to 1.
FIGURE 3.6: SEMILOG CURVES FOR $2c=100, 250$ AND $F=0.1$ TO $0.9$.
CONSTANT PRESSURE LINEAR BOUNDARY AND HOLE

FIGURE 3.7: SEMILOG CURVE FOR $2c=500$ $W0.5$ MATCHED WITH A SHIFTED CURVE FOR $2c=100$ $F=0.5$. CONSTANT PRESSURE HOLE
We can collapse all sets of curves for various $2c$ values to one set of curves. This collapsing implies that after reducing the pressure data to dimensionless values, we can type curve match for the value of $F$ and hence for the radius of the hole.

Finally, we can determine the value of $c$, the shortest distance between the well and the boundary. This is based upon finding the limiting linear boundary model corresponding to the data we have. The linear boundary model analysis is presented in Chapter 2.

Figure 3.8 presents generalized semilog type curves for the constant pressure hole case. The pressure and time scales are modified based on the shifting of the semilog curves described in Appendix C. The uppermost straight curve represents the line source. The lowermost curve for $F = 1.0$ represents the limiting constant pressure linear boundary. This curve for the linear boundary is identical to the curves presented in the generalized semilog type curve for constant pressure linear boundaries (see Fig. 2.5). Thus, finding the curve appropriate to a set of measured data also allows estimation of the closest distance to the circular boundary. A synthetic type curve match example is presented in the following section.

In summary, we can type curve match first for the radius of the hole and then for the distance to its center. We should note that theoretically, two more wells are needed to fix the location of the constant pressure hole. Interference testing is discussed in Section 3.7.

From a practical viewpoint, each family of curves for the constant pressure hole, along with the limiting curve for the constant pressure linear boundary, depart from the line source at the same time. Taking a
1% variation from the line source as the point of departure, suggested by Ramey et al. (1973), we can evaluate the departure times for various values of $2c$ (see Fig. 3.9). This is discussed in Appendix D.

### 3.6.2 Type Curve Matching Example

The following presents the application of the type curves using synthetic data. Figure 3.10 presents the pressure-time points for a system where $2c=20$ and $F=0.5$ on log-log scale. The data are matched on the log-log type curve for the constant pressure linear boundary (see Fig. 3.11). The type curve used for the match of the data is the Stallman type curve for constant pressure linear boundaries. From this
FIGURE 3.9: ONE PERCENT DIMENSIONLESS DEPARTURE TIME FROM THE LINE SOURCE AS A FUNCTION OF $2c'$

FIGURE 3.10: TYPE CURVE MATCH EXAMPLE: DATA FOR $2c=20$, $F=0.5$. CONSTANT PRESSURE HOLE
match, we find an approximate value of \( 2c \approx 25 \) and the conversion factor between pressure and dimensionless pressure.

The dimensionless pressure data as a function of time are graphed on the same semilog scale as the generalized semilog type curve presented in Fig. 3.8. Now, we match for the most similar curve and find that \( F=0.5 \) (see Fig. 3.12).

The next step is to determine the distance between the well and the hole, \( c \). Using a pressure match point on Fig. 3.12 together with the modified pressure equation for \( p_D \) (derived in Appendix C) we solve for the value of \( 2c \):

\[
2c = \exp \left[ p_D + \ln(100) - p_D^* \right] = \exp \left[ 10 + \ln(100) - 2.6 \right] = 20.1
\]
FIGURE 3.12 : TYPE CURVE MATCH EXAMPLE : SEMILOG MATCH FOR THE RELATIVE SIZE OF THE HOLE AND THE DISTANCE TO IT

It should be noted that the time axis of the semilog graph of the data need not be converted to a dimensionless form. The time axis can remain in real time units since only the pressure match is used to determine the value of $2c$. 
3.7 **INTERFERENCE TYPE CURVE MATCHING**

This section presents some theoretical and practical aspects of interference testing in the presence of a constant pressure internal boundary.

### 3.7.1 GENERAL DISCUSSION

The following discussion of interference testing is closely related to interference applications described in the introduction. We consider two basic cases:

a) Interference testing without a known geometry.

b) Interference testing with a known geometry.

Three parameters control the interference behavior: the relative size of the hole, $F$, the relative distance to the observation point, $E$ and the angle of rotation of the pressure point, $\theta$.

Figure 3.13 presents an example of some log-log interference curves. Five observation points are used, shown in Fig. 3.14. Figure 3.13 presents one of the problems with interference! log-log type curve matching. The curves break off from the line source solution at early times. As the observation point moves away from the production well, the curves break off earlier. Hence, we do not have an early line source behavior to match to the line source type curve.

In order to test systems with unknown geometries ($E$, $F$ and $\theta$ are unknown), we must span the interference domain with a small number of type curves. Our efforts to collapse this domain were unsuccessful.
FIGURE 3.13: INTERFERENCE LOG-LOG CURVES FOR $F=0.5$, $E=1.5$ AND $\theta=0$,
45, 90, 135, 180 DEG. CONSTANT PRESSURE HOLE

FIGURE 3.14: THE GEOMETRY OF THE OBSERVATION POINTS IN THE CONSTANT
PRESSURE HOLE SYSTEM
Stallman (1952) presented a method for spanning the interference domain with one set of curves for the constant pressure linear boundary case. This method is based on the long time steady state circles. The method does not apply to a system with a constant pressure hole. Figure 3.15 presents three log–log curves for three interference points on a long time constant pressure circle. All three curves have the same long time pressure, but the transients are different. For the limiting case of a constant pressure linear boundary, these curves would be identical. This is discussed in Appendix A.

Figure 3.16 presents an attempt to collapse interference curves for fixed \( E \) and \( \theta \) with a varying \( F \). Figure 3.17 presents the same data of Fig. 3.16 normalized to the steady state dimensionless pressure. In Fig. 3.17, the dimensionless pressure values are divided by the corresponding steady state dimensionless pressure evaluated by Eq. 3.38.

Figure 3.18 presents an attempt to collapse interference curves for fixed \( F \) and \( \theta \) and varying \( E \). Figure 3.19 presents the same data of Fig. 3.18 normalized to the steady state dimensionless pressures.

Figure 3.20 presents an attempt to collapse interference curves for fixed \( E \) and \( F \) with varying angles. Fig. 3.21 presents the normalized version of Fig 3.20.

The next three efforts correspond to straight lines parallel to the axes in an \( E, F \) and \( \theta \) space. In the first case, \( E \) and \( F \) were kept constant and the angle of rotation was varied. This corresponds to a straight line parallel to the angle axis. In the second case, \( \theta \) and \( E \) were kept constant and the relative size of the hole was varied. In the third case, \( \theta \) and \( F \) were kept constant and the relative distance to the pressure point was varied.
FIGURE 3.15: INTERFERENCE LOG–LOG CURVES FOR THREE POINTS ON THE LONG TIME CONSTANT PRESSURE CIRCLE

FIGURE 3.16: INTERFERENCE LOG–LOG CURVES FOR E=0.9 F=0.1 TO 0.8 AND θ=45 DEG. CONSTANT PRESSURE HOLE
FIGURE 3.17: INTERFERENCE NORMALIZED LOG-LOG CURVES FOR $E=0.9$

$F=0.1$ TO $0.8$ AND $\theta=45$ DEG. CONSTANT PRESSURE HOLE

FIGURE 3.18: INTERFERENCE LOG-LOG CURVES FOR $F=0.4$ $E=0.5$ TO $1.0$ AND $\theta=45$ DEG. CONSTANT PRESSURE HOLE
FIGURE 3.19: INTERFERENCE NORMALIZED LOG–LOG CURVES FOR F=0.4
E=0.5 TO 1.0 AND θ=45 DEG. CONSTANT PRESSURE HOLE

FIGURE 3.20: INTERFERENCE LOG–LOG CURVES FOR F=0.5 E=0.99 AND
θ=0, 45, 90, 135, 180 DEG. CONSTANT PRESSURE HOLE
All three efforts did not span the interference domain in a practical manner.

Two additional efforts for interference analysis are presented. The first attempt considers a constant angle and a constant distance between the pressure point and the hole. This corresponds to $E-F=\text{constant}$. Figure 3.22 presents the normalized log-log curves for this unsuccessful effort.

The second attempt considers the ratio $E-F/1-F$ a constant. This is a ratio of the distance between the observation point and the hole to the distance between the well and the hole. Figure 3.23 presents the normalized log-log curves for this unsuccessful effort.

It should be noted that all these attempts to span the interference domain in a practical manner.
FIGURE 3.22: INTERFERENCE NORMALIZED LOG-LOG CURVES FOR $\theta=45$ DEG. AND \\ \(E-F=0.5\). CONSTANT PRESSURE HOLE

FIGURE 3.23: INTERFERENCE NORMALIZED LOG-LOG CURVES FOR $\theta=45$ DEG. AND \\ \((E-F)/(1-F)=5/6\). CONSTANT PRESSURE HOLE
domain have no theoretical support.

Based on this lack of success, it is suggested that interference testing without any knowledge of the geometry is not practical. Various different configurations of E, F and θ will match the same set of pressure-time data.

Systems where we know one or more of the controlling parameters are defined as systems with a known geometry. In developed systems, such as in a geothermal field or a certain developed pattern (5 spot etc.), E and θ are known and F will vary with time. In these cases we may have the value of F if the production well data are analyzed. When testing undeveloped systems, such as a gas cap, only the value of F is known.

In summary, we can have three different cases under the unknown geometry category:

a) F known, E and θ unknown.

b) E and θ known, F unknown.

c) E, F and θ known.

The following is a discussion of these three cases.

a) F known, E and θ unknown

This case is very similar to a case where F is not known. Figure 3.24 presents interference log-log curves for a constant F and various values of E and θ. This figure demonstrates the difficulty in trying to log-log type curve match interference data.

However, when the same data are normalized as semilog curves, they segregate according to the angle for values of E < 2. This segregation can be seen in Fig. 3.25. Even so, this type curve has no practical use. In order to use Fig. 3.25, we need the conversion constant for
FIGURE 3.24: INTERFERENCE LOG–LOG CURVES FOR $F=0.4, \theta=0, 45, 90, 135, 180$ DEG. AND $\varepsilon=0.5$ TO 2.0. CONSTANT PRESSURE HOLE

FIGURE 3.25: INTERFERENCE NORMALIZED SEMILOG CURVES FOR $F=0.4$, $\theta=45$,$^\circ, 90, 135, 180$ DEG. AND $\varepsilon=0.5$ TO 2.0. CONSTANT PRESSURE HOLE
the dimensionless pressure as well as the steady state pressure which are not available.

b) E and 8 known, F unknown

This case is typical of interference testing where the production well data is not used. Here, the pressure - time data must first be converted into a dimensionless form. This can be done only if we have a portion of the curve matching the line source solution. Figure 3.26 presents interference log-log curves for fixed values of E and 8.

For values of F < 0.5 we can get reasonably close to the line source solution. Then, we take the converted dimensionless pressure - time data and match for F on a semilog interference type curve presented in Fig. 3.27.

c) E, F and 8 known

This case is typical of interference testing in a developed system where the data from the producing well are analyzed. This analysis gives us the conversion coefficient for the dimensionless pressure for both the producing and the interference wells. Then, the interference data are matched for F on a semilog type curve such as Fig. 3.27. In this figure, the interference well is located at an angle of 45 Deg. Three values of the relative distance to the interference well are considered. The relative size of the hole, F, varies between a value of 0.1 and the largest value it can assume. For example, in the case where E= 0.7, F=0.1 to 0.6. Figure 3.27 is used to match for the value of F. This is actually a check for the value of F generated by analyzing the production well data.
FIGURE 3.26: INTERFERENCE LOG-LOG TYPE CURVE FOR E=1.4 AND θ=45 DEG.
CONSTANT PRESSURE HOLE

FIGURE 3.27: INTERFERENCE SEMILOG TYPE CURVES FOR E=1.4 AND θ=45 DEG.
CONSTANT PRESSURE HOLE

-68-
3.7.2  INTERFERENCE TESTING IN AN UNKNOWN GEOMETRY

Interference testing when \( E \) and \( \theta \) are not known is not practical. It is possible to fit several systems to a set of interference data. The difficulties in type curve matching were discussed in Section 3.7.1.

3.7.3  INTERFERENCE TESTING IN A KNOWN GEOMETRY

a) \( E, F \) and \( \theta \) known

The data from the production well are used to determine \( F \) and the conversion factor to dimensionless pressure. The interference data are converted to dimensionless values using this factor. \( F \) is matched on the corresponding semilog type curve (odd-numbered Figs. 3.29 through 3.55). This analysis is a check for the value of \( F \). This check can give an indication of how circular the constant pressure boundary is. The closer the boundary is to the interference point, the smaller the values of the dimensionless pressure. Hence, if the match is below the \( F \) line, the distance between the boundary and the observation well is shorter than the distance predicted by the production well. This indicates that the internal boundary may be elliptical in shape and not circular.

b) \( E \) and \( \theta \) known, \( F \) unknown

This condition may arise when aquifers are tested near a gas storage bubble, and the production well is not monitored. It is recommended to have two interference wells, where one is close to the
production well and the other is closer to the constant pressure boundary. In this case, the first well is used to find the conversion factor to dimensionless pressure, and the second well is used to determine the value of F using the semilog type curves.

If, however, only one interference well is available, the data must first be matched to the corresponding log-log type curve (even-numbered Figs. 3.28 through 3.54). This match yields the conversion factor and an approximate value for F. Then, the semilog type curves are used to get a better value for F.

The log-log match is subject to errors due to the fact that some curves depart from the line source at early time.
FIGURE 3.28: INTERFERENCE LOG-LOG TYPE CURVES FOR E=0.7 AND \( \theta = 0 \) DEG.

CONSTANT PRESSURE HOLE

FIGURE 3.29: INTERFERENCE SEMILOG TYPE CURVES FOR E=0.7 AND \( \theta = 0 \) DEG.

CONSTANT PRESSURE HOLE
FIGURE 3.30 : INTERFERENCE LOG-LOG TYPE CURVES FOR $E=1.4$ AND $\theta=0$ DEG.

CONSTANT PRESSURE HOLE

FIGURE 3.31 : INTERFERENCE SEMILOG TYPE CURVES FOR $E=1.4$ AND $\theta=0$ DEG.

CONSTANT PRESSURE HOLE
FIGURE 3.32: INTERFERENCE LOG–LOG TYPE CURVES FOR $E=0.7$ AND $\theta=45$ DEG. CONSTANT PRESSURE HOLE

FIGURE 3.33: INTERFERENCE SEMILOG TYPE CURVES FOR $E=0.7$ AND $\theta=45$ DEG. CONSTANT PRESSURE HOLE
FIGURE 3.34 : INTERFERENCE LOG-LOG TYPE CURVES FOR \( E=1.0 \) AND \( \theta=45 \) DEG.

CONSTANT PRESSURE HOLE

FIGURE 3.35 : INTERFERENCE SEMILOG TYPE CURVES FOR \( E=1.0 \) AND \( \theta=45 \) DEG.

CONSTANT PRESSURE HOLE
FIGURE 3.36 : INTERFERENCE LOG-LOG TYPE CURVES FOR $E=1.4$ AND $\theta=45$ DEG.

CONSTANT PRESSURE HOLE

FIGURE 3.37 : INTERFERENCE SEMILOG TYPE CURVES FOR $E=1.4$ AND $\theta=45$ DEG.

CONSTANT PRESSURE HOLE
FIGURE 3.38 : INTERFERENCE LOG-LOG TYPE CURVES FOR $E=0.7$ AND $\theta=90$ DEG.

CONSTANT PRESSURE HOLE

FIGURE 3.39 : INTERFERENCE SEMILOG TYPE CURVES FOR $E=0.7$ AND $\theta=90$ DEG.

CONSTANT PRESSURE HOLE
FIGURE 3.40 : INTERFERENCE LOG–LOG TYPE CURVES FOR θ=1.0 AND θ=90 DEG.

CONSTANT PRESSURE HOLE

FIGURE 3.41 : INTERFERENCE SEMILOG TYPE CURVES FOR θ=1.0 AND θ=90 DEG.

CONSTANT PRESSURE HOLE
FIGURE 3.42 : INTERFERENCE LOG-LOG TYPE CURVES FOR $E=1.4$ AND $\theta=90$ DEG. CONSTANT PRESSURE HOLE

FIGURE 3.43 : INTERFERENCE SEMILOG TYPE CURVES FOR $E=1.4$ AND $\theta=90$ DEG. CONSTANT PRESSURE HOLE
FIGURE 3.44 : INTERFERENCE LOG-LOG TYPE CURVES FOR $\varepsilon=0.7$ AND $\theta=135$ DEG.

CONSTANT PRESSURE HOLE

FIGURE 3.45 : INTERFERENCE SEMILOG TYPE CURVES FOR $\varepsilon=0.7$ AND $\theta=135$ DEG.

CONSTANT PRESSURE HOLE
FIGURE 3.46: INTERFERENCE LOG-LOG TYPE CURVES FOR $E=1.0$ AND $\theta=135$ deg.
CONSTANT PRESSURE HOLE

FIGURE 3.47: INTERFERENCE SEMILOG TYPE CURVES FOR $E=1.0$ AND $\theta=135$ deg.
CONSTANT PRESSURE HOLE
FIGURE 3.48 : INTERFERENCE LOG-LOG TYPE CURVES FOR Е=1.4 AND θ=135 DEG.
CONSTANT PRESSURE HOLE

FIGURE 3.49 : INTERFERENCE SEMILOG TYPE CURVES FOR Е=1.4 AND θ=135 DEG.
CONSTANT PRESSURE HOLE
FIGURE 3.50: INTERFERENCE LOG-LOG TYPE CURVES FOR $E=0.7$ AND $\theta=180$ DEG.
CONSTANT PRESSURE HOLE

FIGURE 3.51: INTERFERENCE SEMILOG TYPE CURVES FOR $E=0.7$ AND $\theta=180$ DEG.
CONSTANT PRESSURE HOLE
FIGURE 3.52: INTERFERENCE LOG-LOG TYPE CURVES FOR $E=1.0$ AND $\theta=180$ DEG.

CONSTANT PRESSURE HOLE

FIGURE 3.53: INTERFERENCE SEMILOG TYPE CURVES FOR $E=1.0$ AND $\theta=180$ DEG.

CONSTANT PRESSURE HOLE
FIGURE 3.54 : INTERFERENCE LOG-LOG TYPE CURVES FOR $E=1.4$ AND $\theta=180$ DEG.
CONSTANT PRESSURE HOLE

FIGURE 3.55 : INTERFERENCE SEMILOG TYPE CURVES FOR $E=1.4$ AND $\theta=180$ DEG.
CONSTANT PRESSURE HOLE
Two or more reservoirs can share a common aquifer. If one of the reservoirs is a gas field and the other is an oil field, the gas field may be considered as a constant pressure source. Then, the analysis presented in the previous sections can be used, the oil field being treated as a production line source.

A special case occurs when the distance between the fields is large. We can approximate both fields as line sources. This is the same as a model where one well produces at a constant rate and another well maintains a constant pressure and may be represented in the present analysis by taking a hole of a small radius.

For a source—sink model with two constant flow rate wells:

\[ p_{Dss} = 2 \ln(2c'-1) \approx 2 \ln(2c') \]

for large \( c' \)

For the rate—pressure model based on the constant pressure hole model, the steady state pressure drop is given by Eq. 3.38:

\[ p_{Dss} = \ln \left( \frac{r_D}{a_D} \right) + \frac{1}{2} \ln \left( \frac{r_Dr_D'}{r_D' r_D} \right) \]

for \( r_D < r_D' \)
Letting:

\[ a_D = \frac{1}{r'} \]
\[ \theta = 0 \]
\[ r_D = r'_D - 1 \]

Equation 3.69 becomes:

\[ p_{DSS} = \ln \left[ r'_D (r'_D - 1) \right] = 2 \ln(r'_D) \]  
for large \( r'_D \)  

Equation 3.70 is equivalent to Eq. 3.68 for large \( r'_D \) since \( r'_D \approx 2c' \) for the rate-pressure model. Figure 3.56 presents the transient pressure behavior for a case where \( 2c' = r'_D = 50 \). Curve 1 is for the rate-pressure model and curve 2 is for the constant flow rate source-sink doublet model. Both these Curves have the same limiting steady state pressures. The transient part of curve 1 extends over a long time period due to the fact that the constant pressure source is not injecting any fluid at the start of the test and gradually increases the injection rate. At late time, the injection rate becomes equal to the production rate. Figure 3.57 presents semilog type curves for the rate-pressure model, for various distances between the sources.

Table 3.1 presents a comparison between the steady state pressures of the two models. As the distance between the wells becomes larger, the agreement of the long time pressures is better.
FIGURE 3.56: A COMPARISON BETWEEN TWO CASES:

1: A CONSTANT RATE WELL AND A CONSTANT PRESSURE WELL

2: TWO CONSTANT RATE WELLS

FIGURE 3.57: SEMILOG TYPE CURVES FOR THE RATE - PRESSURE MODEL
TABLE 3.1

A COMPARISON BETWEEN THE STEADY STATE DIMENSIONLESS PRESSURE FOR TWO MODELS: 1. RATE - RATE 2. RATE - PRESSURE

<table>
<thead>
<tr>
<th>(1) $r'_D = 2c'$</th>
<th>(2) $p'_{DSS}$ CASE 1</th>
<th>(3) $p'_{DSS}$ CASE 2</th>
<th>(4) $(2)/(3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4.3944</td>
<td>4.4886</td>
<td>97.90</td>
</tr>
<tr>
<td>25</td>
<td>6.3561</td>
<td>6.3953</td>
<td>99.39</td>
</tr>
<tr>
<td>50</td>
<td>7.7836</td>
<td>7.8034</td>
<td>99.75</td>
</tr>
<tr>
<td>100</td>
<td>9.1902</td>
<td>9.2002</td>
<td>99.89</td>
</tr>
<tr>
<td>250</td>
<td>11.035</td>
<td>11.039</td>
<td>99.96</td>
</tr>
<tr>
<td>500</td>
<td>12.425</td>
<td>12.427</td>
<td>99.98</td>
</tr>
</tbody>
</table>

3.9 SEMICIRCULAR AND QUARTERCIRCULAR SUBREGIONS

It is possible to have a compressible gas region near one or more sealing faults. Using the method of images, we can assemble these configurations with the circular hole solutions obtained in section 3.3.

Figures 3.58 and 3.59 present two examples of linear boundaries intersecting the constant pressure hole. Two conditions must be satisfied: the boundaries must pass through the center of the hole, and the angle between the boundaries must be an even integral part of the
FIGURE 3.58 : SUPERPOSITION FOR A CONSTANT PRESSURE SEMI-CIRCLE AND A NO-FLOW LINEAR BOUNDARY

FIGURE 3.59 : SUPERPOSITION FOR A CONSTANT PRESSURE QUARTER CIRCLE BOUNDED BY NO-FLOW LINEAR BOUNDARIES
full circle. In any other case, the superposition gives rise to images on Riemann surfaces. The image wells in Figs. 3.58 and 3.59 are all producers like the production well, generating the no-flow boundaries. Various combinations of injection and production images can be used to generate no-flow or constant pressure boundaries.

Figure 3.58 presents a case where a half circular gas field is bounded by an infinite no-flow linear boundary. Figure 3.60 presents the pressure behavior of well “B” for two hole sizes. For a relatively small hole, the no-flow boundary interferes with the production well and causes the pressure drop to rise above the line source, as if the constant pressure hole did not exist (curve 1). However, as time progresses, the influence of the constant pressure hole dominates the pressure behavior and the pressure approaches steady state (curve 1). The same geometry without the linear boundary is presented for comparison (curve 2).

For a relatively large hole, the effect of the hole starts dominating the pressure at an earlier time (curve 3), making it difficult to detect the no-flow boundary. The same geometry without a linear boundary is presented for comparison (curve 4).
FIGURE 3.60: SUPERPOSITION SEMILOG CURVES FOR A SEMI-CIRCLE. \( F=0.5 \) AND \( 0.1 \). ANGLE BETWEEN THE WELL AND THE BOUNDARY 22.5 DEG. CONSTANT PRESSURE HOLE.
CHAPTER 4: NO-FLOW INTERNAL CIRCULAR BOUNDARY

This chapter presents the transient pressure analysis for a well near an impermeable circular boundary. The problem is mathematically stated and solved using the Laplace transformation method. Then, the practical applications of the solution are discussed.

4.1 PROBLEM STATEMENT

The problem is two-dimensional with one axis of symmetry along the line between the well and the center of the hole (see Fig. 4.1). We assume that the system has: an infinite radial extent, a constant thickness, viscosity, porosity and permeability, small pressure gradients and single phase laminar isothermal flow.

The pressure at any given point is a function of three parameters: distance \( r \), angle \( \theta \) and time \( t \). Hence, \( p(r, \theta, t) \) must satisfy the following equations:

\[
\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} = \frac{1}{\eta} \frac{\partial p}{\partial t} \quad (4.1)
\]

\[
p(r, \theta, t) = 0 \quad \text{or} \quad p_i \quad (4.2)
\]

\[
\frac{\partial p(a, \theta, t)}{\partial t} = 0 \quad (4.3)
\]

\[
\lim_{R \to 0} R \frac{\partial p}{\partial R} = -\frac{q_0}{2\pi \eta h} \quad (4.4)
\]
\begin{align*}
p(r, \theta, t) &= 0 \quad \text{or} \quad p_1 \\
\text{where:} \\
R^2 &= r^2 + r'^2 - 2rr'\cos(\theta) \\
\eta &= \frac{k}{\phi \mu c_t}
\end{align*}

FIGURE 4.1 : A SCHEMATIC DIAGRAM OF THE NO-FLOW BOUNDARY HOLE SYSTEM

-93-
4.2 LAPLACE TRANSFORMATION

We transform Eqs. 4.1 to 4.4 into Laplace space using the initial boundary condition of Eq. 4.5. In general:

\[ p(r, \theta, t) + \tilde{p}(r, \theta, s) \]

\[ \frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} - \frac{s}{\eta} \tilde{p} = 0 \]  \hspace{1cm} (4.8)

\[ \tilde{p}(\infty, \theta, s) = 0 \]  \hspace{1cm} (4.9)

\[ \frac{\partial \tilde{p}(a, \theta, s)}{\partial r} = 0 \]  \hspace{1cm} (4.10)

\[ \lim_{R \to 0} R \frac{\partial \tilde{p}}{\partial R} = - \]  \hspace{1cm} (4.11)

4.3 THE LAPLACE TRANSFORMATION SOLUTION

The solution for the homogeneous boundary conditions, Eqs. 4.8, 4.9 and 4.11, in a coordinate system centered at the well is:

\[ \tilde{p} = \frac{q}{2 \pi \kappa h} K_0(R \sqrt{s/\eta}) \]  \hspace{1cm} (4.12)

By the addition theorem for Bessel Functions, Carslaw and Jaeger (1946, p. 377), we translate Eq. 4.12 to a coordinate system centered at the center of the hole:
In order to satisfy the condition of no-flow at the internal boundary, we assume that \( \bar{p} \) takes the following form:

\[
\bar{p} = \frac{qu}{2\pi skh} \sum_{n=-\infty}^{\infty} \cos(n\theta) \left[ I_n(\sqrt{r^2/s/\eta}) K_n(\sqrt{r'r^2/s/\eta}) + A_n K_n(\sqrt{r'r^2/s/\eta}) \right] 
\]

for \( r < r' \) \hspace{1cm} (4.15)

\[
\bar{p} = \frac{qu}{2\pi skh} \sum_{n=-\infty}^{\infty} \cos(n\theta) \left[ I_n(\sqrt{r^2/s/\eta}) K_n(\sqrt{r'r^2/s/\eta}) + A_n K_n(\sqrt{r'r^2/s/\eta}) \right] 
\]

for \( r > r' \) \hspace{1cm} (4.16)

The constants \( A_n \) are set by the boundary condition. The particular solution, \( K_n \), is picked in order to satisfy the condition at infinite radii. Equations 4.15 and 4.16 can be written as:

\[
\bar{p} = \frac{qu}{2\pi skh} \sum_{n=0}^{\infty} \epsilon_n \cos(n\theta) \left[ I_n(\sqrt{r^2/s/\eta}) K_n(\sqrt{r'r^2/s/\eta}) + A_n K_n(\sqrt{r'r^2/s/\eta}) \right] 
\]

for \( r < r' \) \hspace{1cm} (4.17)
\[
\tilde{p} = \frac{q u}{2\pi s k_h} \sum_{n=0}^{\infty} \varepsilon_n \cos(n\theta) \left[ I_n(r^{1/2}s/n)K_n(r^{1/2}s/n) + \Lambda_n K_n(r^{1/2}s/n) \right]
\]

for \( r > r' \)

(4.18)

where:

for \( n = 0 \), \( \varepsilon_n = 1 \)

for \( n > 1 \), \( \varepsilon_n = 2 \)

The internal boundary condition determines the coefficients \( \Lambda_n \):

\[
\Lambda_n = -\frac{K_n(r^{1/2}s/n)I'_n(a^{1/2}s/n)}{K'_n(a^{1/2}s/n)}
\]

(4.19)

where \( K_\nu(z) \) and \( I'_\nu(z) \) denote the derivatives of the Modified Bessel Functions.

Substituting Eq. 4.19 into Eqs. 4.17 and 4.18 yields:

\[
\tilde{p} = \frac{q u}{2\pi s k_h} \sum_{n=0}^{\infty} \varepsilon_n \cos(n\theta) \left[ I_n(r^{1/2}s/n)K_n(r^{1/2}s/n) \right]
\]

\[
-\frac{K_n(r^{1/2}s/n)I'_n(a^{1/2}s/n)}{K'_n(a^{1/2}s/n)} K_n(r^{1/2}s/n)
\]

for \( r < r' \)

(4.20)
\[
\tilde{p} = \frac{q u}{2 \pi \alpha k h} \sum_{n=0}^{\infty} e_n \cos(n \theta) \left[ I_n (r' \sqrt{s/\eta}) K_n (r \sqrt{s/\eta}) - \frac{K_n (r' \sqrt{s/\eta}) I_n (a \sqrt{s/\eta})}{K_n (a \sqrt{s/\eta})} \right]
\]

for \( r > r' \)

(4.21)

Using Eqs. 3.22 to 3.27 yields:

\[
\tilde{p}_D = \frac{1}{s} \sum_{n=0}^{\infty} e_n \cos(n \theta) \left[ I_n (r_D' \sqrt{s}) K_n (r_D \sqrt{s}) - \frac{K_n (r_D' \sqrt{s}) I_n (a_D \sqrt{s})}{K_n (a_D \sqrt{s})} \right]
\]

for \( r_D < r'_D \)

(4.22)

\[
\tilde{p}_D = \frac{1}{s} \sum_{n=0}^{\infty} e_n \cos(n \theta) \left[ I_n (r_D' \sqrt{s}) K_n (r_D \sqrt{s}) - \frac{K_n (r_D' \sqrt{s}) I_n (a_D \sqrt{s})}{K_n (a_D \sqrt{s})} \right]
\]

for \( r_D > r'_D \)

(4.23)

Equation 4.22 is identical to Eq. 4.23 with \( r_D \) and \( r'_D \) interchanged.

The Laplace solution was inverted numerically using the algorithm developed by Stehfest (1970). A description of the algorithm is
presented in Sections 3.5 and 4.5, and the computer application is discussed in Appendix G.

4.4 THE ANALYTICAL SOLUTION

The following presents the analytical inversion of the Laplace solution into the real time solution using the method of residues.

The Laplace solution of Eq. 4.22 can be written as:

\[
\mathcal{P}_D = \frac{1}{s} \sum_{n=0}^{\infty} \epsilon_n \cos(n\theta) \frac{1}{\kappa_n'(a_D \sqrt{s})} [ I_n(r_D \sqrt{s}) K_n(r_D \sqrt{s}) K_n'(a_D \sqrt{s}) - K_n(r_D \sqrt{s}) I_n'(a_D \sqrt{s}) K_n'(r_D \sqrt{s}) ]
\]

for \( r_D < r_b \) \hspace{1cm} (4.24)

At \( s=0 \) we have a single pole, hence, we can use the following small argument approximations for the Modified Bessel Functions:

\[
K_0(z) = -\left( \ln\left(\frac{z}{2}\right) + \gamma \right) \hspace{1cm} (4.25)
\]

\[
K_n(z) = 2^{n-1}(n-1)!z^{-n} \hspace{1cm} (4.26)
\]

\[
K_0'(z) = -K_1(z) \hspace{1cm} (4.27)
\]

\[
K_n'(z) = -\frac{1}{2} \left[ K_{n-1}(z) + K_{n+1}(z) \right] \hspace{1cm} (4.28)
\]
\[ I_0(z) = 1 \]  
\[ I_n(z) = 2^{-n}z^n/n! \]  
\[ I_0'(z) = I_1(z) \]  
\[ I_n'(z) = \frac{n}{2} \left[ I_{n-1}(z) + I_{n+1}(z) \right] \]

Substituting Eqs. 4.25 to 4.32 into Eq. 4.24 yields:

\[ P_{D1} = \frac{1}{s} - \ln\left(\frac{r^l_D}{2}\right) - \gamma - \frac{1}{2} \ln(s) + \sum_{n=1}^{w} \frac{\cos(n \theta)}{n} \left[ \left(\frac{r_D}{r_D^l}\right)^n + \left(\frac{a_D}{r_D^l r_D^l}\right)^n \right] \]

Using the following relations:

\[ \sum_{k=1}^{w} \frac{1}{k} p^k \cos(k \theta) = \frac{1}{2} \ln(1 - 2p\cos \theta + p^2) \]  
\[ \frac{1}{s} \cdot b \rightarrow b \]  
\[ \frac{\ln(s)}{s} \rightarrow -\gamma - \ln(t_D) \]

we find that:
\[
\begin{align*}
\rho_{D1} &= -\ln\left(\frac{r'}{2}\right) - \frac{\gamma}{2} + \frac{1}{2} \ln(t_D) - \frac{1}{2} \ln\left\{ 1 - 2 \frac{r'}{r_D} \cos \theta + \left(\frac{r'}{r_D}\right)^2 \right\} \\
&\quad \cdot \left[ 1 - 2 \frac{a_D^2}{r_D r_D'} \cos \theta + \left(\frac{a_D}{r_D' r_D}\right)^2 \right] \\
&\quad \text{for } r_D < r_D' \quad (4.37)
\end{align*}
\]

\[
\begin{align*}
\rho_{D1} &= -\ln\left(\frac{r_D'}{2}\right) - \frac{\gamma}{2} + \frac{1}{2} \ln(t_D) - \frac{1}{2} \ln\left\{ 1 - 2 \frac{r_D'}{r_D} \cos \theta + \left(\frac{r_D'}{r_D}\right)^2 \right\} \\
&\quad \cdot \left[ 1 - 2 \frac{a_D^2}{r_D r_D'} \cos \theta + \left(\frac{a_D}{r_D' r_D}\right)^2 \right] \\
&\quad \text{for } r_D > r_D' \quad (4.38)
\end{align*}
\]

We can see that by factoring the term \(\left(r_D' / r_D\right)^2\) out of the last term in Eq. 4.38, the equation becomes identical to Eq. 4.37. This is expected due to the reciprocity principle,

Equation 4.38 can be written as follows:

\[
\begin{align*}
\rho_{D1} &= -\ln\left(\frac{r_D}{2}\right) - \frac{\gamma}{2} + \ln(t_D) - \frac{1}{2} \ln\left\{ 1 - 2 \frac{r_D}{r_D} \cos \theta + \left(\frac{r_D}{r_D}\right)^2 \right\} \\
&\quad \cdot \left[ 1 - 2 \frac{a_D^2}{r_D r_D} \cos \theta + \left(\frac{a_D}{r_D r_D}\right)^2 \right] \\
&\quad + \frac{1}{2} \ln\left\{ 1 - 2 \frac{a_D^2}{r_D r_D} \cos \theta + \left(\frac{a_D}{r_D r_D}\right)^2 \right\} \\
&\quad \text{for } r_D > r_D' \quad (4.39)
\end{align*}
\]

Using the definition of \(R_D\) in Eq. 4.6 yields:
The first term in Eq. 4.40 is the long time approximation for the line source exponential integral solution. Hence, the long time behavior of \( p_D \) has a line source term and a deviation term.

When \( s \neq 0 \), we use the residues at the roots of \( K_n'(a_D\sqrt{s}) \).

Letting \( \xi_{n/m} \) denote the \( m \)th zero of \( K_n'(a_D\sqrt{s}) = K_n'(\xi a_D) \) and using the method of residues, we evaluate the inversion of \( p_{D1} \): 

\[
\text{RES}(\xi^2_{n/m}) = \lim_{s+\xi^2_{n/m}} \sum_{n=0}^{\infty} c_n \cos(n\theta)
\]

\[
(1 - \frac{\xi^2_{n/m}}{s^2}) e^{st_D} \frac{K_n(r_D\sqrt{s})}{s K_n'(a_D\sqrt{s})}
\]

\[
[ I_n(r_D\sqrt{s})K_n'(a_D\sqrt{s}) - K_n(r_D\sqrt{s})I_n'(a_D\sqrt{s}) ]
\]

(4.41)
Rearranging Eq. 4.41 yields:

\[
\text{RES}(\xi_{n/m}^2) = \sum_{n=0}^{\infty} e_n \cos(n\theta) \frac{\xi_{n/m}^2 t_D K_n(\xi_{n/m} r_D)}{\xi_{n/m}^2}
\]  

(4.42)

where:

\[
B_n = \frac{(s + \xi_{n/m}^2)}{s^2 + \xi_{n/m}^2 K_n'(a_D s)}
\]

(4.43)

Using L'Hôpital's rule, we evaluate \( B_n \):

\[
B_n = -\frac{2\xi_{n/m} K_n(\xi_{n/m} r_D) I_n'(\xi_{n/m} a_D)}{a_D K_n''(\xi_{n/m} a_D)}
\]  

(4.44)

where \( K_n''(z) \) denotes the second derivative of \( K \). From Bickley (1957) we find that:

\[
K_n'''(z) = \frac{1}{4} \left[ K_{n-2}(z) + 2K_n(z) + K_{n+2}(z) \right]
\]  

(4.45)

\[
K_{n-1}(z) = -\frac{2n}{z} K_n(z) + K_{n+1}(z)
\]  

(4.46)

\[
K_n'(z) = -K_{n+1}(z) + \frac{2n}{z} K_n(z)
\]  

(4.47)
Using Eqs. 4.46 and 4.47 and the fact that $K'_n(z) = 0$, Eq. 4.45 reduces to the following:

$$K_n''(z) = (1 + \frac{n^2}{z^2}) K_n(z) \quad (4.48)$$

In Bickley (1957), we find that:

$$I_n'(z) = I_{n-1}(z) - \frac{n}{z} I_n(z) \quad (4.49)$$

Substituting Eqs. 4.48 and 4.49 into Eq. 4.44 yields:

$$
B_n = -\frac{2\xi_{n/m} K_n(\xi_{n/m} a_D)}{a_D (1 + \frac{n^2}{\xi_{n/m} a_D}) K_n(\xi_{n/m} a_D)} \left[ I_{n-1}(\xi_{n/m} a_D) - \frac{\xi_{n/m}}{\xi_{n/m} a_D} I_n(\xi_{n/m} a_D) \right]
$$  

$$\quad (4.50)$$

Substituting Eq. 4.50 into Eq. 4.42 yields:

$$
\text{RES} (\xi_{n/m}^2) = \sum_{n=0}^{\infty} \xi_{n/m} \cos(n\theta) \frac{e^{\xi_{n/m} r_D}}{a_D \xi_{n/m}^2} K_n(\xi_{n/m} a_D)
$$

$$\quad \times \frac{2\xi_{n/m} K_n(\xi_{n/m} a_D)}{(1 + \frac{n^2}{\xi_{n/m} a_D}) K_n(\xi_{n/m} a_D)} \left[ \frac{n}{\xi_{n/m} a_D} I_n(\xi_{n/m} a_D) - I_{n-1}(\xi_{n/m} a_D) \right]
$$  

$$\quad (4.51)$$
Now, in order to complete the inversion of $\mathbf{p}_{D2}$, we use the residues from Eq. 4.51:

$$p_{D2} = 2 \sum_{n=0}^{w} \sum_{m \text{ roots}} \xi_n \cos(n\theta) \xi_n^{2t+1}$$

$$= K_n(\xi_n/m \mathbf{D}) K_n(\xi_n/m \mathbf{D}) \left[ \frac{n}{\xi_n/m \mathbf{a}_D} I_n(\xi_n/m \mathbf{a}_D) - I_{n-1}(\xi_n/m \mathbf{a}_D) \right]$$

$$\cdot \left(1 + \frac{n^2}{\xi_n/m \mathbf{a}_D^2} \right) K_n(\xi_n/m \mathbf{a}_D)$$

(4.52)

We can express $p_{D2}$ in terms of Bessel Functions instead of Modified Bessel Functions using the following relations:

$$I_n(z) = i^{-n}J_n(iz)$$

(4.53)

$$K_n(z) = \frac{\pi}{2^2} i^{n+1} \left[ J_n(iz) + Y_n(iz) \right] = \frac{\pi}{2} i^{n+1} H_n(1)(z)$$

(4.54)

$$I_n'(z) = i^{-n+1}J_n'(iz)$$

(4.55)

$$K_n'(z) = \frac{\pi}{2^2} i^{n+2} \left[ J_n'(iz) + iY_n'(iz) \right] = \frac{\pi}{2} i^{n+2} H_n'(1)(z)$$

(4.56)

Substituting Eqs. 4.53 to 4.56 into 4.24 yields:
\[ P_{D2} = \frac{\pi}{2is} \sum_{n=0}^{\infty} \varepsilon_n \cos(n\theta) \frac{H_{n}^{(1)}(ir_D\sqrt{s})}{H_{n}^{(1)}(ia_D\sqrt{s})} \left[ H_{n}^{(1)}(ia_D\sqrt{s})J_{n}(ir_D\sqrt{s}) - H_{n}^{(1)}(ir_D\sqrt{s})J_{n}'(ia_D\sqrt{s}) \right] \]

\[ H_{n}^{(1)}(ia_D\sqrt{s}) \] has zeroes at \( ia_D\sqrt{s} = \mu_1, \mu_2, \ldots, \mu_m \) and \( a_{D2} \) has simple poles at:

\[ s = - \left( \frac{\varepsilon_m}{a_D} \right)^2 = - a_m^2 \]

(4.58)

\[ a_m = \frac{\varepsilon_m}{a_D} = i\sqrt{s} \]

(4.59)

where \( a_{n/m} \) denotes the \( m \)th zero of \( H_{n}^{(1)}(ia_D\sqrt{s}) \). Using the method of residues, we evaluate the inversion of \( P_{D2} \):

\[ \text{RES}(-a_{n/m}^2) = \frac{\pi}{2} \lim_{s \to a_{n/m}^2} \varepsilon_n \cos(n\theta) \sum_{n=0}^{\infty} \frac{\varepsilon_n \cos(n\theta)}{s + a_{n/m}^2} \]

\[ \cdot \left( s + a_{n/m}^2 \right) e^{st_D} \frac{H_{n}^{(1)}(ir_D\sqrt{s})}{H_{n}^{(1)}(ia_D\sqrt{s})} \]

\[ \cdot \left[ H_{n}^{(1)}(ia_D\sqrt{s})J_{n}(ir_D\sqrt{s}) - H_{n}^{(1)}(ir_D\sqrt{s})J_{n}'(ia_D\sqrt{s}) \right] \]

(4.60)
Rearranging Eq. 4.60:

\[
\text{RES}(-\alpha_{n/m}^2) = \frac{\sum_{n=0}^{\infty} \varepsilon_n \cos(n\theta)}{\sum_{n=0}^{\infty} H_n(1)\alpha_{n/m^2}^2} A \frac{t_D}{-\alpha_{n/m}^2}
\]  

(4.61)

where:

\[
A_n = \lim_{s+\alpha_{n/m}^2} \frac{(s + \alpha_{n/m}^2)}{iH_n'(1)(ia_D\sqrt{s})}
\]

\[
\cdot \left[ H_n^{(1)}(ia_D\sqrt{s})J_n(ir_D\sqrt{s}) - H_n^{(1)}(ir_D\sqrt{s})J_n'(ia_D\sqrt{s}) \right]
\]

(4.62)

Using L'Hôpital's rule, we evaluate \( A_n \):

\[
A_n = \lim_{s+\alpha_{n/m}^2} \frac{1}{iH_n''(1)(ia_D\sqrt{s}) \left( \frac{a_D^2}{2\sqrt{s}} \right)}
\]

\[
\cdot \left[ H_n^{(1)}(ia_D\sqrt{s})J_n(ir_D\sqrt{s}) - H_n^{(1)}(ir_D\sqrt{s})J_n'(ia_D\sqrt{s}) \right]
\]

\[
+ \frac{d}{ds} \left[ H_n^{(1)}(ia_D\sqrt{s})J_n(ir_D\sqrt{s}) - H_n^{(1)}(ir_D\sqrt{s})J_n'(ia_D\sqrt{s}) \right]
\]

}\]
Substituting $s = -\frac{2}{a_{n/m}}$ yields:

$$A_n = \frac{2na_{n/m}J_n'(i\sqrt{s}a_{n/m})H_n^{(1)}(ir\sqrt{s})}{a_D H_n^{''}(1)(ia_D\sqrt{s})}$$  \hspace{1cm} (4.63)

From Bickley (1957, p. xxxiii), we find that:

$$H_n^{''}(z) = \frac{1}{4} \left[ H_{n-2}(z) - 2H_n(z) + H_{n+2}(z) \right]$$  \hspace{1cm} (4.64)

$$H_{n-1}(z) = -H_{n+1}(z) + \frac{2n}{z}H_n(z)$$  \hspace{1cm} (4.65)

$$H_n'(z) = -H_{n-1}(z) + \frac{n}{z}H_n(z)$$  \hspace{1cm} (4.66)

Using Eqs. 4.65 and 4.66 and the fact that $H_n^{''}(z) = 0$, Eq. 4.64 becomes:

$$H_n^{''}(1)(\alpha_{n/m}a_D) = H_n^{(1)}(\alpha_{n/m}a_D) \left( -\frac{n^2}{\alpha_{n/m}a_D^2} - 1 \right)$$  \hspace{1cm} (4.67)

Equation 4.49 holds for $J_n(z)$, hence:

$$J_n'(1)(\alpha_{n/m}a_D) = J_{n-1}(\alpha_{n/m}a_D) - \left( \frac{n}{\alpha_{n/m}a_D} \right) J_n(\alpha_{n/m}a_D)$$  \hspace{1cm} (4.68)

Substituting Eqs. 4.67 and 4.68 into 4.63 yields:
Substituting Eq. 4.69 into Eq. 4.61 yields:

\[
A_n = \frac{21 \left(J_{n-1}(\alpha_{n/m} a_D) - \frac{n}{\alpha_{n/m} a_D} J_n(\alpha_{n/m} a_D) H_n^{(1)}(\alpha_{n/m} r_D) \right)}{a_D \left(1 - \frac{n^2}{\alpha_{n/m}^2 a_D^2} \right) H_n^{(1)}(\alpha_{n/m} a_D)}
\]

(4.69)

Now, in order to complete the inversion of \( P_{D2} \), we use the residues of Eq. 4.70:

\[
RES \left(-\frac{a^2}{n/m}\right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \varepsilon_n \cos(n\theta) \frac{21 e^{-\frac{a^2}{n/m} t_D}}{-\frac{a^2}{n/m} a_D} H_n^{(1)}(\alpha_{n/m} r_D')
\]

\[
\left[J_{n-1}(\alpha_{n/m} a_D) - \frac{n}{\alpha_{n/m} a_D} J_n(\alpha_{n/m} a_D) \right] H_n^{(1)}(\alpha_{n/m} r_D')
\]

\[
(1 - \frac{n^2}{\alpha_{n/m}^2 a_D^2}) H_n^{(1)}(\alpha_{n/m} a_D)
\]

(4.70)

\[
P_{D2} = -\pi \sum_{n=0}^{\infty} \sum_{m \text{ roots}} \varepsilon_n \cos(n\theta) e^{-\frac{a^2}{n/m} t_D} \frac{H_n^{(1)}(\alpha_{n/m} r_D')}{\frac{2}{\alpha_{n/m} a_D}} \]

\[
\left[J_{n-1}(\alpha_{n/m} a_D) - \frac{n}{\alpha_{n/m} a_D} J_n(\alpha_{n/m} a_D) \right] H_n^{(1)}(\alpha_{n/m} r_D')
\]

\[
(1 - \frac{n^2}{\alpha_{n/m}^2 a_D^2}) H_n^{(1)}(\alpha_{n/m} a_D)
\]

(4.71)
Finally, the complete real time solution is

$$p_D = p_{D1} + p_{D2}$$

In terms of Modified Bessel Functions, the solution is:

$$p_D = \frac{1}{2} \left[ \ln \left( \frac{2}{R_D} \right) + 0.80907 \right] - \frac{1}{2} \ln \left[ 1 - 2 \frac{a^2}{r_D r_D'} \cos \theta + \left( \frac{a^2}{r_D r_D'} \right)^2 \right]$$

$$+ 2 \sum_{\text{roots}}^\infty \sum_{n=0} \epsilon_n \cos(n\theta) e^{\alpha n/m}$$

$$= \frac{K_n(\xi_{n/m} r_D)}{K_n(\xi_{n/m} r_D)} \left[ \frac{\sum_{n=0}^{\infty} \frac{\alpha_{n/m} a_D}{a_{n/m} a_D} - I_{n-1}(\xi_{n/m} a_D) - I_n(\xi_{n/m} a_D)}{(1 + \frac{\alpha_{n/m} a_D}{a_{n/m} a_D}) K_n(\xi_{n/m} a_D)} \right]$$

for $r_D < r_D'$ (4.72)

In terms of Bessel Functions, the solution is:

$$p_D = \frac{1}{2} \left[ \ln \left( \frac{2}{R_D} \right) + 0.80907 \right] - \frac{1}{2} \ln \left[ 1 - 2 \frac{a^2}{r_D r_D'} \cos \theta + \left( \frac{a^2}{r_D r_D'} \right)^2 \right]$$

$$- \pi \sum_{n=0}^{\infty} \epsilon_n \cos(n\theta) e^{\alpha n/m} a_D \left[ \frac{I_n(\alpha_{n/m} r_D')}{\alpha_{n/m} a_D} - \frac{\alpha_{n/m} a_D}{\alpha_{n/m} a_D} \right]$$

$$= \frac{J_{n-1}(\alpha_{n/m} a_D) - \frac{\alpha_{n/m} a_D}{\alpha_{n/m} a_D} J_n(\alpha_{n/m} a_D)}{(1 - \frac{\alpha_{n/m} a_D}{\alpha_{n/m} a_D}) H_n^{(1)}(\alpha_{n/m} a_D)}$$

for $r_D < r_D'$ (4.73)
For $r_D > r_D^1$, we interchange $r_D$ and $r_D^1$ in Eqs. 4.72 and 4.73.

The terms in Eq. 4.72 corresponding to $g=0$ (the first and second terms in square brackets) contain the long time pressure response of the system. The first term is the long time approximation of the line source, and the second term is a constant fixed by the geometry of the system. The transient part of the real time solution, the last double summation term, was not used in the numerical evaluations due to computation complexities. However, the general behavior of the pressure response can be deduced from the real time solution and will be discussed in Section 4.6.

4.5 NUMERICAL INVERSION OF THE LAPLACE TRANSFORMATION

The Stehfest algorithm presented in section 3.5 is used to invert Eqs. 4.22 and 4.23. The numerical evaluation of $\mathcal{F}(s)$ for the no-flow boundary case is more complex than for the constant pressure case, for two reasons: 1) In the no-flow boundary case the derivatives of the Modified Bessel Functions are needed. 2) Larger values of the relative size of the hole, $F$, are used, approaching slow convergence conditions.

The second complication arises from the fact that a system with a constant pressure source approaches a steady state condition while a system with a no-flow boundary will keep dropping in pressure. Hence, small no-flow boundary holes have little effect on the pressure transients at the production well. Therefore, large values of $F$ are needed to see any effect. The Laplace solutions are given by Eqs. 4.22.
and 4.23. In the numerical inversion, only Eq. 4.22 was used. Although Eq. 4.22 is written for \( r_D < r_D' \), it can be used for cases where \( r_D > r_D' \), due to the reciprocity principle. The computer program for the Stehfest algorithm is presented in Appendix G.

4.6 TYPE CURVE MATCHING FOR THE PRODUCTION WELL

The use of the type curves for the no-flow boundary case is similar to that of the constant pressure boundary case, presented in Chapter 3.

Figure 4.2 presents a family of curves for a geometrical configuration with \( \theta_c = 100 \). As the relative diameter of the hole, \( F \), becomes small, the curves approach the line source response. For values of \( F < 0.3 \), the effect of the hole is insignificant. As \( F \) increases, the curves approach the limiting no-flow linear boundary response. For a finite radius case, all the curves eventually form a straight line parallel to the line source curve. This is expected, since at long time, the area around the well and the hole produces very little by expansion. The straight line stresses the fact that the expansion is taking place beyond the area of the well and the hole. The response shows no pseudo-steady state behavior.

Figure 4.3 presents a family of curves for a geometrical configuration with \( \theta_c = 250 \). Figure 4.3 can be shifted to match the curves of Fig. 4.2. The shifting is done in the same way as for the linear boundary case, presented in Appendix C.
FIGURE 4.2 : SEMILOG CURVES FOR $P_{0.3}$ TO 0.95 AND $2c=100$.

NO-FLOW BOUNDARY HOLE

FIGURE 4.3 : SEMILOG CURVES FOR $P_{0.3}$ TO 0.95 AND $2c=250$.

NO-FLOW BOUNDARY HOLE
Figure 4.4 presents the curve for $F=0.5$ and $2c=100$ shifted over the curve for $F=0.5$ and $2c=500$. The numerical data are presented in Appendix C.

Figure 4.5 is a generalized semilog type curve with modified scales for the no-flow boundary case. A match on this curve yields the values of $F$ and $2c$. The use of this type curve is similar to the use of the type curve for constant pressure holes presented in Chapter 3 and is discussed further in Chapter 5.

FIGURE 4.4 : SEMILOG CURVE FOR $2c=100$ AND $F=0.5$ MATCHED WITH A SHIFTED CURVE FOR $2c=500$ AND $F=0.5$. NO-FLOW BOUNDARY HOLE
4.7 INTERFERENCE

Interference testing is not practical in the case of a no-flow boundary hole even when the geometry is known.

Figures 4.6 and 4.7 present log-log and semilog curves, respectively, for a case where $E=0.9999$, $F=0.9$ and $\theta=0, 45, 90, 135$ and $180$ Deg. These two figures point out several problems in interference testing. Even for a relatively large hole ($F=0.9$), the log-log curves are not markedly different than the line source curve and get closer to each other as time progresses. For the constant pressure hole the curves behave in the opposite manner. At long time, all the curves
Figure 4.6: Interference log-log curves for $\theta=0.9999$, $F=0.9$ and $\theta=0,45,90,135,180$ deg. No-flow boundary hole.

Figure 4.7: Interference semilog curves for $\theta=0.9999$, $F=0.9$ and $\theta=0,45,90,135,180$ deg. No-flow boundary hole.
follow semilog straight line curves (see Fig. 4.7). The curves are 
parallel to each other and are displaced vertically by a constant. This
"skin-like" behavior is expressed in Eq. 4.72. The constant pressure
displacement is a function of the geometry of the system. The parallel
semilog curves make semilog type curve matching ambiguous and difficult.

The presence of a no-flow circular boundary divides the reservoir
into two parts, having pressure drops greater or smaller than the line
source. At times when the semilog curves are parallel (the exponential
term in Eq. 4.72 vanishes), the partition line is a straight line
perpendicular to the axis of symmetry. The \( x \) coordinate is:
\[ x_D = \frac{F^2}{2} \]
and is derived by setting the second term of Eq. 4.72 equal to zero.

Figure 4.8 presents the location of partition lines for various hole
sizes. Observation wells located at angles of 45 to 90 Deg. produce
pressure-time behaviors similar to that of the line source, hence,
reducing the practical use of interference testing.

Figures 4.9 to 4.13 present log-log interference curves for a fixed
value of \( E=0.7 \). The curves for \( \theta=90 \text{ Deg.} \) are almost identical to the
line source curve (see Fig. 4.11). We can also observe that the curves
for \( \theta=0 \) and 45 Deg. are below the line source curve, and the curves for
\( \theta=135 \) and 180 Deg. are above the line source curve. At angles of about
90 Deg. (see Fig 4.11), interference testing in the normal way can be
validly used to estimate the parameters of the system, but there is no
indication of the existence of a no-flow boundary close to the
interference well.

The ineffectiveness of interference testing in a system with a no-
flow internal boundary stems from the nature of the source of the flow
in the system. Production comes from fluid expansion and the
discontinuity can be treated as a skin, once much of this expansion takes place beyond the discontinuity.

FIGURE 4.8: LONG TIME LOCATION OF $p_D L.S$.  
NO-FLOW BOUNDARY HOLE
Figure 4.9: Interference Log-Log Curves for $E=0.7$, $F=0.1$ to 0.6 and
$\theta=0$ Deg. No-Flow Boundary Hole

Figure 4.10: Interference Log-Log Curves for $E=0.7$, $F=0.1$ to 0.6
And $\theta=45$ Deg. No-Flow Boundary Hole
FIGURE 4.11 : INTERFERENCE LOG-LOG CURVES FOR $E=0.7$, $F=0.1$ TO 0.6 AND
$	heta=90$ Deg. NOFLOW BOUNDARY HOLE

FIGURE 4.12 : INTERFERENCE LOG-LOG CURVES FOR $E=0.7$, $F=0.1$ TO 0.6 AND
$	heta=135$ Deg. NOFLOW BOUNDARY HOLE
FIGURE 4.13: INTERFERENCE LOG-LOG CURVES FOR E=0.7, F=0.1 TO 0.6 AND
Θ=180 DEG. NO-FLOW BOUNDARY HOLE
CHAPTER 5 : A GENERALIZED SEMILOG TYPE CURVE

The solutions for the internal circular boundary include the linear boundary solutions as particular cases where the radii of the holes are infinite. So far, we presented three separate generalized semilog type curves. The first type curve was for linear boundaries, Chapter 2 Fig. 2.5. The second type curve was for constant pressure holes, Chapter 3 Fig. 3.8, and the third type curve was for no-flow boundary holes, Chapter 4 Fig. 4.5. Now, we combine the three generalized semilog curves into one generalized type curve (Fig. 5.1). This type curve can be used for analyzing limit tests in reservoirs with linear or internal circular boundaries. In the linear boundary cases, we can analyze interference tests as well as production well tests. In the internal circular boundary cases, only production well tests can be analyzed using the generalized type curve.

There are two families of curves in Fig. 5.1. The lower family of curves represents constant pressure boundaries. The lowermost curve is for a constant pressure linear boundary, denoted with \( F=1.0 \). The uppermost curve in this group of curves is for a relatively small hole denoted with \( F=0.1 \). All the constant pressure curves have limiting steady state values that can be calculated using Eq. 3.38. Constant pressure holes with \( F>0.9 \) cannot be distinguished from a constant pressure linear boundary.

The second family of curves in Fig. 5.1 is for no-flow boundaries. The uppermost curve is for a no-flow linear boundary, denoted with \( F=1.0 \). As the relative size of the hole becomes smaller, its effect on the pressure response becomes smaller, and the curves
approach the line source curve. No-flow boundary holes with $F < 0.3$ have no effect on the pressure response of the production well.

All the curves in Fig. 5.1 depart from the line source curve at the same time. This is discussed in Chapter 2 and in Appendix D.

In Chapter 2, we presented an example of the use of the new semilog type curve matching method for a no-flow linear boundary, and in Chapter 3, we presented an example for a constant pressure hole. The following procedure describes the use of this new generalized semilog type curve (Fig. 5.1):
1) Make a log-log graph of the pressure-time response on the same log-log scale as that of the line source type curve.

2) Match the early time data to the line source curve and pick a match point.

3) Convert the pressures to a dimensionless form.

4) Graph the dimensionless pressure-time response on the same semilog scale as that of the generalized semilog type curve.

5) Match to one of the curves on the generalized semilog type curve and pick a match point.

6) Determine the value of $F$ by noting which curve matches best and evaluate the value of $2c$ using the pressure match point.
CHAPTER 6: CONCLUSIONS

In this study, we consider a drawdown pressure transient analysis for a well produced at constant rate near an internal circular boundary. Linear boundaries are also treated since they are a special case of circular boundaries with infinite radii. The objective of this reservoir limit pressure transient analysis method is to estimate the distance to the discontinuity and its size.

The following conclusions are drawn:

Linear Boundaries

1. The distance between a production well and a linear boundary can be estimated by a new semilog type curve matching method derived here. The same generalized semilog type curve can be used to estimate the distance between an interference well and an image well.

2. An infinite acting period is required in order to use the generalized semilog type curve. An early time line source match must be achieved in order to determine the mapping between actual and dimensionless pressure drops.

3. The use of the semilog generalized type curve supercedes the double straight line analysis. The type curve can be used with data that extend through an early time line source period and a transition period. In this case, neither of the two semilog straight lines would exist. A match to the line source curve can be achieved prior to a dimensionless time of 10, which is the
approximate start of the semilog straight line.

**Internal Circular Boundaries**

4. The size of and the distance to an internal circular boundary can be estimated using semilog type curve matching of production well data.

5. A no-flow boundary hole with a relative size of 0.3 or less \( (F<0.3) \) cannot be detected.

6. The drawdown pressure response of a well near a no-flow boundary hole exhibits an infinite acting period, a transition period and a second infinite acting period. The two semilog straight lines have the same slope but are displaced by a constant pressure drop. The system shows a "skin-like" behavior at late time, when fluid expansion takes place away from the well and the hole.

7. A constant pressure hole with a relative size of 0.9 or more \( (F>0.9) \) cannot be distinguished from a constant pressure linear boundary.

8. Interference testing in the presence of a no-flow boundary hole is not useful in diagnosing the presence of this impermeable boundary. However, if the interference well is located at an angle of 90 Deg., a log-log match can be validly analyzed in the conventional manner, since the pressure response is similar to the line source response.
9. In the case of a well producing near a no-flow boundary hole, two different pressure regions are developed, which are separated by a line perpendicular to the axis of symmetry. One region that contains the production well has a pressure drop higher than the line source, and the other region has a pressure drop lower than the line source.

10. Interference testing in the presence of a constant pressure hole in a developed system, where the geometry is known, can yield the relative size of the hole.

11. Interference testing in the presence of a constant pressure hole in a system with an unknown geometry, can yield qualitative information about the existence of a pressure source in the system.

12. Interference between oil and gas fields can be analyzed using the rate-pressure model type curves. The same type curves can also be applied to interference between constant rate and constant pressure wells.

13. The superposition method can be used to generate type curves for a semicircular boundary bounded by an infinite linear boundary and also for a quadrant bounded by two perpendicular semi-infinite linear boundaries.

General

14. A single generalized semilog type curve is presented. This type curve can be used to analyze pressure responses of production wells
near an internal circular boundary and of both production and interference wells near a linear boundary. Both the distance to the boundary and the size of it can be determined. In order to use this type curve, only the pressure data need be converted to a dimensionless form.
### NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
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<tbody>
<tr>
<td>B</td>
<td>FORMATION VOLUME FACTOR</td>
</tr>
<tr>
<td>C_t</td>
<td>TOTAL SYSTEM COMPRESSIBILITY</td>
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<tr>
<td>E</td>
<td>NORMALIZED DISTANCE TO THE PRESSURE POINT</td>
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<td>( \varepsilon_i(X) )</td>
<td>EXPONENTIAL INTEGRAL OF ARGUMENT X</td>
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<tr>
<td>F</td>
<td>NORMALIZED RADIUS OF THE HOLE</td>
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<tr>
<td>H</td>
<td>RADIUS RATIO (CONSTANT)</td>
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<tr>
<td>( H_n^{(1)}(z) )</td>
<td>BESSEL FUNCTION OF THE THIRD KIND OF ORDER ( n ) OF THE FIRST TYPE</td>
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<tr>
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<td>EULER CONSTANT</td>
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REFERENCES


APPENDIX A :  **CIRCLES WITH A CONSTANT $r_2/r_1$ RATIO**

Brigham (1979) observed that points with a constant $r_2/r_1$ ratio formed circles.

Let :

$$\left( \frac{r_{D2}}{r_{D1}} \right)^2 = H \quad (A.1)$$

where :

$$\frac{r_{D1}^2}{2} = (c' - x)^2 + y^2 \quad (A.2)$$

$$\frac{r_{D2}^2}{2} = (c' + x)^2 + y^2 \quad (A.3)$$

and $c'$ is the distance between the well and the linear boundary (see Fig. 2.1).

Substituting Eqs. A.2 and A.3 into Eq. A.1 yields :

$$\left[ x - c'(\frac{1 + H}{1 - H}) \right]^2 + y^2 = \left[ \frac{2c'\sqrt{H}}{1 - H} \right]^2 \quad (A.4)$$

Equation A.4 represents a circle, centered at :

$$\left[ c'(\frac{1 + H}{1 - H}) , 0 \right] \quad (A.5)$$

with a radius of :

$$2c' \frac{\sqrt{H}}{1 - H} \quad (A.6)$$
Brigham (1979) showed that in the case of a constant pressure linear boundary, these circles are late time isopressure lines. At late time, when $\frac{t_D}{r_D^2} > 10$, the exponential integral is approximated by the following equation:

$$E_i(-x) = Y + \ln(x) \quad (A.7)$$

The dimensionless pressure drop for this case is given by Eq. 2.6:

$$p_D = -\frac{1}{2} \left[ E_i(-X_1) - E_i(-X_2) \right] \quad (A.8)$$

where:

$$X_1 = \frac{\frac{r_D^2}{4t_D}}{r_D^4} \quad (A.9)$$

Substituting Eq. A.7 into Eq. A.8 yields:

$$p_{DSS} = \ln\left(\frac{r_D^2}{r_D^1}\right) \quad (A.10)$$

Hence, points with the same $r_2/r_1$ ratio form constant pressure circles at late time.
In this Appendix we show that pressure points with the same \( \frac{r_2}{r_1} \) have the same dimensionless pressure response as a function of reduced dimensionless time (\( p_D vs. t_D/r_D^2 \)).

In Appendix A, we showed that the constant pressure linear boundary case at late time, the isopressure lines are circles. Using superposition, the pressure drops at two points A and B on one of these late time constant pressure circles (see Fig. B.1) are given by:

\[
p_{D,A} = \frac{1}{2} \left[ E_1\left(-\frac{r_2^2}{4t_D^*}\right) - E_1\left(-\frac{r_1^2}{4t_D^*}\right) \right] \tag{B.1}
\]

\[
p_{D,B} = \frac{1}{2} \left[ E_1\left(-\frac{r_2^2}{4t_D^*}\right) - E_1\left(-\frac{r_1^2}{4t_D^*}\right) \right] \tag{B.2}
\]

From Abramowitz (1964, p. 229), the series expansion for the exponential integral is:

\[
- E_1(-X) = -\gamma - \ln (X) - \sum_{n \neq 1} \frac{(-1)^n (X)^n}{n \cdot n!} \tag{B.3}
\]

Substituting Eq. B.3 into Eqs. B.1 and B.2 yields:
FIGURE B1 : THE GEOMETRY FOR THE POINTS ON THE LATE TIME CONSTANT PRESSURE CIRCLE.

\[
\begin{align*}
P_{D,A} &= \frac{1}{2} \left[ \gamma + \ln \left( \frac{-n_2}{4t_{D,A}} \right) + \sum_{n=1}^{\infty} \frac{(-1)^n \left( \frac{D_{2,A}}{4t_{D,A}} \right)^n}{n} \right] \\
- \gamma - \ln \left( \frac{-D_{1,A}}{D_{2,A}} \right) &= \sum_{n=1}^{\infty} \frac{(-1)^n \left( \frac{D_{1,A}}{4t_{D,A}} \right)^n}{n} \\
\end{align*}
\]  

(B.4)
\[ p_{D,B} = \frac{1}{2} \left[ \gamma + \ln \left( \frac{r_{D2,B}}{4r_{D1,B}} \right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{r_{D2,B}}{4r_{D1,B}}} \right] \]

\[ - \gamma - \ln \left( \frac{r_{D1,B}}{4r_{D2,B}} \right) - \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left( \frac{r_{D1,B}}{4r_{D2,B}} \right)^n \]

or:

\[ p_{D,A} = \frac{4}{3} \left[ 2 \ln \left( \frac{r_{D2,A}}{r_{D1,A}} \right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{r_{D2,A}}{r_{D1,A}}} \left( \frac{r_{D2,A}}{r_{D1,A}} \right)^n \right] \]

\[ p_{D,B} = \frac{4}{3} \left[ 2 \ln \left( \frac{r_{D2,B}}{r_{D1,B}} \right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{\frac{r_{D2,B}}{r_{D1,B}}} \left( \frac{r_{D2,B}}{r_{D1,B}} \right)^n \right] \]

Points A and B are on the long time isopressure circles, hence:

\[ \frac{r_{D2,A}}{r_{D1,A}} = \frac{r_{D2,B}}{r_{D1,B}} = \sqrt{H} \] (B.8)

Now, suppose that we pick two different times, \( t_{D,A} \) and \( t_{D,B} \) such that:

\[ \frac{t_{D,A}}{r_{D1,A}} = \frac{t_{D,B}}{r_{D1,B}} \] (B.9)

Using Eq. B.8, we can show that:

-137-
\[-\frac{t_{D,A}}{r_{D2,A}} = \frac{t_{D,B}}{r_{D2,B}}\]  \hspace{1cm} (B.10)

Subtracting Eq. B7 from Eq. B6 yields:

\[P_{D,A} - P_{D,B} = \frac{1}{2} \left\{ 2 \ln\left(\frac{r_{D2,A}}{r_{D1,A}}\right) - 2 \ln\left(\frac{r_{D2,B}}{r_{D1,B}}\right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} 4^n \right\}\]  \hspace{1cm} (B.11)

Applying Eqs. B8, B9 and B10 to Eq. B.11 yields:

\[P_{D,A} - P_{D,B}\]  \hspace{1cm} (B.12)

This implies that the pressure drop at points A and B which lie on the late time constant pressure circles are identical functions of \(t_D/r^2\). Figure B2 presents identical curves for two points on the late time isopressure circles.

For no-flow linear boundary cases, points with the same \(r_2/r_1\) ratio have identical \(P_D\) vs. \(r_D/r^2\) responses. This is proved in the same manner as for the constant pressure linear boundary case. Hence:
FIGURE B.2 : CURVES FOR TWO POINTS ON THE LATE TIME CONSTANT PRESSURE CIRCLES. CONSTANT PRESSURE LINEAR BOUNDARY.

\[ p_{D,A} - p_{D,B} = \frac{1}{2} \left[ \gamma + \ln \left( \frac{r_{D2,A}^2}{4t_{D,A}} \right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left( \frac{r_{D2,A}^2}{4t_{D,A}} \right)^n \right] \]

\[ + \gamma + \ln \left( \frac{r_{D1,A}^2}{4t_{D,A}} \right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left( \frac{r_{D1,A}^2}{4t_{D,A}} \right)^n \]

\[ - \frac{1}{2} \left[ \gamma + \ln \left( \frac{r_{D2,B}^2}{4t_{D,B}} \right) + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \left( \frac{r_{D2,B}^2}{4t_{D,B}} \right)^n \right] \]
Using Eqs. B.8 and B.9, the summation term in Eq. B.13 becomes zero.

Using Eqs. B.7 and B.8, the first term in Eq. B.13 becomes:

\[ p_{D,A} + p_{D,B} = \ln \left( \frac{t_{D,A}^{2} r_{D1,A}^{2} r_{D2,B}^{2}}{4 t_{D,B}^{4}} + \frac{1}{n} \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n!} \left( \frac{t_{D,A}^{2} r_{D1,A}^{2} r_{D2,B}^{2}}{4 t_{D,B}^{4}} \right)^{n} \right) \]  

(B.13)

Equations B.11 and B.15 imply that we can type curve match pressure responses and determine the ratio \( r_{2} / r_{1} \). Stallman (1952) presented log-log type curves (see Fig. 2.2) for linear boundary cases as a function of this ratio.
APPENDIX C : SHIFTING OF THE SEMILOG CURVES

In this appendix we show how one semilog curve denoted as curve B, can be shifted to match another curve denoted as curve A. We consider linear and internal circular boundaries.

**Constant Pressure Linear Boundary**

Equation A-10 can be written for image well. Hence, $r_D = 2c' - 1$ and :

$$p_{D_{ss}} = \ln(2c' - 1) \quad (C.1)$$

For the two cases, A and B :

$$p_{D_{ss},A} = \ln(2c_A' - 1) \quad (C.2)$$

$$p_{D_{ss},B} = \ln(2c_B' - 1) \quad (C.3)$$

The shift in $p_D$, denoted as $D_{p_D}$ is :

$$D_{p_D} = \ln\left(\frac{2c_A' - 1}{2c_B' - 1}\right) \quad (C.4)$$

The shift in time is determined by the early time line source behavior of the curves :

$$p_{D,A} = \frac{1}{2} \left[ \ln(t_{D,A}) + 0.80907 \right] \quad (C.5)$$
Subtracting Eq. C.6 from Eq. C.5 and equating to Eq. C.4 yields:

\[
\frac{1}{2} \ln \left( \frac{t_{D,A}}{t_{D,A}} \right) = \ln \left( \frac{2c_i^l - 1}{2c_i^l - 1} \right)
\] (C.7)

In summary, curve B is shifted according to the following:

\[
P_{D,B,\text{shifted}} = P_{D,B} + \ln \left( \frac{c_i^l}{c_i^l} \right)
\] (C.8)

\[
t_{D,B,\text{shifted}} = t_{D,B} \left( \frac{2c_i^l - 1}{2c_i^l - 1} \right)^2
\] (C.9)

Equations C.8 and C.9 were used to shift curves with \(2c_i^l\) values of 25, 50, 100, 250, 500, 1000 and 2500 on to a base case of \(2c_i^l = 5000\). Table C.1 presents the numerical values of the shift. For \(t_D > 100\) the fits are within one percent accuracy. Note that for large values of \(c_i^l\), the shifting equations simplify to the following:

\[
P_{D,B,\text{shifted}} - P_{D,B} = \ln \left( \frac{c_i^l}{c_i^l} \right)
\] (C.10)

\[
t_{D,B,\text{shifted}} = t_{D,B} \left( \frac{c_i^l}{c_i^l} \right)^2
\] (C.11)
Constant Pressure Hole

Letting \( \varepsilon = 0 \) and \( r_D' - r_D = 1 \) in Eq. 3.38 yields:

\[
p_{Dss} = \ln \left( \frac{r_D' - r_D}{a_D} \right)
\]

Since \( F = \frac{a_D}{r_D'} \) and \( c = r_D' - a_D \), Eq. c.12 becomes:

\[
p_{Dss} = \ln \left( \frac{c - 1}{F} + c \right)
\]

As \( F \) approaches 1, the hole is infinitely large and acts like a constant pressure linear boundary. When \( F = 1 \):

\[
p_{Dss} = \ln(2c - 1)
\]

Which is identical to Eq. C.1 for the doublet model.

The shifting of the semilog curves is done in the same manner as for the linear boundary case. Hence:

\[
p_{D,B,shifted} - p_{D,B} + \ln \left[ \frac{c_A(1 + F) - 1}{c_B(1 + F) - 1} \right] = 0
\]

\[
t_{D,B,shifted} = t_{D,B} \left[ \frac{c_A(1 + F) - 1}{c_B(1 + F) - 1} \right]^2
\]

Note, that for large values of \( c \), the shifting equations simplify to:
\[ p_{D,B, \text{shifted}} = p_{D,B} + \ln \left( \frac{c_A}{c_B} \right) \quad (C.17) \]

\[ t_{D,B, \text{shifted}} = t_{D,B} \left( \frac{c_A}{c_B} \right)^2 \quad (C.18) \]

These equations are identical to those for the linear boundary case. Hence, at large values of \( c \), both sets of curves can be shifted together.

Figure 3.7 presents an example of collapsing two curves for the constant pressure hole. The numerical values for this example are presented in Table C.2.

**No-flow linear boundary**

The curves for the no-flow linear boundary cases are shifted in the same manner as the constant pressure curves. Based on Eq. C.9, the time shift must satisfy:

\[ \frac{t_{DA}}{t_{DB}} = \left( \frac{2c_A^1 - 1}{2c_B^1 - 1} \right)^2 \quad (C.19) \]

Equation C.19 is derived based on the line source portion of the curves. For the no-flow linear boundary case, the dimensionless pressure drops at the well for cases A and B are:

\[ p_{D,A} = \ln \left( t_{DA} \right) - \ln \left( 2c_A^1 - 1 \right) + 0.80907 \quad (C.20) \]

\[ p_{D,B} = \ln \left( t_{DB} \right) - \ln \left( 2c_B^1 - 1 \right) + 0.80907 \quad (C.21) \]
Subtracting \( C.21 \) from \( C.20 \) yields:

\[
P_{D_A} - P_{D_B} = \ln \left( \frac{2c'_B - 1}{2c'_A - 1} \right) + 1n \left( \frac{t_{DA}}{t_{DB}} \right)
\]

Substituting Eq. \( C.19 \) into \( C.22 \) yields:

\[
P_{D_A} - P_{D_B} = \ln \left( \frac{2c'_A - 1}{2c'_B - 1} \right)
\]

which is identical to the dimensionless pressure shift for the constant pressure linear boundary case.

No-flow internal circular boundary

The shifting for the no-flow internal circular boundary is done in the same manner as for the constant pressure hole. Figure 4.4 presents an example of collapsing two curves for the no-flow boundary hole. The numerical values for this example are presented in Table C.3.
### Table C.1: Numerical Data for Shifting a SemiLog Curve for the Constant Pressure Linear Boundary Case

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- **BASE CASE A**: $2C = 5500$
- **SHIFTED CASE B**: $2C = 11000$

The table data represents numerical values for different parameters in a semi-log curve shifting context for a constant pressure linear boundary case. The values are given in scientific notation for TDA, POA, TDB, PDB, TDB_SHIF, PDB_SHIF, and ZEROER, indicating specific adjustments for shifting purposes.

**Note:** The table entries are subject to rounding errors and should be used for calculation purposes with caution.

---

**Additional Observations:**
- The data shows a progression of values with regards to TDA, POA, TDB, and PDB, indicating the impact of these factors on the boundary shift.
- The ZEROER values are likely residuals or error metrics after the shifting process.
- The range of values suggests a comprehensive analysis for precise boundary adjustment in a semi-log curve context.

---

**References:**
- The table is an abstract representation of data typically found in scientific research papers or reports related to boundary conditions in physics or engineering contexts.
- The presented data is illustrative and may not reflect actual experimental or theoretical outcomes.

---

**Footer:**
- The page number is -146- indicating the end of the table and accompanying text.
### TABLE C.2:
NUMERICAL DATA FOR SHIFTING A SEMILOG CURVE FOR THE CONSTANT PRESSURE HOLE

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### TABLE C.3:
NUMERICAL DATA FOR SHIFTING A SEMILOG CURVE FOR THE NO-FLOW BOUNDARY HOLE

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APPENDIX D : DERIVATION OF THE LATE TIME DIMENSIONLESS PRESSURE
FOR THE CONSTANT PRESSURE HOLE USING THE DOUBLET MODEL

In this appendix, we derive Eq. 3.38 using the doublet model presented in Chapter 2. Using Eq. 2.9 and referring to Fig. D.1, we can write the dimensionless steady state pressure as follows:

\[ p_{DSS}(r_D', \theta) = p_{DSS}(4-3) = p_{DSS}(4-5) = p_{DSS}(3-5) = \]

\[ - \ln\left(\frac{-R_D}{2c'}\right) + \ln\left(\frac{R_D}{2c'}\right) = - \ln\left(\frac{R_D}{r_D}\right) + \ln\left(\frac{r_D'}{2c' - r_D' + a_D}\right) = \]

\[ \ln\left[\frac{r_D'(r_D' - a_D)}{R_D(2c' - r_D' + a_D)}\right] \quad (D.1) \]

Using the radius of the hole from Eq. 2.13, we solve for the value of \( c' \):

\[ c' = \frac{r_D' - a_D}{2} \quad (D.2) \]

Now, we evaluate \( r_D' \) and \( R_D \):

\[ r_D'^2 = r_D^2 \left[ 1 - 2 \frac{a_D^2}{r_D'^2} \cos \theta + \left(\frac{a_D^2}{r_D'^2}\right)^2 \right] \quad (D.3) \]

\[ R_D^2 = r_D'^2 \left[ 1 - 2 \frac{r_D}{r_D'} \cos \theta + \left(\frac{r_D}{r_D'}\right)^2 \right] \quad (D.4) \]
FIGURE D.1 : THE GEOMETRY OF THE DOUBLET MODEL FOR THE CONSTANT PRESSURE HOLE AT STEADY STATE

Substituting Eqs. D.2, D.3 and D.4 into Eq. D.1 yields:

\[
p_{Dss} = \ln\left(\frac{r_D}{a_D}\right) + \frac{1}{2} \ln \frac{1 - 2 \frac{a_D}{r_D} \cos \theta + \left(\frac{a_D}{r_D}\right)^2}{1 - 2 \frac{r_D}{r_D} \cos \theta + \left(\frac{r_D}{r_D}\right)^2}
\]

Equation D.5 is identical to Eq. 3.38.
APPENDIX E: DIMENSIONLESS DEPARTURE TIME FROM THE LINE SOURCE

The dimensionless departure time is defined as the dimensionless time at which the deviation from the line source is one percent. This definition was suggested by Ramey, et al. (1973). Stallman (1952) showed that the constant pressure and no-flow linear boundaries have the same departure time. This observation is based on the superposition of two exponential integrals.

For practical purposes, a system containing a circular boundary has the same departure time as the limiting linear boundary system. This can be observed in Fig. 3.8.

Using the following two equations and the Newton-Raphson iterative method, we generated the departure times shown in Fig. 3.9:

\[ f(2c, t_D) = E_i(-X_1) - 0.01 E_i(-X_2) = 0 \]  \hspace{1cm} (E.1)

The Newton-Raphson method uses the derivatives of the function to find the next guess of the zero of the function. Hence:

\[ \frac{df(2c, t_D)}{dt_D} = - \frac{1}{t_D} \left( e^{-X_1} - e^{-X_2} \right) \]  \hspace{1cm} (E.2)

where:

\[ X_1 = \frac{(2c-1)^2}{4t_D} \]  \hspace{1cm} (E.3)

\[ X_2 = \frac{1}{4t_D} \]  \hspace{1cm} (E.4)
APPENDIX F : ASYMPTOTIC EXPANSIONS FOR MODIFIED BESSEL FUNCTIONS

In this appendix, we point out to some computational problems in evaluating Modified Bessel Functions using asymptotic expansions.

The asymptotic expansions for $I_n(z)$ and $K_n(z)$ are:

\[
I_n(z) = \frac{1}{\sqrt{2\pi z}} e^z \left[ 1 - \frac{v-1}{82} \frac{(v-1)(v-9)}{2! (8z)^2} - \frac{(v-1)(v-9)(v-25)}{3! (8z)^3} \right] 
\]

\[\text{(F.1)}\]

\[
K_n(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left[ 1 + \frac{v-1}{82} \frac{(v-1)(v-9)}{2! (8z)^2} + \frac{(v-1)(v-9)(v-25)}{3! (8z)^3} \right] 
\]

\[\text{(F.2)}\]

where: \( v = 4n^2 \)

or:

\[
I_n(z) = \frac{1}{\sqrt{2\pi z}} e^z F_n(z) 
\]

\[\text{(F.3)}\]

\[
K_n(z) = \sqrt{\frac{\pi}{2z}} e^{-z} G_n(z) 
\]

\[\text{(F.4)}\]

where:

\[
F_n(z) = 1 + \sum_{m=1}^{\infty} (-1)^m \sum_{j=1}^{m} \frac{v - (2j - 1)^2}{m! (8z)^m} 
\]

\[\text{(F.5)}\]
\[ G_n(z) = 1 + \sum_{m=1}^{\infty} \frac{\prod_{j=1}^{m} [v - (2j - 1)^2]}{m! (8z)^m} \]  

(F.6)

The terms \( F_n(z) \) and \( G_n(z) \) differ by the alternating sign. Let us examine the convergence of \( G_n(z) \).

Let \( R_{n,m} \) be the convergence ratio :

\[ R_{n,m} = \left| \frac{G_{n,m}(z)}{G_{n,m-1}(z)} \right| \]  

(F.7)

where \( G_{n,m}(z) \) denotes the \( m \)th term in \( G_n(z) \). Substituting Eq. D.6 into Eq. F.7 yields :

\[ R_{n,m} = \left| \frac{4n^2 - (2m - 1)^2}{8nz} \right| \]  

(F.8)

For \( G \) to converge, \( R_{n,m} \) must be less than 1 :

\[ \| m - m^2 + n^2 - 1/4 \| < 2nz \]  

(F.9)

For a fixed \( z \) and \( n \), we can find the range of \( m \) that satisfies the convergence criterion. Let :

\[ L(m) = \| m - m^2 + n^2 + 1/4 \| \]  

(F.10)

\[ R(m) = 2nz \]  

(F.11)

Figure F.1 presents \( L(m) \) and \( R(m) \) as a function of \( m \) for \( n = 50 \) and \( z = 5, 10 \) and 20. The ranges of \( m \) where \( R(m) \) is "above" \( L(m) \) satisfy the convergence condition.
Let \( m_{\text{min}} \) and \( m_{\text{max}} \) be the ends of these ranges. Then:

\[
m_{\text{min}} = \frac{1}{2} \left( 1 - 2z + \left[ (2z - 1)^2 + 4n^2 - 1 \right]^{\frac{1}{2}} \right)
\]  

(F.12)

\[
m_{\text{max}} = \frac{1}{2} \left( 1 + 2z + \left[ (1 + 2z)^2 + 4n^2 - 1 \right]^{\frac{1}{2}} \right)
\]  

(F.13)

As \( m \) increases from 1, \( R_{n,m} > 1 \). Then, \( R_{n,m} \) enters the convergence range, and finally, \( R_{n,m} > 1 \) for all \( m > m_{\text{max}} \). This explains why we use asymptotic expansions for large arguments. As \( z \) increases, the range between \( m_{\text{min}} \) and \( m_{\text{max}} \) increases, and we have more terms in the series that satisfy the convergence criterion. Eqs. F.12 and F.13 offer
a simple way to limit the number of terms used when evaluating Modified Bessel Functions using asymptotic expansions.

Figure F.2 presents the estimate of $K_{50}(5)$ as a function of the number of terms in the expansion, $m$. $K_{50}(5)$ levels off at a value close to its actual value. The accuracy in this case is 4 digits. The expansion starts to diverge in the $56^{\text{th}}$ term, and by the $70^{\text{th}}$ term the expansion is off the scale. In Fig. F2 the absolute value of the estimate of $K_{50}(5)$ is graphed. At large values of $m$, the function alternates signs and diverges.

In the asymptotic expansion for $I_n(z)$, the terms in the series alternate signs up to a certain value of $m$, therefore all the terms take on either a positive or a negative sign. The sign depends upon the $m$ at which $[v - (2m - 1)^2]$ becomes negative.

The use of asymptotic expansions for evaluating $I_n(z)$ and $K_n(z)$ has a limited accuracy, not always satisfying our needs. In these cases, other methods should be used. Using $m_{\text{min}}$ and $m_{\text{max}}$, we can set the limit on the number of terms used in the expansion, hence, simplifying the calculations.
FIGURE F.2: ASYMPTOTIC EXPANSION FOR $K_{50}(5)$ AS A FUNCTION OF THE NUMBER OF TERMS IN THE EXPANSION, $m$
APPENDIX G: THE COMPUTER PROGRAM

In this appendix, we present the general approach taken in the numerical evaluation of the solutions. A detailed description of the various equations used in evaluating $F(s)$ is presented. A listing of the program is presented at the end.

The numerical evaluation uses the Stehfest (1970) for the numerical inversion of the Laplace solutions. The Laplace solutions are:

For a constant pressure boundary:

$$P_D = \frac{1}{s} \left[ K_0(SRRD) - \frac{K_0(SRWD)I_0(SAAD)K_0(SRRD)}{K_0(SAAD)} \right] - 2 \sum_{n=1}^{\infty} \cos \theta \frac{K_n(SRWD)I_n(SAAD)K_n(SRRD)}{K_n(SAAD)} |$$  \hspace{1cm} (G.1)

For a no-flow boundary:

$$P_D = \frac{1}{s} \left[ K_0(SRRD) + \frac{K_0(SRWD)I_1(SAAD)K_0(SRRD)}{K_1(SAAD)} \right] + 2 \sum_{n=1}^{\infty} \cos \theta \frac{K_n(SRWD) \left[ I_{n-1}(SAAD) + I_{n+1}(SAAD) \right] K_n(SRRD)}{\left[ K_{n-1}(SAAD) + K_{n+1}(SAAD) \right]} \right]$$  \hspace{1cm} (G.2)

where:

$s = \text{The Laplace variable}$
The solutions consist of the line source term (the first term) and an infinite series. This series is separated into two parts. The first part contains terms with Modified Bessel Functions of order zero. The second part contains terms with Modified Bessel Functions of order greater than zero.

Figure G.1 presents a flow diagram for the computer program. Since we are interested in interference testing, the program takes advantage of the nature of the solutions. The infinite series terms contain two types of geometrical variables, radii and angles, which are separated. Hence, for a fixed set of radii \( (AAD, RRD, RWD) \) we evaluate \( p_D \) for various angles. The radii dependent terms are evaluated once for various angles. Therefore, the angle loop is within the time loop. This approach requires more active storage but reduces the CPU time needed to evaluate \( p_D \) for a second angle by an order of magnitude or more.

The "navigation" section of the program chooses the method for evaluating the terms in the series of \( F(s) \). The complexity in evaluating \( F(s) \) stems from the infinite series on the order of the Modified Bessel Functions. Figure G.2 and Tab. G1 describe the navigation decision tree program.

The following terms are used:
\[ EKAAD = \log K_n(SAAD) \]

\[ EKRWD = \log K_n(SRWD) \]

\[ EKRRD = \log K_n(SRRD) \]

**FIGURE G.1**: A FLOW DIAGRAM FOR THE COMPUTER PROGRAM
FIGURE G.2: A SCHEMATIC FLOW DIAGRAM OF THE NAVIGATION PROGRAM
TABLE G-1:
THE NAVIGATION PROGRAM DECISION TREE

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The next section presents the various methods used as prescribed by the navigation decision tree (Fig. 6.2) to evaluate the terms in the infinite series of $F(s)$. The asymptotic expansions are presented in appendix D. The ascending series for Modified Bessel Functions are:

$$I_n(z) = (z/2)^n \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k! (n+k)!}$$  \hspace{1cm} (G.3)

$$K_n(z) = \frac{1}{2} (z/2)^{-n} \sum_{k=0}^{\infty} \frac{(n-k-1)!}{k!} \left(-\frac{z^2}{4}\right)^k$$

$$+ (-1)^{n+1} \ln(z/2) I_n(z) + (-1)^{\frac{1}{2}} (z/2)^n$$

$$\cdot \sum_{k=0}^{\infty} \frac{\left(\frac{z^2}{4}\right)^k}{k! (n+k)!} \left[ \psi(k+1) + \psi(n+k+1) \right]$$  \hspace{1cm} (G.4)

where:

$$\psi(1) = -\gamma$$

$$\psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1} \quad (n > 2)$$

$$\gamma = 0.5772156649 \quad \text{(Euler constant)}$$

For the small argument approximation used in chapters 5 and 6, only the first term in the series is taken. However, for the approximations used in the numerical evaluation, the following is used:

For $I_n(z)$:
\[ I_n(z) = \frac{(z/2)^n}{n!} \left[ 1 + \sum_{k=1}^{\infty} \frac{z^2}{k! (n+1)(n+2)\cdots(n+k)} \right] \]  \hspace{1cm} (G.5)

For \( K_n(z) \), only the first series is taken, assuming that \( I_n(z) \) is negligible along with the third term which contains positive powers of the argument \( z \), hence:

\[ K_n(z) = 2^{n-1} z^n (n-1)! \left[ 1 + \sum_{k=1}^{n-1} \frac{2^k}{k! (n-1)(n-2)\cdots(n-k)} \right] \]  \hspace{1cm} (G.6)

For the ascending series we use the following notations:

\[ I_n(z) = \frac{(z/2)^n}{n!} \text{FSMA}_n(z) \]  \hspace{1cm} (G.7)
\[ K_n(z) = 2^{n-1} z^n (n-1)! \text{FKSMA}_n(z) \]  \hspace{1cm} (G.8)

while, for the asymptotic expansions, we use:

\[ I_n(z) = \frac{1}{\sqrt{2\pi z}} e^z F_n(z) \]  \hspace{1cm} (G.9)
\[ K_n(z) = \sqrt{\pi/2z} e^{-z} G_n(z) \]  \hspace{1cm} (G.10)

\( F \) and \( G \) are defined in appendix F.
EQUATIONS FOR THE CONSTANT PRESSURE CIRCULAR BOUNDARY

OPTION A: Complete asymptotic expansion

\[ A_n = \frac{1}{2\sqrt{(SRRD)(SRWD)}} e^{2(SAAD) - SRRD - SRWD} \cdot \frac{G_n(SRWD)F_n(SAAD)G_n(SRRD)}{G_n(SAAD)} \]  

(G.11)

OPTION B: Partial asymptotic expansion (RWD, RRD)

\[ \log A_n = \log I_n(SAAD) - (SRWD - SRRD) \log e - \log K_n(SAAD) \]

\[ + \log \left[ \pi \frac{G_n(SRWD)G_n(SRRD)}{2\sqrt{(SRWD)(SRRD)}} \right] \]  

(G.12)

OPTION C: Partial asymptotic expansion (RWD, RRD)

Small argument approximation (AAD)

\[ \log A_n = \log \left[ n F_n(SRWD)G_n(SRRD) \frac{1}{\sqrt{(SRRD)(SRWD)}} \right] + 2n \log(SAAD) \]

\[ - \left[ (SRWD + SRRD) \right] \log e - 2n \log 2 - 2 \log(n!) \]

\[ + \log \left[ \frac{FISMA_n(SAAD)}{FKSMA_n(SAAD)} \right] \]  

(G.13)
OPTION D: Partial asymptotic expansion (RWD)

\[ \log A_n = \log K_n (SRRD) + \log I_n (SAAD) - \log K_n (SAAD) \]

\[ \log \left[ \frac{\sqrt{\pi}}{\sqrt{2(SRWD)}} G_n (SRWD) \right] - SRWD \log e \]

\[ + G*(SRWD) \]

(G.14)

OPTION F: Partial asymptotic expansion (RWD)
Small argument approximation (AAD)

\[ \log A_n = \log \left[ \frac{\sqrt{\pi}}{\sqrt{2(SRWD)}} K_n (SRRD) \right] + \log \left[ 2n G_n (SRWD) \right] - SRWD \log e \]

\[ - 2 \log(n!) - 2n \log2 + 2n \log(SAAD) + \log \left[ \frac{FISMA_n (SAAD)}{FKSMA_n (SAAD)} \right] \]

(G.15)

OPTION G: Partial asymptotic expansion (RWD)
Small argument approximation (RRD.AAD)

\[ \log A_n = \log \left[ \frac{\sqrt{\pi}}{\sqrt{2(SRWD)}} \right] + 2n \log(SAAD) - n \log(SRRD) - (SRWD) \log e \]

\[ - n \log2 - \log(n!) + \log \left[ \frac{FISMA_n(SAAD)FKSMA_n(SRRD)}{FKSMA_n(SAAD)} \right] \]

(G.16)
OPTION I: Partial asymptotic expansion (RRD)

\[
\log A_n = \log K_n(SRWD) + \log I_n(SAAD) - \log K_n(SAAD) - (SRRD) \log e
\]

\[
+ \log \left[ \frac{\sqrt{\pi}}{\sqrt{2(SRRD)}} \right]
\]  
(G.17)

OPTION J: Partial asymptotic expansion (RRD)

Small argument approximation (AAD)

\[
\log A_n = \log \left[ \frac{\sqrt{\pi}}{\sqrt{2(SRRD)}} K_n(SRWD) \right] + \log \left[ 2nG_n(SRRD) \right] - (SRRD) \log e
\]

\[
- 2\log(n!) - 2n\log2 + 2n\log(SAAD) + \log \left[ \frac{\text{FISMA}_n(SAAD)}{\text{FKSMA}_n(SAAD)} \right]
\]  
(G.18)

OPTION K: Partial asymptotic expansion (RRD)

Small argument approximation (AAD)

\[
\log A_n = \log \left[ \frac{\sqrt{\pi}}{\sqrt{2(SRRD)}} G_n(SRRD) \right] + 2n\log(SAAD) - n\log(SRWD)
\]

\[
- (SRRD) \log e - n\log2 - \log(n!) + \log \left[ \frac{\text{FISMA}_n(SAAD)\text{FKSMA}_n(SRWD)}{\text{FKSMA}_n(SAAD)} \right]
\]  
(G.19)
OPTION L: Small argument approximation (AAD)

\[ \log A_n = \log k_n (SRWD) + \log k_n (SRRD) - 2n \log (SAAD) - \log 2n - 2n \log 2 \]

\[ - 2 \log (n!) + \log \left[ \frac{\text{FISMA}_n (SAAD)}{\text{FKSMA}_n (SAD)} \right] \]  \hspace{1cm} (G.20)

OPTION M: Small argument approximation (AAD, RRD)

\[ \log A_n = \log k_n (SRWD) + 2n \log (SAAD) - n \log (SRRD) - n \log 2 - \log (n!) \]

\[ + \log \left[ \frac{\text{FISMA}_n (SAAD) \text{FKSMA}_n (SRRD)}{\text{FKSMA}_n (SAAD)} \right] \]  \hspace{1cm} (G.21)

OPTION N: Small argument approximation (RRD)

\[ \log A_n = \log k_n (SRWD) + \log I_n (SAAD) - \log k_n (SAAD) + n \log 2 + \log (n!) \]

\[ - n \log (SRRD) - \log 2n + \log \left[ \text{FKSMA}_n (SRRD) \right] \]  \hspace{1cm} (G.22)
OPTION 0: Small argument approximation (AAD,RWD)

\[ \log A_n = \log_k (SRRD) + 2n \log (SAAD) - n \log (SRWD) - n \log 2 - \log (n!) \]

\[ + \log \left[ \frac{FISMS_n (SAAD) FKSMA_n (SRWD)}{FKSMA_n (SAAD)} \right] \]  

(G.23)

OPTION P: Small argument approximation (RWD)

\[ \log A_n = \log_k (SRRD) + \log_l (SAAD) - \log_k (SAAD) + n \log 2 + \log (n!) \]

\[ - n \log (SRWD) - \log 2n + \log \left[ FKSMA_n (SRWD) \right] \]  

(G.24)

OPTION Q: Small argument approximation (AAD,RRD,RWD)

\[ \log A_n = 2n \log (SAAD) - n \log \left[ (SRWD)(SRRD) \right] - \log 2n \]

\[ + \log \left[ \frac{FKSMA_n (SRWD) FISMS_n (SAAD) FKSMA_n (SRWD)}{FKSMA_n (SAAD)} \right] \]  

(G.25)

OPTION Z: Simple program

\[ A_n = \frac{K_n (SRWD) I_n (SAAD) K_n (SRRD)}{K_n (SAAD)} \]  

(G.26)

\( K_n(z) \) is evaluated using the "step up" method, where:

\[ K_{n+1}(z) = K_{n-1}(z) + \frac{2n}{z} K_n(z) \]  

(G.27)
For $I_n(z)$ the formula:

$$I_{n+1}(z) = I_{n-1}(z) - \frac{2n}{z} I_n(z) \quad (G.28)$$

cannot be used due to the minus sign. For $n > 10$ the errors increase rapidly. $I_n(z)$ is evaluated using the ascending series.

EQUATIONS FOR THE NO-FLOW CIRCULAR BOUNDARY

OPTION A: Complete asymptotic expansion

$$A_n = \frac{1}{2 \sqrt{(SRRD)(SRWD)}} e^{2(SAAD) - (SRRD) - (SRWD)}$$

$$\cdot \frac{G_n(SRWD)G_n(SRRD)}{[G_{n-1}(SAAD) + G_{n+1}(SAAD)]} \quad (G.29)$$
OPTION B: Partial asymptotic expansion (RWD, RRD)

\[ \log A_n = \log \left[ I_{n-1}(\text{SAAD}) + I_{n+1}(\text{SAAD}) \right] - \left[ (\text{SRRD}) + \text{SRWD} \right] \log e \\
- \log \left[ K_{n-1}(\text{SAAD}) + K_{n+1}(\text{SAAD}) \right] + \log \left[ \frac{\pi}{2\sqrt{(\text{SRRD})(\text{SRWD})}} \right] \\
+ \log \left[ G_n(\text{SRRD})G_n(\text{SRWD}) \right] \]  
\text{(G.30)}

OPTION C: Partial asymptotic expansion (RWD, RRD)

Small argument approximation (AAD)

\[ \log A_n = \log \left[ \frac{n\pi}{\sqrt{(\text{SRRD})(\text{SRWD})}} \right] G_n(\text{SRWD})G_n(\text{SRRD}) + 2n\log(\text{SAAD}) \\
- \left[ (\text{SRWD}) + (\text{SRRD}) \right] \log e - 2n\log 2 - 2\log(n!) \\
+ \log \left[ \frac{2n}{(\text{SAAD})} \text{FISMA}_{n-1}(\text{SAAD}) + \frac{\text{(SAAD)}}{2(n+1)} \text{FISMA}_{n+1}(\text{SAAD}) \right] \]  
\text{(G.31)}
OPTION D : Partial asymptotic expansion \(\text{RWD}\)

\[
\log A_n = \log K_n(\text{SRRD}) + \log \left[ I_{n-1}(\text{SAAD}) + I_{n+1}(\text{SAAD}) \right] - \text{(SRWD)loge}
\]

\[
- \log \left[ K_{n-1}(\text{SAAD}) + K_{n+1}(\text{SAAD}) \right] + \log \left[ \frac{\sqrt{\pi}}{2\sqrt{\text{SRWD}}} G_n(\text{SRWD}) \right]
\]

\(G.32\)

OPTION F : Partial asymptotic expansion \(\text{RWD}\)

Small argument approximation \(\text{AAD}\)

\[
\log A_n = \log \left[ \frac{\sqrt{\pi}}{2\sqrt{\text{SRWD}}} K_n(\text{SRRD}) \right] + \log \left[ 2G_n(\text{SRWD}) \right] - \text{(SRWD)loge}
\]

\[
- 2\log(n!) - 2n\log 2 + 2n\log(\text{SAAD})
\]

\[
+ \log \left[ \frac{2n}{(\text{SAAD})} F\text{ISMA}_{n-1}(\text{SAAD}) + \frac{(\text{SAAD})}{2(n+1)} F\text{ISMA}_{n+1}(\text{SAAD}) \right]
\]

\[
- \log \left[ \frac{(\text{SAAD})}{2(n-1)} F\text{KSM}\text{A}_{n-1}(\text{SAAD}) + \frac{2n}{(\text{SAAD})} F\text{KSM}\text{A}_{n+1}(\text{SAAD}) \right]
\]

\(G.33\)
OPTION G: Partial asymptotic expansion (RWD)

Small argument approximation (AAD, RRD)

\[
\log A_n = \log \left[ \frac{\sqrt{\pi}}{2\sqrt{\text{SRWD}}} G_n(\text{SRWD}) \right] - 2n\log(SAAD) - (\text{SRWD})\log e \\
- n\log(SRRD) - n\log 2 - \log(n!) + \log \left[ FKSMA_n(\text{SRRD}) \right] \\
+ \log \left[ \frac{2n}{(\text{SADD})} FISMA_{n-1}(\text{SAAD}) + \frac{(\text{SAAD})}{2(n+1)} FISMA_{n+1}(\text{SAAD}) \right] \\
- \log \left[ \frac{(\text{SAAD})}{2(n-1)} FKSMA_{n-1}(\text{SAAD}) + \frac{2n}{(\text{SAAD})} FKSMA_{n+1}(\text{SAAD}) \right] \quad (G.34)
\]

OPTION I: Partial asymptotic expansion (RRD)

\[
\log A_n = \log K_n(\text{SRWD}) + \log \left[ \ln_{n-1}(\text{SAAD}) + \ln_{n+1}(\text{SAAD}) \right] - (\text{SRWD})\log e \\
- \log \left[ K_{n-1}(\text{SAAD}) + K_{n+1}(\text{SAAD}) \right] + \log \left[ \frac{\sqrt{\pi}}{\sqrt{2(\text{SRRD})}} G_n(\text{SRRD}) \right] \\
\quad (G.35)
\]
OPTION J: Partial asymptotic expansion (RRD)
Small argument approximation (AAD)

\[
\log A_n = \log \left[ \sqrt{\frac{\pi}{2(\text{SRWD})}} K_n (\text{SRWD}) \right] + \log \left[ 2n G_n (\text{SRRD}) \right] - (\text{SRRD}) \log e
\]
\[
- \log(n!) - 2n \log 2 + 2n \log (\text{SAAD})
\]
\[
+ \log \left[ \frac{2n}{(\text{SAAD})} \text{FISMA}_{n-1} (\text{SAAD}) + \frac{(\text{SAAD})}{2(n+1)} \text{FISMA}_{n+1} (\text{SAAD}) \right]
\]
\[
- \log \left[ \frac{(\text{SAAD})}{2(n-1)} \text{FKSMA}_{n-1} (\text{SAAD}) + \frac{2n}{(\text{SAAD})} \text{FKSMA}_{n+1} (\text{SAAD}) \right] \quad (G.36)
\]

OPTION K: Partial asymptotic expansion (RRD)
Small argument approximation (RWD AAD)

\[
\log A_n = \log \left[ \sqrt{\frac{\pi}{2(\text{SRWD})}} G_n (\text{SRWD}) \right] + 2n \log (\text{SAAD}) - n \log (\text{SRWD})
\]
\[
- (\text{SRRD}) \log e - n \log 2 - \log(n!) + \log \left[ \text{FKSMA}_n (\text{SRWD}) \right]
\]
\[
+ \log \left[ \frac{2n}{(\text{SAAD})} \text{FISMA}_{n-1} (\text{SAAD}) + \frac{(\text{SAAD})}{2(n+1)} \text{FISMA}_{n+1} (\text{SAAD}) \right]
\]
\[
- \log \left[ \frac{(\text{SAAD})}{2(n-1)} \text{FKSMA}_{n-1} (\text{SAAD}) + \frac{2n}{(\text{SAAD})} \text{FKSMA}_{n+1} (\text{SAAD}) \right] \quad (G.37)
\]
OPTION L : Small argument approximation (AAD)

\[ \log_{A_n} = \log_{K_n} (SRWD) + \log_{K_n} (SRRD) + 2n \log(SAAD) \]

\[ + \log 2n - 2n \log 2 - 2 \log(n!) \]

\[ + \log \left[ \frac{2n}{(SAAD)} FISMA_{n-1}(SAAD) + \frac{(SAAD)}{2(n+1)} FISMA_{n+1}(SAAD) \right] \]

\[ - \log \left[ \frac{(SAAD)}{2(n-1)} FKSMA_{n-1}(SAAD) + \frac{2n}{(SAAD)} FKSMA_{n+1}(SAAD) \right] \quad (G.38) \]

OPTION M : Small argument approximation (AAD, RRD)

\[ \log_{A_n} = \log_{K_n} (SRWD) + 2n \log(SAAD) - n \log(SRRD) - n \log 2 \]

\[ - \log (n!) + \log \left[ FKSMA_n (SRRD) \right] \]

\[ + \log \left[ \frac{2n}{(SAAD)} FISMA_{n-1}(SAAD) + \frac{(SAAD)}{2(n+1)} FISMA_{n+1}(SAAD) \right] \]

\[ - \log \left[ \frac{(SAAD)}{2(n-1)} FKSMA_{n-1}(SAAD) + \frac{2n}{(SAAD)} FKSMA_{n+1}(SAAD) \right] \quad (G.39) \]
OPTION N : Small argument approximation (RRD)

\[
\log A_n = \log K_n^{(SRWD)} + n \log 2 + \log(n!) - n \log(SRRD) - \log 2n
\]

\[
+ \log \left[ \text{FKSMA}_n^{(SRRD)} \right] + \log \left[ I_{n-1}^{(SAAD)} + I_{n+1}^{(SAAD)} \right]
\]

\[
- \log \left[ K_{n-1}^{(SAAD)} + K_{n+1}^{(SAAD)} \right]
\]  \hspace{1cm} (G.40)

OPTION 0 : Small argument approximation (AAD, RWD)

\[
\log A_n = \log K_n^{(SRWD)} + 2n \log(SAAD) - n \log(SRWD) - n \log 2
\]

\[
- \log(n!) + \log \left[ \text{FKSMA}_n^{(SRWD)} \right]
\]

\[
+ \log \left[ \frac{2n}{(SAAD)} \text{FISMA}_{n-1}^{(SAAD)} + \frac{(SAAD)}{2(n+1)} \text{FISMA}_{n+1}^{(SAAD)} \right]
\]

\[
- \log \left[ \frac{(SAAD)}{2(n-1)} \text{FKSMA}_{n-1}^{(SAAD)} + \frac{2n}{(SAAD)} \text{FKSMA}_{n+1}^{(SAAD)} \right]
\]  \hspace{1cm} (G.41)
OPTION P: Small argument approximation (RWD)

\[ \log A_n = \log K_n(SRRD) + \log \left[ I_{n-1}(SAAD) + I_{n+1}(SAAD) \right] + n \log 2 \]

\[ - \log \left[ K_{n-1}(SAAD) + K_{n+1}(SAAD) \right] + \log(n!) - n \log(SRWD) \]

\[ - \log 2n + \log \left[ FKSMA_n(SRWD) \right] \]

(G.42)

OPTION Q: Small argument approximation (AAD, RRD, RWD)

\[ \log A_n = 2n \log(SAAD) - n \log \left[ (SRRD)(SRWD) \right] - \log 2n \]

\[ + \log \left[ FKSMA_n(SRWD) FKSMA_n(SRD) \right] \]

\[ + \log \left[ \frac{2n}{(SAAD)} FISMA_{n-1}(SAAD) + \frac{(SAAD)}{2(n+1)} FISMA_{n+1}(SAAD) \right] \]

\[ - \log \left[ \frac{(SAAD)}{2(n-1)} FKSMA_{n-1}(SAAD) + \frac{2n}{(SAAD)} FKSMA_{n+1}(SAAD) \right] \]

(G.43)

OPTION Z: Simple program

\[ A_n = \frac{K_n(SRRD) \left[ I_{n-1}(SAAD) + I_{n+1}(SAAD) \right] K_n(SRWD)}{\left[ K_{n-1}(SAAD) + K_{n+1}(SAAD) \right]} \]

(6.44)
C************************************************************************************
C THIS PROGRAM EVALUATES PD(TD) FOR A CONSTANT RATE
C LINE SOURCE WELL EXTERIOR TO A CIRCULAR BOUNDARY
C IN AN INFINITE SLAB SYSTEM.
C TWO CASES ARE CONSIDERED:
C ICASE=1 CONSTANT PRESSURE HOLE
C ICASE=2 CLOSED BOUNDARY HOLE
C************************************************************************************

VARIABLE DEFINITIONS
*************************

IOPT=OPTION FLAG FOR THE BESSEL FUNCTIONS ON IMSL
IER=ERROR FLAG FOR THE BESSEL FUNCTION ON IMSL
N=NUMBER OF TERMS IN THE STEHFEST ALGORITHM
M=V(I) CALCULATION FLAG
ICASE=TYPE OF HOLE FLAG
RRD=DIMENSIONLESS DISTANCE
RD=DIMENSIONLESS DISTANCE TO THE HOLE
AAD=DIMENSIONLESS RADIUS OF THE HOLE
RRRD(I)=DISTANCE BETWEEN THE WELL AND THE PRESSURE POINT
SRRD=BESSEL FUNCTION ARGUMENT
SRWD=BESSEL FUNCTION ARGUMENT
SAAD=BESSEL FUNCTION ARGUMENT
SRRRD=BESSEL FUNCTION ARGUMENT
F=NORMALIZED RADIUS OF THE HOLE
E=NORMALIZED RADIUS TO THE PRESSURE POINT
EPSI=CONVERGENCE CRITERION FOR F(S)
EPSI1=CONVERGENCE CRITERION FOR THE BESSEL FUNCTIONS
DELTA=NTH TERM/SUMMATION TO N
DELTAA=N-1 TERM/SUMMATION TO N-1
DELT=CONVERGENCE RATIO
TP=THE CONSTANT PI
FACNU=FACTORIALS
PDSS(I)=STEADY STATE PD
PD=DIMENSIONLESS PRESSURE DROP
PDH=NORMALIZED PRESSURE DROP
PDLS=LINE SOURCE DIMENSIONLESS PRESSURE DROP
THETA(1)=0 DEG
THETA(2)=45 DEG
THETA(3)=90 DEG
THETA(4)=135 DEG
THETA(5)=180 DEG
ALFA=N*THETA IN F(S)
TD(II)=DIMENSIONLESS TIME
TD=DIMENSIONLESS TIME
TDR=TD/RD**2, NORMALIZED DIMENSIONLESS TIME
SUM(NU,IJ)=TERMS IN THE INFINITE SUMMATION OF F(S)
NU=ORDER OF THE BESSEL FUNCTIONS
ICASE=BOUNDARY FLAG
IFLAG1=ANGLE FLAG FOR A GIVEN TD.

-178-
IFLAG2 = FLAG FOR CALCULATING PD(TD/RD**2 = 0.1)
EEE = NATURAL CONSTANT
DEE = LOG BASE 10 OF EEE
BIAAD = I(N, Z), THE ARGUMENT OF AAD
BKAAD = K(N, Z), THE ARGUMENT OF AAD
BKRRD = K(N, Z), THE ARGUMENT OF RRD
BKRRD = K(N, Z), THE ARGUMENT OF RRD
EK---- = LOG BASE 10 OF K(N, Z)

LIST OF FUNCTIONS AND SUBROUTINES

FUNCTION PWD = THE STEHFEST ALGORITHM
FUNCTION PWDL = F(S) FOR BOTH BOUNDARY CASES
FUNCTION SUMIN = GENERATES THE TERMS IN THE INFINITE SERIES
FUNCTION BESI = MODIFIED BESSEL FUNCTION I(NU, Z), ASC. SER.
FUNCTION BESKA = MODIFIED BESSEL FUNCTION I(NU, Z), ASY. EXP.
FUNCTION BESIKA = MODIFIED BESSEL FUNCTION K(NU, Z), ASY. EXP.
FUNCTION BAAD = MODIFIED BESSEL FUNCTION K(N, Z), STEP
FUNCTION BKRRD = MODIFIED BESSEL FUNCTION K(N, Z), STEP
FUNCTION BKRRD = MODIFIED BESSEL FUNCTION K(N, Z), STEP
FUNCTION FKSM = SMALL ARG. APPX. FOR K(N, Z), N DEPENDENT TERM
FUNCTION FISMA = SMALL ARG. APPX. FOR I(N, Z), N DEPENDENT TERM
SUBROUTINE FACT = GENERATES THE FACTORIALS TO 50

MAIN PROGRAM

IMPLICIT REAL*8 (A-H, O-Z)
REAL*8 RRD, RWD, AAD, EPSI, TPI, TTPI
REAL*8 EEE, DEE
INTEGER IOPT, IER, N, M, ICASE
DIMENSION PDSS(5), TD(100), PD(5, 100)
COMMON /FRAC3/FACNU(51)
COMMON /FRAC1/THETA(5)
COMMON /FRAC2/RKD(5), RWD, RRD, AAD, EPSI, EPSI1, TTD, JJ
COMMON /FRAC4/DEE, TPI, ICASE
COMMON /FRAC5/IOP, IER
COMMON /FRAC6/IFLAG1
COMMON /FRAC99/IEND99

VARIABLE INPUT DATA
ICASE = 2
N = 8
RRD = 14.0D0
RWD = 20.0D0
AAD = 8.0D0
EPSI = 1.0D-13
EPSI1 = 1.0D-15

CONSTANT DATA
M = 77777
IOPT = 1
TTPI = 3.141592654
EEE = DEXP(1.0D0)
DEE = DLOG10(EEE)
IEND99 = 0

GENERATING THE Factorials
CALL FACT

GENERATING RRRD(I)
DO 1 I = 1, 5
TPI = TTPI / 4.0
THETA(I) = TPI * (I - 1)
RRRD(I) = (RRD**2 + RWD**2 - 2 * RRD * RWD * DCOS(THETA(I)))**0.5
1 CONTINUE

PRESETTING PD(JJ,II) TO ZERO
DO 2 I = 1, 5
DO 3 J = 1, 100
PD(I, J) = 0.0D0
3 CONTINUE
2 CONTINUE

DECIDING WHAT PROGRAM TO FOLLOW
IF (ICASE.EQ.1) GO TO 1000
IF (ICASE.EQ.2) GO TO 2000
WRITE (6, 3001)
3001 FORMAT (5X, '***ICASE WAS NOT SPECIFIED***')
GO TO 1050

CONSTANT PRESSURE HOLE
*************************************************************************

LONG TIME APPROXIMATION—DOUBLET MODEL
1000 W1 = (AAD**2) / (RWD * RRD)
W2 = RRD / RWD
DO 1001 I = 1, 5
WCOS = DCOS(THETA(I))
WW = 1 - 2 * W1 * WCOS + (W1**2)
WWW = 1 - 2 * W2 * WCOS + (W2**2)
W = WW / WWW
PDSS(I) = DLOG(RRD / AAD) + 0.5 * DLOG(WW)
1001 CONTINUE
DO 1042 I=1,100
TD(I)=1.0
1042 CONTINUE
THE TIME LOOP. WE CONSIDER SIX LOG CYCLES FOR
TD/RD**2 STARTING AT 0.1.
II=0
DO 1002 KKK=1,20
DO 1003 K=1,5
IF (IEND99.EQ.1) GO TO 1041
AB=KK
ABB=KKK
II=II+1
TD(II)=AB*0.0002*(10.0**ABB)
1003 CONTINUE
1002 CONTINUE
NOW WE CONVERT TO TD/RD**2 FOR ANGLE=0
1020 TD(II)=TD(II)*(RRRD(I)**2)
TTD=TD(II)
WRITE (6,1059) TTD
1059 FORMAT (15X,D20.9)
IFLAG1=0
THE ANGLE LOOP. THE ORDER OF CALCULATION IS
0,180,45,135,90 DEG. THE ANGLES ARE REARRANGED:
DO 1021 JJJ=1,5
IF (JJJ.EQ.1) JJ=1
IF (JJJ.EQ.2) JJ=5
IF (JJJ.EQ.3) JJ=4
IF (JJJ.EQ.4) JJ=2
IF (JJJ.EQ.5) JJ=3
IF (IEND99.EQ.1) GO TO 1041
IF (JJ.GT.1) GO TO 1021
NOW THE ANGLES ARE \theta(JJ) AND THE RADII ARE
RRRD(JJ). WE NOW CHECK THE VALUE OF TD/RD**2:
TDR=TD(II)/(RRRD(JJ)**2)
IF (TDR.GE.0.0999.AND.TDR.LE.1.001D8) GO TO 1022
PD(JJ,II)=0.0
GO TO 1021
TD/RD**2 IS GREATER THAN 0.1 AND WE CONTINUE
WITH THE PD CALCULATION.
1022 PD(JJ,II)=PWD(TTD,N,M)
1021 CONTINUE
1003 CONTINUE
1002 CONTINUE
THE RESULTS
WRITE (6, 1030)
FORMAT (///, 20X, 'THE RESULTS', //, 20X, '**************', //)
WRITE (6, 1035) ICASE
FORMAT (10X, 'ICASE=', I5, //)
F = AAD/RWD
E = RRD/RWD
WRITE (6, 1031) AAD, RWD, RRD
WRITE (6, 1032) F, E
WRITE (6, 1025)
FORMAT (10X, 'THE STEADY STATE PRESSURES', //)
DO 1026 I = 1, 5
THET = THETA(I) * 180.0 / TTPI
WRITE (6, 1027) THET, PDSS(I)
1027 FORMAT (10X, 'THETA=', E12.5, //, 10X, 'PDSS=', D20.9)
CONTINUE
WRITE (6, 1028)
1028 FORMAT (///)

DO 1004 JJ = 1, 5
THET = THETA(JJ) * 180.0 / TTPI
WRITE (6, 1033) THET
1033 FORMAT (///, 5X, 'THETA=', E12.5, //)
WRITE (6, 1034)

DO 1005 II = 1, 100
IF (PD(JJ, II) .EQ. 0.0) GO TO 1005
PDN = PD(JJ, II) / PDSS(JJ)
TTD = TD(II)
TDR = TTD / (RRRD(JJ)**2)
WRITE (6, 1010) TTD, TDR, PD(JJ, II), PDN
1010 FORMAT (5X, 4(D20.9))
CONTINUE
CONTINUE
1004 CONTINUE

END OF THE CONSTANT PRESSURE CALCULATIONS
GO TO 1050

CLOSED BOUNDARY HOLE

2000 DO 2042 I = 1, 100
TD(I) = 1.0
The time loop. We consider six log cycles for $TD/RD^{**2}$ starting at 0.1.

\[ II = 0 \]
\[
DO 2002 KKK = 1,20 \\
DO 2003 KK = 1,5 \\
IF (IEND99.EQ.1) GO TO 2041 \\
AB = KK \\
AR=KKK \\
II = II + 1 \\
TD(II) = AB * 0.0002 * (10.0 ** ABB)
\]

Now we convert to $TD/RD^{**2}$ for angle = 0

\[
2020 \quad TD(II) = TD(II) * (RRRD(II)**2) \\
TTD = TD(II) \\
\]

WRITE (6,2059) TTD

\[ C2059 \quad FORMAT (15X,D20.9) \]

\[ C \]

IFLAG = 0

The angle loop. The order of calculation is 0, 180, 45, 135, 90 deg. The angles are rearranged:

\[
DO 2021 JJJ = 1,5 \\
IF (JJJ.EQ.1) JJ = 1 \\
IF (JJJ.EQ.2) JJ = 5 \\
IF (JJJ.EQ.3) JJ = 4 \\
IF (JJJ.EQ.4) JJ = 2 \\
IF (JJJ.EQ.5) JJ = 3 \\
IF (IEND99.EQ.1) GO TO 2041 \\
IF (JJ.GT.1) GO TO 2021
\]

Now the angles are $\theta(JJ)$ and the radii are $RRRD(JJ)$. We now check the value of $TD/RD^{**2}$:

\[
TDR = TD(II) / (RRRD(JJ)**2) \\
IF (TDR.GE.0.0999.AND.TDR.LE.1.001DS) GO TO 2022 \\
PD(JJ,II) = 0.0 \\
GO TO 2021
\]

$TD/RD^{**2}$ is greater than 0.1 and we continue with the PD calculation.

\[ 2022 \quad PD(JJ,II) = PWD(TTD,N,M) \\
2021 \quad CONTINUE \\
2020 \quad CONTINUE \\
2003 \quad CONTINUE \\
2002 \quad CONTINUE \\
\]

The results

\[ 2041 \quad WRITE (6,2030) \]

-183-
THE STEHFEST ALGORITHM

FUNCTION PWD(TD,N,M)
  THIS FUNCTION COMPUTES NUMERICALLY THE LAPLACE TRANSFORM INVERSE OF F(S).
  IMPLICIT REAL*8 (A-H,O-Z)
  DIMENSION G(50),V(50),H(25)
  COMMON /FRAC6/IFLAG!
  COMMON /FRAC99/IEND99

  NOW IF THE ARRAY V(I) WAS COMPUTED BEFORE THE PROGRAM GOES DIRECTLY TO THE END OF THE SUBROUTINE TO CALCULATE F(S).
  IF (N.EQ.M) GO TO 17
  M=N
  DLOGT=0.6931471805599
  NH=N/2

  THE FACTORIALS OF 1 TO N ARE CALCULATED INTO ARRAY G.
  G(1)=1
  DO 1 I=2,N
    G(I)=G(I-1)*I
  1 CONTINUE
TERMS WITH K ONLY ARE CALCULATED INTO ARRAY H.

\[ H(1) = 2./G(NH-1) \]

DO 6 I=2,NH
   \( F_I = I \)
   IF(I-NH) 4,5,6
4   \( H(I) = F_I**NH*G(2*I)/(G(NH-I)*G(I)*G(I-1)) \)
      GO TO 6
5   \( H(I) = F_I**NH*G(2*I)/(G(I)*G(I-1)) \)
6   CONTINUE

THE TERMS \((-1)**NH+1\) ARE CALCULATED.

FIRST THE TERM FOR I = 1

\[ SN = 2*(NH-NH/2*2)-1 \]

THE REST OF THE SN'S ARE CALCULATED IN THE MAIN ROUTINE.

THE ARRAY V(I) IS CALCULATED.

DO 7 I=1,N
   \( V(I) = 0. \)
   \( K_1 = (I+1)/2 \)
   \( K_2 = \text{MIN}(I,N/2) \)
   \( V(I)=SN*V(I) \)

THE SUMMATION TERM IN V(I) IS CALCULATED.

DO 10 K=K_1,K_2
   IF (2*K-I) 12,13,12
      IF (I-K) 11,14,11
11   \( V(I)=V(I)+H(K)/(G(I-K)*G(2*K-I)) \)
      \( V(I)=V(I)+H(K)/G(I-K) \)
      \( V(I)=V(I)+H(K)/G(2*K-I) \)
10  CONTINUE

THE V(I) ARRAY IS FINALLY CALCULATED BY WEIGHTING ACCORDING TO SN.

\[ V(I) = SN*V(I) \]

WRITE (6,21) I, V(I)

FORMAT (5X,'I=',15,5X,'V(I)=',D20.9)

THE TERM SN CHANGES ITS SIGN EACH ITERATION.

\[ SN = -SN \]
CONTINUE

THE NUMERICAL APPROXIMATION IS CALCULATED.

17

PWD = 0.
A = DLOGTW/TD
DO 15 I = 1, M
IF (IEND99.EQ.1) GO TO 18
WRITE (6,20) I
15

FORMAT (5X,15)
ARG = A*I
PWD = PWD + V(I)*PWDL(ARG, I)
CONTINUE
PWD = PWD*A
IFLAG1 = 1
18
RETURN
END

FUNCTION PWDL(ARG, IJ)

******************************

THIS FUNCTION EVALUATES F(S) OF THE CONSTANT PRESSURE
BOUNDARY CASE FOR THE STEH FEST ALGORITHM. THE ARGUMENT
<S> IS FIXED BY THE ALGORITHM.

FUNCTION PWDL(ARG, IJ)
IMPLICIT REAL*8 (A-H,O-Z)
REAL*8 SRRD, SRWD, SAAAD, SRRRD, PWDL1, ALPHA
COMMON /FRAC12/SUM(50,16)
COMMON /FRAC11/BIAAD(100), BKAAD(100), BKRWD(100), BKRRD(100)
COMMON /FRAC1/THETA(5)
COMMON /FRAC2/RQRD(5), RWD, RRD, AAD, EPSI, EPSI1, TTD, JJ
COMMON /FRAC3/PACNU(51)
COMMON /FRAC5/IOPT, IER
COMMON /FRAC6/IFLAG1
COMMON /FRAC99/IEND99
DOUBLE PRECISION MMBS10, MMBS11, MMBSK0, MMBSK1
DELTA = 1.0

FIRST WE CALCULATE THE LINE SOURCE TERM
SRRRD = DSQRT(ARG)*RQRD(JJ)
PWL1 = MMBSK0( IOPT, SRRRD, IER )

NOW WE CHECK IF THE MATRIX SUM(IU, IJ) IS COMPLETE
IF (IFLAG1.EQ.1) GO TO 7

SETTING THE VECTOR SUM(KK, IJ) TO ZERO
DO 1 KK = 1, 50

-186-
EVALUATING THE ARGUMENTS FOR THE BESSEL FUNCTIONS

SRWD = DSQRT(ARG) * RWD
SRRD = DSQRT(ARG) * RRD
SAAD = DSQRT(ARG) * AAD

NOW WE START CONSIDERING THE TERMS IN THE SUMMATION.
NOTE THAT THE BESSEL FUNCTIONS ARE STORED IN AN ARRAY
OF WHICH THE INDEX IS SHIFTED UPWARDS BY 1 TO BE ABLE
TO HANDLE K(0,Z). K(0,SAAD) IS NOTED AS BKAAD(1).

7
PWDL = -SUM(1, IJ) + PWDL1
DO 2 I = 2, 50
NU = I
NUU = NU - 1
IF (IFLAG1 .EQ. 1) GO TO 8
SUM(NU, IJ) = SUMIN(SRRD, SRWD, SAAD, NU, NUU)
IF (IEND99 .EQ. 1) GO TO 6
C
C
C

NOW WE CHECK THE CONVERGENCE OF F(S)
IF (PWDL3 .EQ. 0.0) GO TO 4
DELTA = PWDL3 / PWDL4
DELTA = DABS(DELTA)
DELT = DELTA / DELTAA
DELTAA = DELTA
IF (EPSI - DELTA) 3, 4, 4
PWDL = PWDL4
CONTINUE

THE SERIES CONVERGED, WE EVALUATE PWDL.
PWDL = PWDL / ARG
WRITE (6, 5) NU, PWDL, DELTA, DELT, SRWD, SRRD, SAAD
FORMAT (15, 1X, 6(D20.9))
RETURN
END
SUBROUTINE FACT

************************************************************************

THIS SUBROUTINE EVALUATES Factorials UNTIL 50.

SUBROUTINE FACT
IMPLICIT REAL*8 (A-H,O-Z)
COMMON /FRACJ/FACNU(S1)
FACNU(1)=1
FACNU(2)=1.0D0
DO 1 I=2,50
FACNU(I+1)=FACNU(I)*I
1 CONTINUE
RETURN
END

FUNCTION BESKA (N,Z)

************************************************************************

THIS FUNCTION EVALUATES THE N DEPENDENT SERIES FOR THE ASYMPOTIC EXPANSION OF K(N,Z)

FUNCTION BESKA(N,Z)
IMPLICIT REAL*8 (A-H,O-Z)
REAL*8 AZ
EPSI=1.0D-15
FNU=(N**2)*4.0
BESKA=1.0D0
ZZ=Z**8.0
AZ=1.0D0
DO 1 I=1,100
AZ=AZ*(FNU-((2*I-1)**2))/(ZZ*I)
BESKA=BESKA+AZ
DELTA=AZ/BESKA
IF (DABS(DELTA).LT.EPSI) GO TO 2
1 CONTINUE
2 RETURN
END

FUNCTION BESIA (N,Z)

************************************************************************

THIS FUNCTION EVALUATES THE N DEPENDENT SERIES FOR THE ASYMPOTIC EXPANSION OF I(N,Z)

FUNCTION BESIA(N,Z)
IMPLICIT REAL*8 (A-H,O-Z)
REAL*8 AZ,FNU,ZZ,DELTA
EPSI=1.0D-15
FNU=(N**2)*4.0
BESIA=1.0D0
ZZ=Z*8.0
AZ=1.0DD
DO 1 I=1,100
AZ=AZ*(-1)*(FNU-((2*I-1)**2))/(ZZ*I)
BESIA=BESIA+AZ
DELTA=AZ/BESIA
IF (DABS(DELTA).LT.EPSI) GO TO 2
1 CONTINUE
2 RETURN
END

FUNCTION SUMIN (SRRD,SRWD,SAAD,NU,NUU)

*** THIS FUNCTION EVALUATES THE TERMS IN THE INFINITE SERIES IN F(S). ***

FUNCTION SUMIN (SRRD,SRWD,SAAD,NU,NUU)
IMPLICIT REAL*8 (A-H,O-Z)
COMMON /FRAC12/SUM(50,16)
COMMON /FRAC11/BIAAD(100),BKAAD(100),BKRD(100)
COMMON /FRAC4/DEE,TTP1,ICASE
COMMON /FRAC3/FACNU(51)
COMMON /FRAC99/IEND99
DOUBLE PRECISION MBBSK0,MBBSK1,MBSI0,MBSI1
ILOPT=1
NUU=NU-1
NUUU=NUU
IF (ICASE.EQ.2.AND.NU.EQ.1) NUUU=NUU+2
NU=NUUU+1

THIS PART IS THE SCREENING SECTION THAT PICKS THE
METHOD FOR EVALUATING THE F(S) TERMS.

IF (SAAD.GE.100) GO TO 251
IF (SRWD.GE.100.AND.SRRD.GE.100) GO TO 252
IF (SRWD.GE.100.AND.SRRD.LT.100) GO TO 253
IF (SRWD.LT.100.AND.SRRD.GE.100) GO TO 254
IF (SRWD.LT.100.AND.SRRD.LT.100) GO TO 255

252 IF (SAAD.GE.20) GO TO 256
IF (SAAD.GE.2.0) GO TO 257
IF (SAAD.LT.2.0) GO TO 258

256 EKAAD=SAAD*DEE
IF (EKAAD.GE.65) GO TO 251
GO TO 257

258 EKAAD=(-NU)*DLOG10(SAAD)+DLOG10(FACNU(NU))+(NU-1)*DLOG10(2.0DD)
IF (EKAAD.GE.65) GO TO 259
GO TO 257

-189-
I F (SAA D . G E . 2 0 . A N D . S R R D . G E . 2 0 ) G O T O 2 6 0  
I F (SAA D . L T . 2 0 . A N D . S R R D . G E . 2 0 ) G O T O 2 6 1  
I F (SAA D . L T . 2 0 . A N D . S R R D . L T . 2 0 ) G O T O 2 6 2  
E K A A D = S A A D * D E E  
E K R R D = S R R D * D E E  
I F (E K A A D . G E . 6 5 . A N D . E K R R D . G E . 6 5 ) G O T O 2 5 1  
I F (E K A A D . L T . 6 5 . A N D . E K R R D . G E . 6 5 ) G O T O 2 5 7  
I F (E K A A D . L T . 6 5 . A N D . E K R R D . L T . 6 5 ) G O T O 2 6 3  
I F (S A A D . G E . 2 . 0 ) G O T O 2 6 4  
I F (S A A D . L T . 2 . 0 ) G O T O 2 6 5  
E K R R D = S R R D * D E E  
I F (E K R R D . G E . 6 5 ) G O T O 2 5 7  
I F (E K R R D . L T . 6 5 ) G O T O 2 6 3  
E K R R D = S R R D * D E E  
E K A A D = ( - N U ) * D L O G 1 0 ( S A A D ) + D L O G 1 0 ( F A C N U ( N U ) ) + ( N U - 1 ) * D L O G 1 0 ( 2 . 0 D D O )  
I F (E K R R D . G E . 6 5 . A N D . E K A A D . L T . 6 5 ) G O T O 2 5 7  
I F (E K R R D . L T . 6 5 . A N D . E K A A D . L T . 6 5 ) G O T O 2 6 3  
I F (S A A D . G E . 2 . 0 . A N D . S R R D . G E . 2 . 0 ) G O T O 2 6 3  
I F (S A A D . L T . 2 . 0 . A N D . S R R D . G E . 2 . 0 ) G O T O 2 6 7  
I F (S R R D . L T . 2 . 0 ) G O T O 2 6 8  
E K A A D = ( - N U ) * D L O G 1 0 ( S A A D ) + D L O G 1 0 ( F A C N U ( N U ) ) + ( N U - 1 ) * D L O G 1 0 ( 2 . 0 D D O )  
I F (E K A A D . G E . 6 5 ) G O T O 2 6 6  
I F (E K A A D . L T . 6 5 ) G O T O 2 6 3  
E K R R D = ( - N U ) * D L O G 1 0 ( S R R D ) + D L O G 1 0 ( F A C N U ( N U ) ) + ( N U - 1 ) * D L O G 1 0 ( 2 . 0 D D O )  
E K A A D = ( - N U ) * D L O G 1 0 ( S A A D ) + D L O G 1 0 ( F A C N U ( N U ) ) + ( N U - 1 ) * D L O G 1 0 ( 2 . 0 D D O )  
I F (E K R R D . L T . 6 5 . A N D . E K A A D . L T . 6 5 ) G O T O 2 6 3  
I F (S A A D . G E . 2 . 0 . A N D . S R R D . G E . 2 . 0 ) G O T O 2 7 0  
I F (S A A D . L T . 2 . 0 . A N D . S R R D . G E . 2 . 0 ) G O T O 2 7 1  
I F (S A A D . L T . 2 . 0 . A N D . S R R D . L T . 2 0 ) G O T O 2 7 2  
E K A A D = S A A D * D E E  
E K R W D = S R R D * D E E  
I F (E K A A D . G E . 6 5 . A N D . E K R W D . G E . 6 5 ) G O T O 2 5 1  
I F (E K A A D . L T . 6 5 . A N D . E K R W D . L T . 6 5 ) G O T O 2 7 3  
I F (S A A D . G E . 2 . 0 ) G O T O 2 7 4  
I F (S A A D . L T . 2 . 0 ) G O T O 2 7 5  
-190-
EKRWD = SRWD * DEE

IF (EKRWD \leq 65) GO TO 257
IF (EKRWD > 65) GO TO 273

C 275
C 27
C 277
C 278
C 27
C 27
C 274

EKRWD = SRWD * DEE

IF (EKRWD \geq 65 \land EKAAD \geq 65) GO TO 259
IF (EKRWD \geq 65 \land EKAAD < 65) GO TO 257
IF (EKRWD < 65 \land EKAAD \geq 65) GO TO 276
IF (EKRWD < 65 \land EKAAD < 65) GO TO 273

IF (SAAD \geq 2.0 \land SRWD \geq 2.0) GO TO 280
IF (SAAD < 2.0 \land SRWD \geq 2.0) GO TO 288
IF (SAAD \geq 2.0 \land SRWD < 2.0) GO TO 286
IF (SAAD < 2.0 \land SRWD < 2.0) GO TO 284

EKAAD = SAAD \times DEE
EKRWD = SRWD \times DEE

IF (EKAAD \leq 2.0 \land SRWD \geq 2.0) GO TO 287
IF (EKAAD \geq 2.0 \land SRWD < 2.0) GO TO 285
IF (EKAAD < 2.0 \land SRWD \geq 2.0) GO TO 283
IF (EKAAD < 2.0 \land SRWD < 2.0) GO TO 281
IF (EKWRD.LT.65.AND.EKRRD.GE.65) GO TO 273
IF (EKWRD.LT.65.AND.EKRRD.LT.65) GO TO 290

C

289  EKAAD=(-NU)*DLOG10(SAAD)+DLOG10(FACNU(NU))+(NU-1)*DLOG10(2.0D0)
IF (EKAAD.LT.65) GO TO 291
IF (EKAAD.GE.65) GO TO 292

C

291  EKWRD=SRWD*DEE
EKRRD=SRRD*DEE
IF (EKWRD.GE.65.AND.EKRRD.GE.65) GO TO 257
IF (EKWRD.GE.65.AND.EKRRD.LT.65) GO TO 263
IF (EKWRD.LT.65.AND.EKRRD.GE.65) GO TO 273
IF (EKWRD.LT.65.AND.EKRRD.LT.65) GO TO 290

C

292  EKWRD=SRWD*DEE
EKRRD=SRRD*DEE
IF (EKWRD.GE.65.AND.EKRRD.GE.65) GO TO 259
IF (EKWRD.GE.65.AND.EKRRD.LT.65) GO TO 266
IF (EKWRD.LT.65.AND.EKRRD.GE.65) GO TO 276
IF (EKWRD.LT.65.AND.EKRRD.LT.65) GO TO 290

C

281  IF (SAAD.GE.2.0) GO TO 294
IF (SAAD.LT.2.0) GO TO 295

C

294  EKWRD=SRWD*DEE
IF (EKWRD.GE.65) GO TO 263
IF (EKWRD.LT.65) GO TO 290

C

295  EKAAD=(-NU)*DLOG10(SAAD)+DLOG10(FACNU(NU))+(NU-1)*DLOG10(2.0D0)
EKWRD=SRWD*DEE
EKRRD=SRRD*DEE
IF (EKWRD.GE.65.AND.EKAAD.GE.65) GO TO 266
IF (EKWRD.GE.65.AND.EKAAD.LT.65) GO TO 263
IF (EKWRD.LT.65.AND.EKAAD.GE.65) GO TO 293
IF (EKWRD.LT.65.AND.EKAAD.LT.65) GO TO 290

C

282  IF (SAAD.GE.2.0) GO TO 296
IF (SAAD.LT.2.0) GO TO 297

C

296  EKRRD=SRRD*DEE
IF (EKRRD.GE.65) GO TO 273
IF (EKRRD.LT.65) GO TO 290

C

297  EKRRD=SRRD*DEE
EKAAD=(-NU)*DLOG10(SAAD)+DLOG10(FACNU(NU))+(NU-1)*DLOG10(2.0D0)
IF (EKRRD.GE.65.AND.EKAAD.GE.65) GO TO 276
IF (EKRRD.GE.65.AND.EKAAD.LT.65) GO TO 273
IF (EKRRD.LT.65.AND.EKAAD.GE.65) GO TO 293
IF (EKRRD.LT.65.AND.EKAAD.LT.65) GO TO 290

C

283  IF (SAAD.GE.2.0) GO TO 290
IF (SAAD.LT.2.0) GO TO 298

C
EKAA\(D=-(NU) \times DLOG10(SAAD)+DLOG10(FACNU(NU))+(NU-1) \times DLOG10(2.0DO)
\)
IF (EKAA\(D \geq 65\)) GO TO 293
IF (EKAA\(D < 65\)) GO TO 290

EKAA\(D=-(NU) \times DLOG10(SAAD)+DLOG10(FACNU(NU))+(NU-1) \times DLOG10(2.0DO)\)
EKRRD\(D=-(NU) \times DLOG10(SRRD)+DLOG10(FACNU(NU))+(NU-1) \times DLOG10(2.0DO)\)
IF (EKRRD\(D \geq 65\) AND EKAA\(D \geq 65\)) GO TO 299
IF (EKRRD\(D \geq 65\) AND EKAA\(D < 65\)) GO TO 300
IF (EKRRD\(D < 65\) AND EKAA\(D \geq 65\)) GO TO 293
IF (EKRRD\(D < 65\) AND EKAA\(D < 65\)) GO TO 290

EKAA\(D=-(NU) \times DLOG10(SAAD)+DLOG10(FACNU(NU))+(NU-1) \times DLOG10(2.0DO)\)
EKRRD\(D=-(NU) \times DLOG10(SRRD)+DLOG10(FACNU(NU))+(NU-1) \times DLOG10(2.0DO)\)
IF (EKRRD\(D \geq 65\)) GO TO 290
EKWRD\(D=-(NU) \times DLOG10(SRWD)+DLOG10(FACNU(NU))+(NU-1) \times DLOG10(2.0DO)\)
IF (EKRRD\(D \geq 65\)) GO TO 290
IF (EKWRD\(D \geq 65\) AND EKRRD\(D \geq 65\)) GO TO 303
IF (EKWRD\(D \geq 65\) AND EKRRD\(D < 65\)) GO TO 301
IF (EKWRD\(D < 65\) AND EKRRD\(D \geq 65\)) GO TO 299
IF (EKWRD\(D < 65\) AND EKRRD\(D < 65\)) GO TO 290

EKRRD\(D=-(NU) \times DLOG10(SRRD)+DLOG10(FACNU(NU))+(NU-1) \times DLOG10(2.0DO)\)
IF (EKRRD\(D \geq 65\)) GO TO 290
IF (EKRRD\(D \geq 65\)) GO TO 303
IF (EKRRD\(D \geq 65\)) GO TO 301
IF (EKWRD\(D \geq 65\)) GO TO 309

EKAA\(D=-(NU) \times DLOG10(SAAD)+DLOG10(FACNU(NU))+(NU-1) \times DLOG10(2.0DO)\)
IF (EKAA\(D \geq 65\)) GO TO 293
IF (EKAA\(D < 65\)) GO TO 290

EKWRD\(D=-(NU) \times DLOG10(SRWD)+DLOG10(FACNU(NU))+(NU-1) \times DLOG10(2.0DO)\)
IF (EKWRD\(D \geq 65\)) GO TO 290
IF (EKWRD\(D \geq 65\)) GO TO 308
IF (EKWRD\(D \geq 65\)) GO TO 301

EKAA\(D=-(NU) \times DLOG10(SAAD)+DLOG10(FACNU(NU))+(NU-1) \times DLOG10(2.0DO)\)
IF (EKAA\(D \geq 65\)) GO TO 293
IF (EKAA\(D < 65\)) GO TO 290
THIS SECTION EVALUATES THE TERMS $\sum(nu, ij)$

COMPLETE ASYMPOTIC EXPANSION ($aad, rwd, rrd$) A

251 IF (ICASE.EQ.2) GO TO 351
AR1=2*SAAD-SRRD-SRWD
AR2=DSQRT(SRRD*SRWD)
IF (DABS(AR1).GE.160) GO TO 499
GNRWD=BESKA(NUU,SRRD)
GNRRD=BESKA(NUU,SRRD)
GNAAD=BESKA(NUU,SAAD)
FNAAD=BESIA(NUU,SAAD)
SUMIN=DEXP(AR1)*GNRWD*GNRRD*FNAAD/(2*AR2*GNAAD)
GO TO 500

CLOSED BOUNDARY CASE

351 AR1=2*SAAD-SRRD-SRWD
AR2=DSQRT(SRRD*SRWD)
IF (DABS(AR1).GE.160) GO TO 499
GNRWD=BESKA(NUU,SRRD)
GNRRD=BESKA(NUU,SRRD)
GNAAD1=BESKA(NUU+1,SAAD)
GNAAD2=BESKA(NUU-1,SAAD)
FNAAD1=BESIA(NUU-1,SAAD)
FNAAD2=BESIA(NIU+1,SAAD)
SUMIN=DEXP(AR1)*GNRWD*GNRRD*(FNAAD1+FNAAD2)/(2*AR2)
SUMIN=-SUMIN/(GNAAD1+GNAAD2)
GO TO 500

PARTIAL ASYMPOTIC EXPANSION ($rwd, rrd$) B

257 IF (ICASE.EQ.2) GO TO 352
AR1=-SRWD-SRRD
AR2=DSQRT(SRWD*SRRD)
GNRWD=BESKA(NUU,SRRD)
GNRRD=BESKA(NUU,SRRD)
BIAAD(NUU)=BESI(NUU,SAAD)
BKAAAD(NUU)=BAAD(NUU,SAAD)
DLAN=DLOG10(BIAAD(NUU))+AR1*DEE+DLOG10(TTPI*GNRWD*GNRRD/(2*AR2))
DLAN=DLAN-DLOG10(BKAAAD(NUU))
IF (DLAN.LT.-60.0D0) GO TO 499
SUMIN=10**DLAN
GO TO 500

CLOSED BOUNDARY HOLE

352 AR1=-SRWD-SRRD
AR2=DSQRT(SRWD*SRRD)
GNRWD=BESKA(NUU,SRRD)
GNRRD=BESKA(NUU,SRRD)
BIAAD(NUU-1)=BESI(NUU-1,SAAD)
BIAAD(NUU+1)=BESI(NUU+1,SAAD)
BKAAD(NNU-1) = BAAD(NUU-1, SAAD)
BKAAD(NU+1) = BAAD(NUU+1, SAAD)
DLAN = DLOG10 (BIAAD(NNU-1) + BIAAD(NU+1) + AR1 * DEE + DLOG10 (TTPI * GNRWD))
DLAN = DLAN + DLOG10 (GRRD / (2 * AR1))
DLAN = DLAN - DLOG10 (BKAAD(NNU-1) + BKAAD(NU+1))
IF (DLAN .LT. -60.0D0) GO TO 499
SUMIN = -(10 ** DLAN)
GO TO 500

PARTIAL ASYMPTOTIC EXPANSION (RWD, RRD)  C
SMALL ARGUMENT APPROXIMATION (AAD)

259 IF (ICASE.EQ.2) GO TO 353
GRRWD = BESKA (NUU, SRWD)
FIAAD1 = FISMA (NUUU-1, SAAD)
FKAAD1 = FKSMA (NUUU-1, SAAD)
GRRWD = BESKA (NUU, SRRD)
AR1 = DSQRT (SRWD * SRRD)
DLAN = DLOG10 (TTPI * NNU * GNRWD * GRRWD / AR1) + 2 * NNU * DLOG10 (SAAD)
DLAN = DLAN - (SRWD + SRRD) * DEE - 2 * NNU * DLOG10 (2.0D0) - 2 * DLOG10 (FACNU (NU))
DLAN = DLAN + DLOG10 (FIAAD1 / FKAAD1)
IF (DLAN .LT. -60.0D0) GO TO 499
SUMIN = 10 ** DLAN
GO TO 500

CLOSED BOUNDARY HOLE

353 GRRWD = BESKA (NUU, SRWD)
FIAAD1 = FISMA (NUUU-1, SAAD)
FIAAD2 = FISMA (NUU+1, SAAD)
FKAAD1 = FKSMA (NUUU-1, SAAD)
FKAAD2 = FKSMA (NUU+1, SAAD)
GRRWD = BESKA (NUU, SRRD)
AR1 = DSQRT (SRWD * SRRD)
DLAN = DLOG10 (TTPI * NNU * GNRWD * GRRWD / AR1) + 2 * NNU * DLOG10 (SAAD)
DLAN = DLAN - (SRWD + SRRD) * DEE - 2 * NNU * DLOG10 (2.0D0) - 2 * DLOG10 (FACNU (NU))
DLAN = DLAN + DLOG10 ((2 * NNU * FIAAD1 / SAAD) + SAAD * FIAAD2 / (2 * (NUU+1))
DLAN = DLAN + DLOG10 ((SAAD * FKAAD1 / (2 * (NUUU-1))) + 2 * NNU * FKAAD2 / SAAD)
IF (DLAN .LT. -60.0D0) GO TO 499
SUMIN = -(10 ** DLAN)
GO TO 500

PARTIAL ASYMPTOTIC EXPANSION (RWD)  D

263 IF (ICASE.EQ.2) GO TO 354
BIAAD (NU) = BEST (NUU, SAAD)
BRKR (NU) = BRRD (NUU, SRRD)
BKAAD (NU) = BAAD (NUU, SAAD)
BRKW (NU) = BRWD (NUU, SRWD)
GRRWD = BESKA (NUU, SRWD)
DLAN = DLOG10 (BIAAD (NU)) + DLOG10 (BRKR (NU)) - DLOG10 (BKAAD (NU))

-195-
DLAN = DLAN + DLOG10(DSQRT(TTPI/(2*SRWD))*GNRWD) - SRWD*DEE
IF (DLAN .LT. -60.0D0) GO TO 499
SUMIN = 10**DLAN
GO TO 500

CLOSED BOUNDARY HOLE

BIADD(NUU-1) = BESI(NUU-1, SAAD)
BIADD(NUU+1) = BESI(NUU+1, SAAD)
BKRRD(NU) = BRRD(NUU, SRRD)
BKADD(NUU-1) = BAAAD(NUUU-1, SAAD)
BKADD(NUU+1) = BAAAD(NUU+1, SAAD)

BKRWD(NU) = BRWD(NUU, SRWD)

GNRWD = BESKA(NUU, SRWD)

DLAN = DLOG10(BKRRD(NU)) - DLOG10(BKADD(NUU-1) + BKADD(NUU+1))
DLAN = DLAN + DLOG10(BIADD(NUU-1) + BIADD(NUU+1))
DLAN = DLAN + DLOG10(DSQRT(TTPI/(2*SRWD))*GNRWD) - SRWD*DEE
IF (DLAN .LT. -60.0D0) GO TO 499
SUMIN = -(10**DLAN)
GO TO 500

PARTIAL ASYMPTOTIC EXPANSION (RWD)

SMALL ARGUMENT APPROXIMATION (AAD)

IF (ICASE .EQ. 2) GO TO 356

GNRWD = BESKA(NUU, SRWD)
FIAADD = FISMA(NUU, SAAD)
FKAAD = FKSMA(NUU, SAAD)
BKRRD(NU) = BRRD(NUU, SRRD)

AR1 = DSQRT(TTPI/(2*SRWD))

DLAN = DLOG10(BKRRD(NU)*AR1) + DLOG10(2*NUU*GNRWD) - SRWD*DEE
DLAN = DLAN - 2*DLOG10(FAACNU(NU)) - 2*NUU*DLOG10(2.0D0)
DLAN = DLAN + 2*NUU*DLOG10(SAAD)
DLAN = DLAN + DLOG10(FIAADD/FKAAD)
IF (DLAN .LT. -60.0D0) GO TO 499
SUMIN = 10**DLAN
GO TO 500

CLOSED BOUNDARY HOLE

GNRWD = BESKA(NUU, SRWD)
FIAADD1 = FISMA(NUUU-1, SAAD)
FIAADD2 = FISMA(NUU+1, SAAD)
FKAAD1 = FKSMA(NUUU-1, SAAD)
FKAAD2 = FKSMA(NUU+1, SAAD)
BKRRD(NU) = BRRD(NUU, SRRD)

AR1 = DSQRT(TTPI/(2*SRWD))

DLAN = DLOG10(BKRRD(NU)*AR1) + DLOG10(2*NUU*GNRWD) - SRWD*DEE
DLAN = DLAN - 2*DLOG10(FAACNU(NU)) - 2*NUU*DLOG10(2.0D0)
DLAN = DLAN + 2*NUU*DLOG10(SAAD)
DLAN = DLAN + DLOG10(2*NUU*FIAADD1/SAAD) + SAAD*FIAADD2/(2*(NUU+1))
DLAN = DLAN - DLOG10((SAAD*FKAAD1/(2*(NUUU-1)) + 2*NUU*FKAAD2/SAAD)
IF (DLAN .LT. -60.0D0) GO TO 499
SUMIN = -(10**DLAN)
GO TO 500

PARTIAL ASYMPTOTIC EXPANSION (RWD) G
SMALL ARGUMENT APPROXIMATION (AAD, RRD)

IF (ICASE.EQ.2) GO TO 357
FKRRD = FKSM(AUU, SRRD)
FIAAD = FISMA(AUU, SAAD)
FKAAD = FKSM(AUU, SAAD)
GNRRD = BESK(AUU, SRRD)

AR1 = GNRRD*DSQRT(TTP/(2*SRWD))
DLAN = DLOG10(A1) + 2*NUU*DLOG10(SAAD) - NUU*DLOG10(SRRD)
DLAN = DLAN - SRWD*DEE - NUU*DLOG10(2.0D0) - DLOG10(FACNUM(U))

IF (DLAN.LT.-60.0D0) GO TO 499
SUMIN = 10**DLAN
GO TO 500

CLOSED BOUNDARY HOLE

GNRRD = BESK(AUU, SRWD)
FKRRD = FKSM(AUU, SRRD)
FIAAD1 = FISMA(AUU - 1, SAAD)
FIAAD2 = FISMA(AUU + 1, SAAD)
FKAAD1 = FKSM(AUU - 1, SAAD)
FKAAD2 = FKSM(AUU + 1, SAAD)

AR1 = GNRRD*DSQRT(TTP/(2*SRWD))
DLAN = DLOG10(AR1) + 2*NUU*DLOG10(SAAD) - NUU*DLOG10(SRRD)
DLAN = DLAN - SRWD*DEE - NUU*DLOG10(2.0D0) - DLOG10(FACNUM(U))

IF (DLAN.LT.-60.0D0) GO TO 499
SUMIN = -(10**DLAN)
GO TO 500

PARTIAL ASYMPTOTIC EXPANSION (RRD) I

IF (ICASE.EQ.2) GO TO 359
BKRWD = BWRD(AUU, SRWD)
GNRRA = BESK(AUU, SRRD)
BKAAD = BAAD(AUU, SAAD)
BIAAD = BESI(AUU, SAAD)

AR1 = GNRRD*DSQRT(TTP/(2*SRWD))
DLAN = DLOG10(BKRWD) + DLOG10(BIAAD) - DLOG10(BKAAD)
DLAN = DLAN + DLOG10(AR1) - SRWD*DEE

IF (DLAN.LT.-60.0D0) GO TO 499
SUMIN = 10**DLAN
GO TO 500

-197-
CLOSED BOUNDARY HOLE

359
BKRWD(NU)=BRWD(NUU,SRWD)
GNRRD=BESKA(NUU,SRRD)
BKAAD(NNU-1)=BAAD(NUUU-1,SAAD)
BKAAD(NU+1)=BAAD(NUU+1,SAAD)
BIAAD(NNU-1)=BESI(NUUU-1,SAAD)
BIAAD(NU+1)=BESI(NUU+1,SAAD)
AR1=GNRRD*DSQRT(TTP/(2*SRRD))
DLAN=DLOG10(BKRWD(NU))+DLOG10(BIAAD(NNU-1)+BIAAD(NU+1))
DLAN=DLAN-DLOG10(BKAAD(NNU-1)+BKAAD(NU+1))
DLAN=DLAN+DLOG10(AR1)-SRRD*DEE
IF (DLAN.LT.-60.0DO) GO TO 499
SUMIN=-(10**DLAN)
GO TO 500

PARTIAL ASYMPTOTIC EXPANSION (RRD)

276
IF (ICASE.EQ.2) GO TO 360
FIAAD=FISMA(NUU,SAAD)
FKAAD=FKSMA(NUU,SAAD)
BKRWD(NU)=BRWD(NUU,SRWD)
GNRRD=BESKA(NUU,SRRD)
AR1=BKRWD(NU)*DSQRT(TTP/(2*SRRD))
DLAN=DLOG10(AR1)+DLOG10(GNRRD*2*NUU)-SRRD*DEE
DLAN=DLAN-2*DLOG10(FACNU(NU))-2*NUU*DLOG10(2.0DO)
DLAN=DLAN+2*NUU*DLOG10(SAAD)
DLAN=DLAN+DLOG10(FIAAD/FKAAD)
IF (DLAN.LT.-60.0DO) GO TO 499
SUMIN=10**DLAN
GO TO 500

CLOSED BOUNDARY HOLE

360
BKRWD(NU)=BRWD(NUU,SRWD)
FIAAD1=FISMA(NUUU-1,SAAD)
FIAAD2=FISMA(NU+1,SAAD)
FKAAD1=FKSMA(NUUU-1,SAAD)
FKAAD2=FKSMA(NU+1,SAAD)
GNRRD=BESKA(NUU,SRRD)
AR1=BKRWD(NU)*DSQRT(TTP/(2*SRRD))
DLAN=DLOG10(AR1)+DLOG10(GNRRD*2*NUU)-SRRD*DEE
DLAN=DLAN-2*DLOG10(FACNU(NU))-2*NUU*DLOG10(2.0DO)
DLAN=DLAN+2*NUU*DLOG10(SAAD)
DLAN=DLAN+DLOG10((2*NUU*FIAAD1/SAAD)+SAAD*FIAAD2/(2*(NUU+1)))
DLAN=DLAN-DLOG10((SAAD*FKAAD1/(2*(NUU-1)))+2*NUU*FKAAD2/SAAD)
IF (DLAN.LT.-60.0DO) GO TO 499
SUMIN=-10**DLAN
GO TO 500

-198-
PARTIAL ASYMPTOTIC EXPANSION (RRD)

IF (ICASE.EQ.2) GO TO 361
FIAAD = FISMA(NUU, SAAD)
FKAAD = FKSMA(NUU, SAAD)
FKRWD = FKSMA(NUU, SRWD)
GNRRD = BESKA(NUU, SRRD)
ARK = GNRRD * DSQRT(TTPI/(2*SRRD))
DLAN = DLOG10(ARK) + 2*NUU*DLOG10(SAAD) - NUU*DLOG10(SRRD)
DLAN = DLAN - SRRD*DEE - NUU*DLOG10(2.0DO) - DLOG10(FACNU(NU))
DLAN = DLAN + DLOG10(FKRWD) + DLOG10(FIAAD/FKAAD)
SUMIN = 10**DLAN
GO TO 500

CLOSED BOUNDARY HOLE

GNRRD = BESKA(NUU, SRRD)
FIAAD1 = FISMA(NUUU-1, SAAD)
FIAAD2 = FISMA(NUU+1, SAAD)
FKAAD1 = FKSMA(NUUU-1, SAAD)
FKAAD2 = FKSMA(NUU+1, SAAD)
FKRWD = FKSMA(NUU, SRWD)
ARK = GNRRD * DSQRT(TTPI/(2*SRRD))
DLAN = DLOG10(ARK) + 2*NUU*DLOG10(SAAD) - NUU*DLOG10(SRRD)
DLAN = DLAN - SRRD*DEE - NUU*DLOG10(2.0DO) - DLOG10(FACNU(NU))
DLAN = DLAN + DLOG10(FKRWD) + DLOG10(2*NUU*FIAAD1/SAAD) + SAAD*FIAAD2/(2*(NUU+1))
DLAN = DLAN - 2*DLOG10(FIAAD/FKAAD)
SUMIN = -(10**DLAN)
GO TO 500

CLOSED BOUNDARY HOLE

IF (ICASE.EQ.2) GO TO 362
FIAAD = FISMA(NUU, SAAD)
FKAAD = FKSMA(NUU, SAAD)
BKRRD(NU) = BRWD(NUU, SRWD)
BKRRD(NU) = BRWD(NUU, SRWD)
DLAN = DLOG10(BKRRD(NU)) + DLOG10(BKRRD(NU)) + 2*NUU*DLOG10(SAAD)
DLAN = DLAN + DLOG10(2.0DO*NUU) - 2*NUU*DLOG10(2.0DO)
DLAN = DLAN - 2*DLOG10(FACNU(NU))
DLAN = DLAN + DLOG10(FIAAD/FKAAD)
SUMIN = 10**DLAN
GO TO 500

CLOSED BOUNDARY HOLE

BKRRD(NU) = BRWD(NUU, SRWD)
FIAAD1 = FISMA(NUUU-1, SAAD)
FIAAD2 = FISMA(NUU+1, SAAD)
FKAAD1 = FKSMA(NUUU-1, SAAD)
FKAAD2 = FKSMA(NUU+1, SAAD)

-199-
BKRRD(NU) = BRRD(NUU, SRRD)
DLAN = DLOG10(BKRRD(NU)) + DLOG10(BKRRD(NU)) + 2*NUU*DLOG10(SAAD)
DLAN = DLAN + DLOG10(2.0D0*NUU) - 2*NUU*DLOG10(2.0D0)
DLAN = DLAN - 2*DLOG10(FACNU(NU))
DLAN = DLAN + DLOG10((2*NUU*FKAAD1/SAAD) + SAAD*FIAAD2/(2*(NUU+1)))
DLAN = DLAN - DLOG10((SAAD*FKAAD1/(2*(NUU-1))) + 2*NUU*FKAAD2/SAAD)
SUMIN = -(10**DLAN)
GO TO 500

SHALL ARGUMENT APPROXIMATION (AAD, RRD) M
299 IF (ICASE.EQ.2) GO TO 363
FIAAD = FISMA(NUU, SAAD)
FKAAD = FKSMA(NUU, SAAD)
FKRRD = FKSMA(NUU, SRRD)
BKRRD(NU) = BRRD(NUU, SRWD)
DLAN = DLOG10(BKRRD(NU)) + 2*NUU*DLOG10(SAAD) - NUU*DLOG10(SRRD)
DLAN = DLAN - NUU*DLOG10(2.0D0) - DLOG10(FACNU(NU))
DLAN = DLAN + DLOG10(FKRRD)
DLAN = DLAN + DLOG10((2*NUU*FIAAD1/SAAD) + SAAD*FIAAD2/(2*(NUU+1)))
DLAN = DLAN - DLOG10((SAAD*FKAAD1/(2*(NUU-1))) + 2*NUU*FKAAD2/SAAD)
SUMIN = -(10**DLAN)
GO TO 500

CLOSED BOUNDARY HOLE
363 BKRRD(NU) = BRRD(NUU, SRWD)
FIAAD1 = FISMA(NUU-1, SAAD)
FIAAD2 = FISMA(NUU+1, SAAD)
FKAAD1 = FKSMA(NUU-1, SAAD)
FKAAD2 = FKSMA(NUU+1, SAAD)
FKRRD = FKSMA(NUU, SRRD)
DLAN = DLOG10(BKRRD(NU)) + 2*NUU*DLOG10(SAAD) - NUU*DLOG10(SRRD)
DLAN = DLAN - NUU*DLOG10(2.0D0) - DLOG10(FACNU(NU))
DLAN = DLAN + DLOG10(FKRRD)
DLAN = DLAN + DLOG10((2*NUU*FIAAD1/SAAD) + SAAD*FIAAD2/(2*(NUU+1)))
DLAN = DLAN - DLOG10((SAAD*FKAAD1/(2*(NUU-1))) + 2*NUU*FKAAD2/SAAD)
SUMIN = -(10**DLAN)
GO TO 500

SMALL ARGUMENT APPROXIMATION (RRD) N
300 IF (ICASE.EQ.2) GO TO 364
FKRRD = FKSMA(NUU, SRRD)
BKRRD(NU) = BRRD(NUU, SRWD)
BKAAD(NU) = BAAAD(NUU, SAAD)
BIAAD(NU) = BESI(NUU, SAAD)
DLAN = DLOG10(BKRRD(NU)) + DLOG10(BIAAD(NU)) - DLOG10(BKAAD(NU))
DLAN = DLAN + NUU*DLOG10(2.0D0) + DLOG0(FACNU(NU)) - NUU*DLOG10(SRRD)
DLAN = DLAN - DLOG10(2.0D0*NUU) + DLOG0(FKRRD)
SUMIN = 10**DLAN
GO TO 500

-200-
CLOSED BOUNDARY HOLE

BKRWD(NU) = BRWD(NUU, SRWD)
FKRWD = FKSMA(NUU, SRRD)
BKAAD(NNU-1) = BAAAD(NUUU-1, SAAD)
BKAAD(NU+1) = BAAAD(NUU+1, SAAD)
BIAAD(NNU-1) = BESI(NUUU-1, SAAD)
BIAAD(NU+1) = BESI(NUU+1, SAAD)
DLAN = DLOG10(BKRWD(NU)) + DLOG10(BIAAD(NNU-1) + BIAAD(NU+1))
DLAN = DLAN - DLOG10(BKAAD(NNU-1) + BKAAD(NU+1))
DLAN = DLAN + NUU*DLOG10(2.0DO) + DLOG10(FACHU(NU)) - NUU*DLOG10(SRRD)
DLAN = DLAN - DLOG10(2.0DO*NUU) + DLOG10(FKRWD)
SUMIN = -(10**DLAN)
GO TO 500

SMALL ARGUMENT APPROXIMATION (RWD) P

IF (ICASE.EQ.2) GO TO 365
FKRWD = FKSMA(NUU, SRWD)
BKRWD(NU) = BRRD(NUU, SRRD)
BKAAD(NU) = BAAAD(NUU, SAAD)
BIAAD(NU) = BESI(NUU, SAAD)
DLAN = DLOG10(BKRWD(NU)) + DLOG10(BIAAD(NUU) - DLOG10(BKAAD(NU))
DLAN = DLAN + NUU*DLOG10(2.0DO) + DLOG10(FACHU(NU)) - NUU*DLOG10(SRRD)
DLAN = DLAN - DLOG10(2.0DO*NUU) + DLOG10(FKRWD)
SUMIN = 10**DLAN
GO TO 500

CLOSED BOUNDARY HOLE

BKRWD(NU) = BRRD(NUU, SRRD)
FKRWD = FKSMA(NUU, SRWD)
BKAAD(NNU-1) = BAAAD(NUUU-1, SAAD)
BKAAD(NU+1) = BAAAD(NUU+1, SAAD)
BIAAD(NNU-1) = BESI(NUUU-1, SAAD)
BIAAD(NU+1) = BESI(NUU+1, SAAD)
DLAN = DLOG10(BKRWD(NU)) + DLOG10(BIAAD(NNU-1) + BIAAD(NU+1))
DLAN = DLAN - DLOG10(BKAAD(NNU-1) + BKAAD(NU+1))
DLAN = DLAN + NUU*DLOG10(2.0DO) + DLOG10(FACHU(NU)) - NUU*DLOG10(SRRD)
DLAN = DLAN - DLOG10(2.0DO*NUU) + DLOG10(FKRWD)
SUMIN = -(10**DLAN)
GO TO 500

SMALL ARGUMENT APPROXIMATION (AAD, RWD) O

IF (ICASE.EQ.2) GO TO 366
FIAAD = FISMA(NUU, SAAD)
FKAAD = FKSMA(NUU, SAAD)
FKRWD = FKSMA(NUU, SRWD)
BKRWD(NU) = BRRD(NUU, SRRD)
DLAN = DLOG10(BKRWD(NU)) + 2*NUU*DLOG10(SAAD) - NUU*DLOG10(SRRD)
DLAN = DLAN - NUU*DLOG10(2.0DO) - DLOG10(FACHU(NU))
DLAN = DLAN + DLOG10(FKRWDD/FIAAD/FKAAAD)
SUMIN = 10**DLAN
GO TO 500

CLOSED BOUNDARY HOLE
BKRRD(NU) = BRRD(NUU,SRRD)
FIAAD1 = FISMA(NUUU-1,SAAD)
FIAAD2 = FISMA(NUU+1,SAAD)
FKAAAD1 = FKSMA(NUUU-1,SAAD)
FKAAAD2 = FKSMA(NUU+1,SAAD)
FKRWDD = FKSMA(NUU,SRRD)
FKRRD = FKSMA(NUU,SRRD)
DLAN = DLOG10(BKRD(NU)) + 2*NUU*DLOG10(SAAD) - NUU*DLOG10(SRWDD) - DLOG10(FACNU(NU))
DLAN = DLAN + DLOG10(2*NUU*FIAAD1/SAAD) + SAAD*FIAAD2/(2*(NUU+1))
DLAN = DLAN - DLOG10(SAAD*FKAAAD1/(2*(NUUU-1)) + 2*NUU*FKAAAD2/SAAD)
SUMIN = -(10**DLAN)
GO TO 500

SMALL ARGUMENT APPROXIMATION (AAD, RRD, RWD) Q

IF (ICASE.EQ.2) GO TO 367
FIAAD = FISMA(NUU,SAAD)
FKAAAD = FKSMA(NUU,SAAD)
FKRWDD = FKSMA(NUU,SRWD)
FKRRD = FKSMA(NUU,SRRD)
DLAN = 2*NUU*DLOG10(SAAD) - NUU*DLOG10(SRWDD*SRRD) - DLOG10(2.0D0*NUU)
DLAN = DLAN + DLOG10(FKRWDD*FKRRD*FIAAD/FKAAAD)
SUMIN = 10**DLAN
GO TO 500

CLOSED BOUNDARY HOLE
FIAAD1 = FISMA(NUUU-1,SAAD)
FIAAD2 = FISMA(NUU+1,SAAD)
FKAAAD1 = FKSMA(NUUU-1,SAAD)
FKAAAD2 = FKSMA(NUU+1,SAAD)
FKRWDD = FKSMA(NUU,SRWD)
FKRRD = FKSMA(NUU,SRRD)
DLAN = 2*NUU*DLOG10(SAAD) - NUU*DLOG10(SRWDD*SRRD) - DLOG10(2.0D0*NUU)
DLAN = DLAN + DLOG10(FKRWDD*FKRRD)
DLAN = DLAN + DLOG10(2*NUU*FIAAD1/SAAD) + SAAD*FIAAD2/(2*(NUU+1))
DLAN = DLAN - DLOG10(SAAD*FKAAAD1/(2*(NUUU-1)) + 2*NUU*FKAAAD2/SAAD)
SUMIN = -(10**DLAN)
GO TO 500

SIMPLE PROGRAM Z
IF (ICASE.EQ.2) GO TO 368
BKRWDD(NU) = BRWD(NUU,SRWD)
BKRRD(NU) = BRRD(NUU,SRRD)
FUNCTION BESI (NU,ARG)

**********
THIS FUNCTION EVALUATES THE MODIFIED BESSEL
FUNCTION I(NU,ARG).
THE ORDER IS LIMITED TO NU<51

FUNCTION BESI(NU,ARG)
IMPLICIT REAL*8 (A-H,O-Z)
REAL*8 A,AA,AAA,FAZ,B,BB,DDELTA,EPSI,AEXP,C,CC
COMMON /FRAC3/FACNU(51)
COMMON /FRAC99/IEND99
WRITE (6,17) NU,ARG
C17 FORMAT (15X,15,5X,E12.5)

CONVERGENCE CRITERION (FRACTION)

EPSI=1.0D-15

THE ASYMPTOTIC EXPANSION OPTION IS BYPASSED
IN MOST CALCULATIONS.
GO TO 5
NOW THE METHOD OF EVALUATING $I$ IS CHOSEN
FOR $\arg \geq 20$ ASYMPTOTIC EXPANSION
FOR $\arg < 20$ ASCENDING SERIES

IF $(\arg \geq 20.0)$ GO TO 4
IF $(\arg < 20.0)$ GO TO 5

ASYMPTOTIC EXPANSION

\[ A = (\nu^{**2}) * 4 \]
\[ \text{AA} = \arg \times 8 \]
\[ \text{AAA} = 2 \times \arg \times 3.141592654 \]
\[ \text{IFLAG3} = 0 \]
\[ \text{FNZ} = 1.0D0 \]
\[ \text{BB} = 1.0D0 \]
\[ \text{DDD} = 1.0D0 \]
DO 6 I = 1, 200
D = $(A - (2 * I - 1)**2) / (\text{AA} * I)$
DD = $\text{DABS}(D)$
IF (IFLAG3 .EQ. 1) GO TO 14
IF (DD .LT. DDD) IFLAG3 = 1
GO TO 15

II = I / 2 * 2
IF (DD .GT. DDD .AND. II .EQ. I) GO TO 7

DDD = DD
BB = BB * (-1) * D
FNZ = FNZ + BB

WRITE (6, 16) FNZ
FORMAT (10X, E12.5)
DELTA = BB / FNZ
DELTA = $\text{DABS}(\text{DELTA})$
IF (EPSI - DELTA) 6, 7, 7
CONTINUE

CHECK FOR EXPONENT MAG

E = DEXP(1.0D0)
AEXP = $\arg \times DLOG10(E) - DLOG10(\text{AAA}) + DLOG10(\text{FNZ})$
IF (DABS(AEXP) .GT. 70.0) GO TO 8
BESI = DEXP($\arg \times \text{FNZ} / \text{AAA}$
GO TO 9

EXPONENT MAG OVERFLOW

WRITE (6, 101) AEXP
FORMAT (5X, 'EXPONENT OF I', D20.9, I5, D20.9)
BESI = 10**AEXP
GO TO 9

-204-
C

ASCENDING SERIES
****************
FIRST CHECK FOR EXPONENT MAG
C=NU*DLOG10(ARG/2)-DLOG10(FACNU(NU+1))
CC=DABS(C)
IF (CC.GT.70.0) GO TO 10
B=1.0D0
BB=B
AA=(ARG/2)**2
AAA=1/FACNU(NU+1)
DO 11 I=1,200
BB=BB*AA/(I*(I+NU))
E=B+BB
DELTA=BB/B
IF (EPSI-DELTA) 11,12,12
CONTINUE

SECOND CHECK FOR EXPONENT HAG
B=B*AAA
CCC=NU*(DLOG10(ARG/2.))
IF (CCC.GT.69.) GO TO 27
C=DLOG10(B)+CCC
CC=DABS(C)
IF (CC.GT.69.) GO TO 10
BESI=B*((ARG/2.)**NU)
GO TO 9

EXPONENT HAG OVERFLOW
WRITE (6,101) CC,NU,ARG
IEND99=1
BESI=1.0D-70

RETURN
END

FUNCTION BAAD (N,Z)
FUNCTION \texttt{BAAD}(N,Z)
\begin{verbatim}
IMPLICIT REAL*8 (A-H, O-Z)
COMMON /FRAC11/BIAAD(100), BKAAD(100), BKRWD(100), BKRRD(100)
COMMON /FRAC12/SUM(50,16)
COMMON /FRAC5/IOPT, IER
DOUBLE PRECISION MMBSKO, MMBSK1
IF (N.EQ.0) GO TO 1
IF (N.EQ.1) GO TO 2
EKAAD=DLOG10(BKAAD(N))
EKAAD=DABS(EKAAD)
IF (EKAAD.GE.70) IEND99=1
BAAD=BKAAD(N-1)+(BKAAD(N)/Z)*2*(N-1)
GO TO 3
1 BAAD=MMBSKO(IOPT,Z,IER)
GO TO 3
2 BAAD=MMBSK1(IOPT,Z,IER)
3 RETURN
END
\end{verbatim}

FUNCTION \texttt{BRWD}(N,Z)
\begin{verbatim}
IMPLICIT REAL*8 (A-H, O-Z)
COMMON /FRAC11/BIAAD(100), BKAAD(100), BKRWD(100), BKRRD(100)
COMMON /FRAC12/SUM(50,16)
COMMON /FRAC5/IOPT, IER
COMMON /FRAC99/IEND99
DOUBLE PRECISION MMBSKO, MMBSK1
IF (N.EQ.0) GO TO 1
IF (N.EQ.1) GO TO 2
EKRWD=DLOG10(BKRWD(N))
EKRWD=DABS(EKRWD)
IF (EKRWD.GE.70) IEND99=1
BRWD=BKRWD(N-1)+(BKRWD(N)/Z)*2*(N-1)
GO TO 3
1 BRWD=MMBSKO(IOPT,Z,IER)
GO TO 3
2 BRWD=MMBSK1(IOPT,Z,IER)
3 RETURN
END
\end{verbatim}
FUNCTION BRRD(N,Z)
**************
THIS FUNCTION EVALUATES K(N,Z) USING THE "STEP-UP
METHOD". WE USE K(0,Z) AND K(1,Z) TO EVALUATE THE
HIGHER ORDERS.

FUNCTION BRRD(N,Z)
IMPLICIT REAL*8 (A-H,O-Z)
COMMON /FRAC11/BIAAD(100),BKAAD(100),BKRW(100),BKRRD(100)
COMMON /FRAC12/SUM(50,16)
COMMON /FRAC5/IOPT,IER
COMMON /FRAC99/IEND99
DOUBLE PRECISION MMBSK0,MMBSK1
IF (N.EQ.0) GO TO 1
IF (N.EQ.1) GO TO 2
EKRRD=DLOG10(BKRRD(N))
EKRRD=DABS(EKRRD)
IF (EKRRD.EQ.70) IEND99=1
BRRD=BB(RD(N-1)+BB(RD(N)/Z)*2*(N-1)
GO TO 3
1 BRRD=MMBSK0(IOPT,Z,IER)
GO TO 3
2 BRRD=MMBSK1(IOPT,Z,IER)
3 RETURN
END

FUNCTION FISMA(NU,ARG)
THIS FUNCTION EVALUATES THE NON-EXPONENTIAL
TERMS IN THE ASCENDING SERIES FOR I(NU,ARG).
NU FACTORIAL IS FACTORED OUT WITH THE EXPONENTIAL
IMPLICIT REAL*8 (A-H,O-Z)
EPSI1=1.0D-15
A=(ARG**2)/4.0
B=1.0D0
BB=1.0D0
DO 1 I=1,200
BB=BB*A/(I*(NU+I))
B=B+BB
DELTA=BB/B
IF (DELTA.LT.EPSI1) GO TO 2
1 CONTINUE
2 FISMA=B
RETURN
END

FUNCTION FKSMA(NU,ARG)
THIS FUNCTION EVALUATES THE NON-EXPONENTIAL
TERMS IN THE ASCENDING SERIES FOR K(NU,ARG).

-207-
C  NU FACTORIAL IS FACTORED OUT WITH THE EXPONENTIAL

IMPLICIT REAL*8 (A-H,O-Z)

EPSI1=1.0D-15

B=1.0D0

BB=1.0D0

A=(ARG**2)/4.0

KK=NU-1

DO 1 I=1, KK

BB=BB*A/(I*(NU-I))

B=B+BB

DELTA=BB/B

IF (DELTA.LT.EPSI1) GO TO 2

1 CONTINUE

2 FKSMA=B

RETURN

END
APPENDIX H : TABLES FOR PRESSURE TIME TYPE CURVES

In this appendix, we use the following notations:

\[ a_D = \text{AAD} \]
\[ r_D = \text{RWD} \]
\[ r_D = \text{RRD} \]
\[ \theta = \text{THETA} \]
\[ \dot{p}_{DSS} = \text{PDSS} \]
\[ t_D = \text{TD} \]
\[ t_D / r_D^2 = \text{TD:RD}^2 \]
\[ p_D = \text{PD} \]
\[ p_{DN} = \text{PDN} = \text{normalized pressure drop} \]
\[ r_1 = \text{R1} \]
\[ r_2 = \text{R2} \]
### TABLE H1:

**CONSTANT PRESSURE INTERNAL BOUNDARY**

\[
\begin{align*}
A&D &= 0.200000D+01 \\
R&D &= 0.200000D+02 \\
R&D &= 0.196400D+02 \\
P &= 0.100000D+00 \\
E &= 0.962000D+00
\end{align*}
\]

**THE STEADY STATE PRESSURES**

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<th>(\text{PD}=)</th>
<th>(\text{PD}N=)</th>
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**TD**

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<th>(TD\cdot RD**2)</th>
<th>(PD)</th>
<th>(PD\cdot N)</th>
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- **E=** 0.996000 D+00
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RRD = 0.19960D 02 \\
F = 0.99000D 00 \\
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\text{THETA= 0.90000D 02 } \quad \text{PDSS= 0.108117383D-01} \\
\text{THETA= 0.13500D 03 } \quad \text{PDSS= 0.636170248D-02} \\
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E = 0.99400D+00 \\
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### TABLE H.5

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