TEMPERATURE TRANSFER IN A CONVECTION–DOMINANT, NATURALLY FRACTURED GEOTHERMAL RESERVOIR UNDERGOING FLUID INJECTION

By

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This study considers the heat and fluid flow characteristics of an infinite, naturally fractured geothermal reservoir in which forced convection is the only form of heat transfer. For simplicity, it is assumed that there is only one injector well and no producer wells in the system. Further, primary porosity is neglected and the fracture porosity is assumed to be constant throughout the reservoir.

With these specifications, the governing equations are derived from an energy balance, and solved using dimensionless parameters and the Laplace transform. Both numerical inversion and analytical inversion are then used, though only the latter appears to give a reliable solution. The results are plotted as dimensionless temperature versus dimensionless volume swept (called dimensionless radius), and the velocity of the thermal front in the rock and water determined.
1. INTRODUCTION

Owing to the analogous nature of many petroleum and geothermal reservoir phenomena, studies of naturally fractured petroleum reservoirs are often of great value in the study of geothermal systems. This effort draws upon one such study by Warren and Root (1962), and a later paper by Mavor (1978), which present analytical flow models for the analysis of pressure-production response. A somewhat similar flow problem for geothermal reservoirs is presented in this report, in order to examine the heat transfer during cold water injection.

The focus of this study is to determine the location of the thermal front in the reservoir, as well as examine its form and behaviour. These results should then yield the velocity of the thermal front, in both the rock and water, relative to that of the injected fluid. While their validity is limited by certain simplifying assumptions, the results should also represent a useful step toward solving related problems of greater complexity.
2. ASSUMPTIONS

1. Single well in an infinite reservoir, radial flow.

This paper considers only the case of one injector well in an infinite system. Superposition is therefore not used.

2. Steady state flow of injected fluid.

The solution to this problem will not be valid for short times during which transient fluid flow occurs.

3. Only convection heat transfer, conduction not considered.

Conduction of heat in the reservoir rock is assumed to be small compared to forced convection. This assumption should be good for all but very long time.

4. Initial reservoir temperature the same throughout system.

5. Only secondary or fracture porosity

The effects of primary porosity are not considered in this study. It is assumed that this porosity is small enough to be neglected or that the model must be adjusted to account for its effect on the transfer of heat and fluid flow in the reservoir.
6. Reservoir uniformly fractured and fracture width constant. This problem does not consider significant variation in fracture width anywhere in the reservoir. Vertically, fracture length is limited by the height of the producing interval. Orientation of the fracture need not be specified as it is implicit in the effective porosity.

7. Effective porosity constant throughout reservoir. In order that convection heat transfer occur, the reservoir rock must be swept by the injected fluid. The effective porosity constraint requires that each fracture be oriented such that it conducts the injected fluid radially. Concentric and other fractures orthogonal to the direction of flow are not included in this porosity. The fracture porosity used for this model therefore is less than the total fracture porosity of the system. Figure 1 presents an element volume of such a reservoir, in which neither the degree of fracturing nor the fracture width changes.

All of the above constraints are introduced so as to keep the reservoir flow model simple but at the same time still realistic.
\[ \delta = \text{FRACTURE WIDTH} \]

\[ T_w - \frac{\partial T_w}{\partial r} \Delta r \]

FIG. 1
3. ENERGY BALANCE

An element volume of the naturally fractured reservoir is shown as Fig. 1. An energy balance on this element yields the following two equations:

\[(1) \quad T_w \Delta r - T_w = h_c A_w (T_w - T_R) = \frac{\partial T_w}{\partial t}\]

\[(2) \quad h_c A_w (T_w - T_R) = \frac{\partial T_R}{\partial t}\]

(The symbols used are defined in APPENDIX 1: Nomenclature and Typical Values)

Note that while the rock and water volumes of the element are different, they share the same surface area and have identical convection terms. Their combined volumes give the total element volume. In addition, the water volume and surface area is equal to the fracture volume and surface area of the element. Finally, note that the storage term in (1) is negative.

Temperature in the reservoir is a function of the distance from the injection well and the time since the start of injection. Given any particular radius and time, the solution of these two equations will give both the rock and water temperature.
Rewriting the above equations with their appropriate initial conditions and boundary conditions,

\[
\begin{align*}
-\rho q \frac{\partial T_w}{\partial r} + h A_e (T_w - T_R) &= \rho C_v \frac{\partial T_w}{\partial t} \\
h A_e (T_w - T_R) &= \rho C_v \frac{\partial T_R}{\partial t}
\end{align*}
\]  

(1)

(2)

Initial Conditions: \( T_w(r,0) = T_R(r,0) = T_{\text{inj}} \)

(3a)

Boundary Condition: \( T_w(r_{\text{w},t}) = T_{\text{inj}} \)

(3b)

The total volume of the element is given by:

\[
V_e = 2 \pi r dr dl
\]

(4)

While for assumptions 5, 6, and 7 of section 2 to hold, the water and rock element volumes also must be given by:

\[
V_w = \phi_2 V_e \\
V_R = (1 - \phi_2) V_e
\]

(4c)

(4b)
Alternatively, the total volume can be expressed as:

\[ V_e = V_w + V_R \]  

(5)

And the rock surface area for heat exchange can be written as:

\[ A_{RW} = \frac{2V_w}{\delta} \]  

(6)

Where:

\[ \phi_2 = \text{fracture porosity} \]

\[ \delta = \text{fracture width} \]

Using equation (5), and substituting (4), and (6) into equations (1) and (2) yields:

\[ -\frac{B}{r} \frac{\partial T_w}{\partial r} - D(T_w - T_R) = M_w \frac{\partial T_w}{\partial t} \]  

(7)

\[ D(T_w - T_R) = M_R \frac{\partial T_R}{\partial t} \]  

(8)
Where the new constants are defined as:

\[ B = \frac{\rho \sigma_w C_{vw}}{2\pi \delta} \]

\[ D = \frac{2h C_2^2}{\delta} \]  \hspace{1cm} (8b)

\[ M_w = \rho_w C_w \phi_2 \]

\[ M_R = \rho_R C_r (1 - \phi_2) \]  \hspace{1cm} (8c)
Before solving equations (7) and (8) for $T_w(r,t)$ and $T_r(r,t)$, it is convenient to re-state these equations using dimensionless parameters:

Dimensionless Time:

$$ t_D = \frac{D t}{M_R} \quad \text{or} \quad t_D = \frac{2 h \phi \cdot t_c}{\rho_c C_v R (1 - \phi)} $$

Dimensionless radius:

$$ r_D = \frac{D(r^2 - r_w^2)}{2B} \quad \text{or} \quad r_D = \frac{2\pi f(r^2 - r_w^2) h \phi}{\rho_w C_v w q_w} $$

Dimensionless temperature:

$$ \frac{T_w}{\Delta T_w} = \frac{T_w - T_{wi}}{T_{inj} - T_{wi}} \quad \text{and} \quad \frac{T_r}{\Delta T_r} = \frac{T_r - T_{wi}}{T_{inj} - T_{wi}} $$
Note that the dimensionless radius term is actually dimensionless volume and must not be confused with the \( r/r_w \) used in pressure-production problems.

Re-stated in dimensionless form, equations (7) and (8) become:

\[
\begin{align*}
- \frac{3T_w}{\partial r_D} - (c_w - T_R) &= \frac{M_w}{M_R} \frac{3T_w}{\partial t_D} \\
(T_w - T_R) &= \frac{3T_R}{\partial t_D}
\end{align*}
\]

(10)

Initial Condition: \( T_w(r_D, 0) = T_R(r_D, 0) = 0 \) \hspace{1cm} (12a)

Boundary Condition: \( T_w(0, t_D) = 1 \) \hspace{1cm} (12b)
5. SOLUTION BY THE LAPLACE TRANSFORM

Using the Laplace transform, equations (10) and (11) are transformed in time and solved simultaneously in Laplace space to produce an ordinary differential equation in terms of \( \overline{T}_w(r_D, S) \), where \( S \) is the Laplace transformed variable. This equation is solved using the boundary condition and \( \overline{T}_r(r_D, S) \) is then easily determined. The details of this procedure are given in Appendix B.

The solutions are found to be:

\[
\overline{T}_w(r_D, S) = \frac{1}{S} \exp \left\{ -r_D \left( 1 + \frac{M_w}{M_R} S - \frac{1}{S+1} \right) \right\} \quad (13a)
\]

\[
\overline{T}_r(r_D, S) = \frac{1}{S(S+1)} \exp \left\{ -r_D \left( 1 + \frac{M_w}{M_R} S - \frac{1}{S+1} \right) \right\} \quad (13b)
\]

The asymptotic forms of equations (13a) and (13b) are given in Appendix 3.
It is now possible to invert both equations (13a) and (13b) and arrive at an analytical solution to the problem. While it was necessary to first determine the water temperature solution (13b) using the boundary condition, the order of inversion makes no difference.

Inversion of Equation for Rock Temperature, $T_R(r_D, S)$:

Re-writing equation (13b) in a more convenient order for inverting,

$$
\overline{T}_R(r_D, S) = \exp(-r_D) \exp\left(-r_D \frac{M_w}{M_R} S \frac{1}{S+1} \exp\left(\frac{r_D}{S+1}\right) \right)
$$

(14)

Step 6

Applying the shifting theorem,

$$
T_R(r_D, t_D^*) = u(t_D^*) \exp(-r_D) \mathcal{L}^{-1}\left\{ \frac{1}{S} \frac{1}{S+1} \exp\left(\frac{r_D}{S+1}\right) \right\}
$$

(15)

where,

$$
t_D^* = t_D - \left( t_D^* - \frac{M_w}{M_R} r_D \right)
$$

(15a)
and $u(t_D)$ is the step function:

\[
\begin{align*}
\left\{ \begin{array}{ll}
  u(t_D) = 0 & \text{for } t_D < 0 \\
  u(t_D) = 1 & \text{for } t_D > 0
\end{array} \right.
\end{align*}
\]  

(15b)

Application of the convolution theorem gives,

\[
T_R(t) = u(t) \exp(-tD) f_1(t_D) * f_2(t_D)
\]  

(16)

Where,

\[
\begin{align*}
  f_1(t_D) * f_2(t_D) &= t_D \\
  &\quad f_1(t_D) f_2(t_D) e^{-\tau} \quad \text{for } 0 \leq \tau
\end{align*}
\]  

(16a)

**Step #2**

Equation (16) is reduce further using the following relations:

\[
\begin{align*}
  F_1(S) &= \frac{1}{S} \\
  F_2(S+1) &= \frac{1}{S+1} e^{\frac{t_D}{S+1}}
\end{align*}
\]  

(17)

(17.1)
Inverting,

$$\mathcal{F}^{-1}\{ F(S) \} = \mathcal{F}^{-1}\{ 1 \} = 1 \quad (17.2)$$

And,

$$\mathcal{F}^{-1}\{ F(S + 1) \} = \exp(-t^*_D) \mathcal{F}^{-1}\{ F(S) \} \quad (17.3)$$

Where,

$$\mathcal{F}^{-1}\{ F(S) \} = \mathcal{F}^{-1}\{ \exp \left( \frac{r_D}{s} \right) \} \quad (17.4)$$

From Abramowitz and Stegun, the inverse of equation (17.4) is found to be:

$$\mathcal{F}^{-1}\{ \frac{1}{s} \exp \left( \frac{A}{s} \right) \} = I_0 \left( 2 \sqrt{A t} \right) \quad (17.5)$$

and if,

$$V = (t_D - \tau), \quad A = r_D \quad (17.6)$$

then the rock temperature solution becomes:

$$T_R(r_D, t_D) = u(t^*_D) \left[ \int_{-r_D}^{t_D} \exp(-V) I_0 \left( 2 \sqrt{r_D V} \right) dV \right]$$

$$-16-$$
Although the integral in equation (18) is not easily found in many tables of integrals, an excellent discussion of it appears in Luke, 'Integrals of Bessel Functions' \(^5\). This book also contains some valuable references for further investigation. After some rearranging, the integral is:

\[
\int_0^X \text{Exp}(-v) \, I_0(2 \sqrt{y \, v}) \, dv = \text{Exp}(y) \left\{ 1 - J(x, y) \right\}
\]  \hspace{1cm} (19)

where \(J(x, y)\), a function defined in terms of the above integral, is expanded below.

If we let,

\[
y = r_D, \quad x = t_D^*\]

the rock and water temperature (from appendix 4) then become:

\[
T_R^*(r_D, t_D) = u(t_D) \left\{ 1 - J(t_D^*, r_D) \right\}
\]  \hspace{1cm} (19.2)

\[
T_w^*(r_D, t_D) = u(t_D) \, \text{Exp} \left\{ - \left( t_D + t_D^* \right) \right\} \, I_0\left( 2 \sqrt{r_D \, t_D^*} \right) + T_R(r_D, t_D^*)
\]  \hspace{1cm} (19.3), (A4.4)
This analytical solution is checked for the initial and boundary values, as well as the asymptotic forms in Appendix 3.

The expansion of the function $J(x,y)$ has two different forms:

Case #1

$$\sqrt{\frac{y}{x}} < 1 :$$

$$J(x,y) = \text{Exp} \left\{ -(x + y) \right\} \sum_{k=0}^{\infty} \sqrt{\frac{y}{x}} I_k(2\sqrt{xy})$$  \hspace{1cm} (20)

Case #2

$$\sqrt{\frac{y}{x}} > 1 :$$

$$J(x,y) = 1 - \text{Exp} \left\{ -(x + y) \right\} \sum_{k=1}^{\infty} \sqrt{\frac{y}{x}} I_k(2\sqrt{xy})$$  \hspace{1cm} (21)

At this point in the derivation, the reader is referred to Appendix 4, where both the rock and water temperature expansions are derived in greater detail.
From Appendix 4, the analytical results are:

Case #1

\[ \frac{r_D}{t_D^*} < 1 \]

**Rock Temperature**:

\[
T_R(r_n, t_n^*) = u(t_D^*) \left( 1 + \exp \left\{ -\left( r_D + t_D^* \right) \right\} \sum_{k=0}^{\infty} \frac{r_D^k}{t_D^*} I_k \left( 2 \sqrt{r_D \cdot t_D^*} \right) \right) 
\]

(22)

**Water Temperature**:

\[
T_W(r_D, t_D^*) = u(t_D^*) \left( 1 + \exp \left\{ -\left( r_D + t_D^* \right) \right\} \sum_{k=1}^{\infty} \frac{r_D^k}{t_D^*} I_k \left( 2 \sqrt{r_D \cdot t_D^*} \right) \right) 
\]

(23)

Case #2

\[ \frac{r_D}{t_D^*} > 1 \]
Rock Temperature:

\[ T_R(r_D, t_D^*) = u(t_D^*) \exp \left\{ - \left( r_D + t_D^* \right) \right\} \sum_{k=1}^{\infty} \frac{r_D}{t_D^*} I_k \left( 2 \sqrt{r_D t_D^*} \right) \]

(24)

Water Temperature:

\[ T_w(r_D, t_D^*) = u(t_D^*) \exp \left\{ - \left( r_D + t_D^* \right) \right\} \sum_{k=0}^{\infty} \frac{r_D}{t_D^*} I_k \left( 2 \sqrt{r_D t_D^*} \right) \]

(25)

In both of the above cases, the convection term has the form:

\[ T_w(r_D, t_D^*) - T_R(r_D, t_D^*) = u(t_D - \frac{M_w}{M_R} r_D) \exp \left\{ - \left( r_D + t_D^* \right) \right\} I_0 \left( 2 \sqrt{r_D t_D^*} \right) \]

(26)

The above expansions are also checked at the initial and boundary values, as well as the limiting forms, in Appendix 3.
The expansions for both cases described can be evaluated numerically using the following approximations for the given modified Bessel functions.  

For small values of \( x \)

\[
I_0(x) = \frac{1}{2} x^v + \frac{(4v^2 - 1)(4v^2 - 9)}{21(8x)} - \frac{(4v^2 - 1)(4v^2 - 9)(4v^2 - 25)}{31(8x)^3} + \ldots
\]

or more simply,

\[
I_0(x) = \frac{\exp(x)}{\sqrt{2\pi x}} f(v, x)
\]

For large values of \( x \)

\[
I_0(x) = \frac{\exp(x)}{\sqrt{2\pi x}} \left[ 1 - \frac{4v^2}{8v} \right] + \frac{(4v^2 - 1)(4v^2 - 9)}{21(8x)} - \frac{(4v^2 - 1)(4v^2 - 9)(4v^2 - 25)}{31(8x)^3} + \ldots
\]

Since the dimensionless terms for radius and time are large for almost
all times, the approximation for large values is used in expanding the solutions.

If we let,

\[ a = 2 \sqrt{r_D t_D^*} \]  \hspace{1cm} (27) \hspace{1cm} \beta = \left< r_D^* t_D \right> \]  \hspace{1cm} (28)

the solutions become:

**Case 1**

\[ T_R(r_D, t_D^*) = u(t_D^*) \left[ 1 - \frac{\text{Exp}(\alpha - \beta)}{\sqrt{2\pi\alpha}} \sum_{k=0}^{\infty} \frac{r_D^*}{\sqrt{t_D}} f(k, \chi) \right] \]  \hspace{1cm} (30)

\[ T_w(r_D, t_D^*) = u(t_D^*) \left[ 1 - \frac{\text{Exp}(\alpha - \beta)}{\sqrt{2\pi\alpha}} \sum_{k=1}^{\infty} \frac{r_D^*}{\sqrt{t_D}} f(k, \chi) \right] \]  \hspace{1cm} (31)

**Case 2**

\[ T_R(r_D, t_D^*) = u(t_D^*) \frac{\text{Exp}(\alpha + \beta)}{\sqrt{2\pi\alpha}} \sum_{k=1}^{\infty} \frac{r_D^*}{\sqrt{t_D}} f(k, \chi) \]  \hspace{1cm} (32)

\[ T_w(r_D, t_D^*) = u(t_D^*) \frac{\text{Exp}(\alpha + \beta)}{\sqrt{2\pi\alpha}} \sum_{k=0}^{\infty} \frac{r_D^*}{\sqrt{t_D}} f(k, \chi) \]  \hspace{1cm} (33)
ANALYTICAL SØLUTJØN FOR TD = 4

WATER TEMP: SOLID
RØCK TEMP: DASHES

FIG. 2
INVERTER AND ANALYTICAL RESULT, TD = 15

FIG. 3
INVERTER AND ANALYTICAL RESULT, TD=15

FIG. 4
INVERTER AND ANALYTICAL RESULT, TD=40

FIG. 5
INVERTER AND ANALYTICAL RESULT, TD=80

FIG. 6
ANALYTICAL SOLUTION FOR TD=800

WATER TEMP: SOLID
ROCK TEMP: DASHES

DIMENSIONLESS RÖCK AND WATER TEMP

DIMENSIONLESS RADIUS RD
ANALYTICAL SOLUTION FOR TD=80, 800, 1600

FIG. 8
8. DISCUSSION OF ANALYTICAL RESULTS

The analytical results described are checked at the initial condition and boundary condition in Appendix 3. The asymptotic forms are also checked. For small times, equations (A3.2) and (A3.4) indicate an exponential form of solution, and Figures 2 and 3 show this effect. However, the numerical approximation of the modified Bessel function applies for only large values of x, and must be changed for smaller values. Furthermore, there exists a middle range where neither solution is well behaved. The Stehfest inverter is well behaved in this range, however, and it is used to match the analytical solution for the relatively small dimensionless time of 15. At this value of dimensionless time, real time is about 6 minutes and the thermal front is developing from a simple step function through this exponential modification process. At a dimensionless time of 40, as shown in Fig. 5, the general symmetric form of the front is apparent and does not change until much later. Also, the Stehfest routine is shown to no longer match and is abandoned at this point. In Appendix 5, Figures 11 through 15 indicate the result using the inverter for later times.

Figures 5 through 9 are plots of the water and/or rock temperature for later times. These plots show that the front spreads as time increases but maintains its symmetric form. Figure 2 shows the relatively large difference in rock and water temperature at early dimensionless times. In Figures 9 through 11, this temperature difference is plotted at various times and is seen to decrease as time increases. For any given time, the maximum temperature difference occurs at the center of the front as expected. To the left of center, both the rock and water temperatures approach one (the injection temperature), while to the right of center, they approach zero (initial reservoir temperature). Equation (11) indicates that this plot is also that of the derivative of the rock temperature with respect to time, and that this has a maximum value at the center of the front, as one would expect. Further, these figures provide a simple measure of the spread of the front, as a rock-to-water temperature difference occurs only along the front.
The reason for the symmetry of the front can be seen from equations (22) through (24). The upper portion of the curve is generated by case #1, which requires that the dimensionless radius to shifted-time ratio be less than or equal to one. At the center, $r_D$ equals $t_D$ and the dimensionless temperature equals 0.5. The lower portion of the curve is next generated by case #2 as $r_D$ increases. If the dimensionless radius is divided by this ratio and plotted with dimensionless temperature, the center of the front is always located at one, for any dimensionless time. Figures 4 and 5 are two examples of this plot while Figure 8 plots the dimensionless water temperature for various times. From the latter plot it is seen that the curve steepens for larger times, indicating that the velocity of each point on the front approaches a constant value. This means that the rate of spread stops increasing and reaches a constant rate. This spreading change is also seen in Figs. 9, 10 and 11. For $t_D = 80$, approximately 95% of the area is covered over a dimensionless distance of 60, while for $t_D = 800$, the distance is only 100, and for $t_D = 1600$, it is 200. The former implies an increasing rate of spread, while the latter two indicate a constant rate of spread.

For times significantly larger than those discussed above, the front does not behave in the same way. For a dimensionless time of 8000 and a constant velocity of each point on the front, one would expect a spread-distance of 1000. The actual value of this spread is about 350, much less than anticipated. Although the velocity of the center of the front (or average front velocity) does not change, the velocity of other points on the front must be decreasing. For times much larger, one might expect (given the results of Appendix 3), the following sequence of events:
(a). All points on the front approach the same velocity and the front no longer spreads.

(b). The front begins to shrink as the frontal velocities decrease at different rates.

(c). The front as such disappears and is replaced by the step function. The average radial velocity of the thermal front has now changed.
10. VELOCITY OF THE THERMAL FRONT

(1) Short and Intermediate Times

From plots such as Fig. 11, one can determine the velocity of any point on the front since the location of the center of the front is fixed. Typical values of the plotted ratio

\[ \frac{r_D}{t_D} - \frac{M_w}{M_R} r_D \]

range from 0.5 to 1.5. Rearranging this ratio, eliminating dimensionless terms, and differentiating with respect to time, gives the location and velocity of any point on the front for all but long times. The result for the short and intermediate times is:

\[ q_w t = \pi \left( r^2 - r_w^2 \right) \left\{ \frac{R C v_R (1 - \phi_w)}{\phi_w C} + \phi_w^2 \right\} \]  \hspace{1cm} (32)

where,

\[ \eta = \frac{r_D}{t_D} \]  \hspace{1cm} (33)

\( r \) is the radial distance to that point on the front, \( q_w \) is the flow rate of water injected into the reservoir, and \( q_w t \) is the total reservoir volume of injected water.
Re-writing equation (32), we have,

$$ q_w t = \pi \left( t^2 - r_w^2 \right) \zeta $$

(34)

where,

$$ \zeta = \left\{ \frac{\rho_w c_{vR}}{\rho_w c_{vw}} \frac{(1- \phi_w)}{\eta} + \phi_w \right\} $$

(34.1)

Differentiating (32) gives

$$ \frac{q_w}{2\pi r^2 \phi_w} = \frac{\zeta}{\phi_w} \frac{3r}{\phi} \frac{\partial r}{\partial t} $$

(35)

Equation (35) can also be written in terms of radial velocities as:

$$ v_h = \frac{\phi_w}{\zeta} v_w $$

(36)

Where, $v_h$ is the radial velocity of a given point on the thermal front, and $v_w$ is the interstitial radial velocity of the water.

Note that for $\phi_w = 1$, equation (32) gives the location of the center of the thermal front and equation (34) gives its velocity, which is also the average velocity of the front.
(11) **Long Times**

For long times the front exists as the step function $u(t_D)$ and the above equations are replaced by:

$$q_w t = \pi (r^2 - r_w^2) \xi \phi_2$$  \hspace{1cm} (37)

$$\frac{\partial r}{\partial t} = \frac{q_w}{2\pi r \xi \phi_2}$$  \hspace{1cm} (38)

or,

$$v_h = v_w$$  \hspace{1cm} (38)

Equations (37) and (38) respectively correspond to equations (35) and (36) for $\zeta = \phi_2$. 
When the dimensionless terms are eliminated as shown above, the original energy balance relationships become clear. Dimensionless radius at the front becomes the total volume swept by that part of the thermal front, while the \( q \) \( t \) term represents the total volume of injected fluid. Equations (32) and (34) also include the relative volumes of rock and water swept by the front, as the weighted sum of these terms is proportional to the total amount of injected heat. For large times, incremental volumes of injected water advance the thermal front by smaller increments of real radius or radial distance. In the limit, this increment of real radius is infinitesimally small and the thermal front must become step-like with respect to the real radius. Eventually the front would also appear step-like with respect to the volume swept, as this convergence is proportional to \( r \Delta t \). While this report does not examine plots of dimensionless temperatures versus real radius, this type of plot is easily made and clarifies the behaviour of the thermal front with respect to the real radial distance. It can also be used to estimate the time for using the velocity equations given above. In any case, the form and velocity of the front approaches that of the injected fluid.
ANALYTICAL SOLUTION CONVECTION TERM

FIG. 9
ANALYTICAL SOLUTION CONVECTION TERM

FIG. 11

DIMENSIONLESS TEMPERATURE DIFFERENCE (T - TR)

DIMENSIONLESS RADIUS RD

TD = 1600
12. CONCLUSIONS

Assuming one injector well in an infinite, naturally fractured geothermal reservoir, the results of this study apply for those times during which convection is the dominant form of heat transfer and fluid flow is steady state. The period therefore excludes very long and very short times.

Upon injection of water into the system, a thermal front of the form discussed quickly develops, and moves through the reservoir at a constant average rate. Its velocity is less than that of the injected fluid for relatively early time. Although at first the front spreads at an increasing rate, spreading gradually slows and eventually reverses itself. The rock and water temperature fronts become equal at a dimensionless time of about 1000 and move at the same rate thereafter. For times much greater than this, the front moves with the fluid, limited to step-like displacement of heat in the reservoir. As there is no conduction, the radial velocity of the thermal front decreases with the radial velocity of the fluid, and in the limit approaches zero.

As a final note, the problem presented in this report can be solved in a straightforward way using the Laplace transform. While numerical inversion routines are often of great value, analytical inversion appears to be the most reliable approach to a solution of this problem.
## APPENDIX 1

### Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Typical Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{vR}$</td>
<td>Specific heat, rock, $\frac{btu}{lb \cdot ^\circ F}$</td>
<td>0.228</td>
</tr>
<tr>
<td>$C_{vw}$</td>
<td>Specific heat, water, $\frac{btu}{lb \cdot ^\circ F}$</td>
<td>1.00</td>
</tr>
<tr>
<td>$\rho_R$</td>
<td>Density, rock, $\frac{lb}{m^3 \cdot ft}$</td>
<td>170.00</td>
</tr>
<tr>
<td>$\rho_w$</td>
<td>Density, water, $\frac{lb}{ft^3}$</td>
<td>60.00</td>
</tr>
<tr>
<td>$h_c$</td>
<td>Convection constant, water, $\frac{btu}{hr \cdot ft^3 \cdot ^\circ F}$</td>
<td>50.00</td>
</tr>
<tr>
<td>$q_w$</td>
<td>Flow rate, water, $\frac{ft^3}{hr}$</td>
<td>50.00</td>
</tr>
<tr>
<td>$M_w$</td>
<td>$\rho_w C_{vw} \phi_2 \left( \frac{btu}{ft^3 \cdot ^\circ F} \right)$</td>
<td>12.0</td>
</tr>
<tr>
<td>$M_R$</td>
<td>$\rho_R C_{vR} (1-\phi_2) \left( \frac{btu}{ft^3 \cdot ^\circ F} \right)$</td>
<td>31.0</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>Fracture porosity, dimensionless</td>
<td>0.20</td>
</tr>
<tr>
<td>$\delta$</td>
<td>Fracture width, ft</td>
<td>0.002</td>
</tr>
</tbody>
</table>
\[ \lambda \quad \text{Reservoir height, ft} \quad 25.00 \]

\[ r_w \quad \text{Wellbore radius, ft} \quad 0.75 \]

\[ T_{wi} \quad \text{Initial reservoir temperature, } ^\circ\text{F} \quad 400 \]

\[ T_{inj} \quad \text{Temperature of injection fluid, } ^\circ\text{F} \quad 60 \]

\[ r_D \quad \text{Dimensionless radius} \]
\[
\text{(actually a dimensionless volume term)}
\]

\[ t_D \quad \text{Dimensionless time} \]

\[ T_R, T_w \quad \text{Dimensionless temperature of rock, water} \]

\[ v_h, v_f \quad \text{Radial velocity of heat, fluid, ft} \]
Transforming equations (10) and (11) implies:

\[-\frac{\partial T_w}{\partial \delta} - (T_w - T_R) = \frac{M_w}{M_R} s T_w\]  \hspace{1cm} \text{(A 2.1)}

\[(T_w - T_R) = s T_R\]  \hspace{1cm} \text{(A 2.2)}

Solving (A 2.2) for \(T_R\) and substituting it in (A 2.1) produces the following ordinary differential equation:

\[-\frac{\partial T_w}{\partial \delta} - (1 + \frac{M_w}{M_R} s \frac{1}{s+1}) T_w = 0\]  \hspace{1cm} \text{(A 2.3)}

Equation (13a) is derived by solving this ordinary differential equation for the water temperature and determining the constant of integration using the boundary condition. Equation (13b) is derived by substituting this result into equation (A 2.2).
APPENDIX 3 : Checks and Asymptotic Forms

Check of Initial and Boundary Conditions :

Initial Condition : $T_w(D,0) = T_r(D,0) = 0$ \hspace{1cm} (12a)

Case 2 applies for the initial condition above and from equations (24) and (25) it is seen that the step function ensures this for both the rock and water. As initially no heat (or cold) has been injected into the reservoir, one would expect the reservoir rock and water to be at their initial temperature.

Boundary Condition : $T_w(0,t_0) = 1$ \hspace{1cm} (12b)

Case 1 applies for the above boundary condition, and it is easily seen from equation (23) that this condition is satisfied. Note, however, that the rock temperature should not satisfy this boundary condition, as it is dependent on the water temperature. This implicit condition is also satisfied.
Limiting Forms

Another useful check for the analytical result is to examine the short and long time behaviour of the Laplace space equation. The limiting forms of this equation are then inverted and compared with the analytical solution for these times.

Short Time Solution

Water Temperature:

For short times, $s$ is large and equation (13a) approaches the form:

$$T_w(r_D,s) = \exp(-r_D) \int \frac{M_w}{M_R} S$$

(A3.1)

After inverting,

$$T_w(r_D,t^*) = \exp(-r_D) u$$

(A3.2)

Therefore one would expect the short time solution for the water temperature to have an exponential form of front.
Rock Temperature Solution:

For large $S$, equation (13a) has the form:

$$\mathcal{T}_R(r_D, S) = \exp(-r_D) \frac{L}{S^2} \exp(-r_D \frac{M}{M^*} S)$$  \hspace{2cm} (A3.3)

After inverting, this equation becomes:

$$\mathcal{T}_R(r_D, t_D) = \exp(-r_D) \frac{t_D}{t_D} \mathcal{T}_R(t_D)$$  \hspace{2cm} (A3.4)

As $t_D + 0$, $t_D > 0$, and it is seen that the rock temperature approaches zero or is set to zero by the step function. The latter would be the case for all $r_D$ if the time became zero as in the initial condition.

As another check, the asymptotic form of the inverted transform can be examined. The inverse transform of equation (13a) is equation (19.1).

$$\mathcal{T}_R(r_D, t_D) = u(t_D) \{ 1 - J(t_D, r_D) \}$$  \hspace{2cm} (19.1)

and the limiting forms are:

$$\lim_{y \to 0} J(x, y) = 1, \quad \lim_{x \to 0} J(x, y) = \exp(-x)$$  \hspace{2cm} (A3.4.5)
which is equivalent to:

\[
\lim_{t_D \to 0} J(t_D, r_D) = \exp(-t_D),
\]

Substituting into (19.1), we have the same result for the rock temperature at short times.

If these limiting forms of the rock temperature for short time are also substituted into equation (A4.4), it is seen that as

\[
t_D^* \to 0, \quad I_0(2 \sqrt{r_D r_D^*}) + 1, \quad \text{and the water temperature solution becomes:}
\]

\[
T_w(r_D, t_D^*) = u(t_D^*) \exp(-t_D^*)
\]

which was obtained previously.

**Long Time Solution**

For long time, both the rock and water temperature solutions have the same form. For a given dimensionless radius (ie. volume), equations (13a) and (13b) approach the form:

\[
\frac{T_w(r_D, S)}{T_R(r_D, S)} = \frac{1}{S} \exp\left(-\frac{M_w}{M_R} S\right)
\]

(A3.7)
After inverting, this equation becomes,

\[ T_w(r_D, t_D^*) = T_R(r_D, t_D^*) = u(t_D^*) \]  \hspace{1cm} (A3.8)

Therefore, one would expect the analytical solution to approach the form of the given step function for relatively long times.

Examining the inverted form of the solution again at the other limit:

\[ \lim_{y \to \infty} J(x, y) = 1 \, \quad \lim_{x \to -} J(x, y) = 0 \]

which is equivalent to,

\[ \lim_{t_D^* \to \infty} J(t, r_D^*) = 0 \, \quad \lim_{r_D^* \to \infty} J(t, t_D^*) = 1 \]  \hspace{1cm} (A3.9)

Substituting the first limiting form of (A3.10) into (19.1) gives the step function (A3.8). Substituting this result for the rock temperature into (19.2) gives the same result for the water temperature. Equation (A3.8) is the expected water and rock solution form for long time as both are equal and step functions.

The same results are also easily seen using case #1 of the expansion.

The same procedure can also be used to verify the analytical solution for large and small \( t_D \).
APPENDIX 4: Complete Derivation of Analytical Solution

Rock Temperature Solution:

The rock temperature solution for each case is found by substituting equations (20) or (21) into equation (19). The only other required substitution is allowing:

\[ x = t_D, \quad y = r_D \]  \hspace{1cm} (M.1)

After cancelling some terms, equations (22) and (24) are derived.

Water Temperature Solution:

\[ T_w(r_D, s) = \frac{S}{s} \exp \left\{ -r_D \left( \frac{M_w}{M_d} s - \frac{1}{s+1} \right) \right\} \]  \hspace{1cm} (13a)

Re-writing equation (13a), as the sum of two parts,

\[ T_w(r_D, t_D) = \exp(-r_D \frac{M_w}{M_d} s) \exp(-r_D) \left[ \frac{1}{s+1} \exp\left( \frac{r_D}{s+1} \right) + \frac{1}{s(s+1)} \exp\left( \frac{r_D}{s+1} \right) \right] \]  \hspace{1cm} (A4.2)

Inverting equation (A4.2) yields,

\[ T_w(r_D, t_D) = u(t_D) \exp(-r_D) f^{-1}\left\{ \frac{1}{s+1} \exp\left( \frac{r_D}{s+1} \right) \right\} + T_r(r_D, t_D) \]  \hspace{1cm} (A4.3)
Using the same procedure outlined in equations (17) through (17.5), this becomes:

\[
T_w(r_D^*, t_D^*) = u(t_D^*) \exp \left\{ \left( r_D^* + t_D^* \right) \right\} I_0 \left( 2 \sqrt{r_D^* t_D^*} \right) + T_R(r_D^*, t_D^*)
\]

(A4.4)

Substituting into (A4.4) the rock temperature solution for each case gives the water temperature solution for each case. Note that in both cases, the difference between the rock and water temperature solutions is the same function given in equation (26).
This appendix contains the results obtained by inverting equations (13a) and (13b) numerically using the Stehfest routine.

For early dimensionless times up to 40, a good match of the analytical solution is made. For times greater than this, however, the inverter is not well behaved and is not considered reliable. Figures 3 through 6 are plots of the numerical and analytical result for various dimensionless times. Note that the match is lost after 40 and worsens for later times. It appears that the match is lost when the Stehfest routine computes values of dimensionless temperature greater than one and less than zero. In addition, the thermal front as shown in Figures (12) through (16) would spread at increasing rates and never converge, a phenomenon which is physically impossible for this system. The Stehfest algorithm does prove useful however, for those early and mid-ranges of time during which the analytical solution is slowly convergent and/or the approximating equations (27a) and (27b) are not well behaved.
STEHFEST INVERTER RESULT FOR TD=80

WATER TEMP: SOLID
ROCK TEMP: DASHES

DIMENSIONLESS WATER AND ROCK TEMP

DIMENSIONLESS RADIUS, R/

FIG. 12
STEHFEST INVERTER RESULT FOR TD=800

DIMENSIONLESS WATER AND ROCK TEMP

DIMENSIONLESS RADIUS, RD

WATER TEMP: SOLID
ROCK TEMP: DASHES

FIG. 13
STEHFEST INVERTER Result For TD=1600

WATER TEMP: SOLID
ROCK TEMP: DASHES

FIG. 14
STEHEFEST INVERTER RESULT, TD=80, 800, 1600

FIG. 16
REFERENCES


PROGRAM FOR ANALYTICAL SOLUTION

1. // JOB
2. // EXEC WAT
3. IMPL CIT REAL*8 (A-H,O-Z)
4. COMM N TINJ, TWI, RW
5. DOUBE PRECISION TD, RD, TUR, TUW
6. QF=5.0
7. DENW=60.0
8. CVW=0.00
9. HT=2.00
10. DENR=170.00
11. CVR=0.2280
12. DELT=0.004
13. HC=5.00
14. PHI2=0.20
15. RW=750
16. TWI=30.00
17. TINJ=50.00
18. C
19. C ASSIGN INSTANTS
20. C
21. B=DE2J*QF*CVW/(2.0*3.1415927*HT)
22. D=2.0*HC*PHI2/DELT
23. SGN=0.00
24. SGN=0.00
25. 250 FORM: (5X,3(3X,E18.11))
26. C
27. C USE DIMENSIONLESS PARAMETERS
28. C
29. TD=8.00
30. PRINT ' ',
31. PRINT ' ',
32. C
33. DO 90 LM=1,24
34. C
35. RD=5.0*(D+75.0*(LM-1))
36. STEP:TD-SIGW*RD/SGM
37. IF(SI:PL.LT.0.0)GO TO 25
38. CALL BES (STEP1,RD,1,TW1)
39. 25 IF(SI:PL.LT.0.0)TR1=0.0
40. IF(SI:PL.LT.0.0)TW1=0.0
41. PRINT ' ',
42. WRITE 6,250 RD,TW1
43. PRINT ' ',
44. C
45. 90 CONTINUE
46. C
47. WRITE 90,100 TD=8.00
48. C
49. STOP
50. END
51. C
52. SUBRC TINE IBES(TD,RD,18,TE)
53. IMPLI IT REAL*8 (A-H,O-Z)
54. C3MMO TINJ, TWI, RW
55. DOUBE PRECISION TE
56. C
57. C THIS SUBROUTINE COMPUTES SUM OF BESSEL FNS I(Z)
58. C
59. ADD=0
60. Z=2.0*(RD*TD)**.5
61. VX:=(R/ TD)**.5
62. C
63. DO 80 K=1,235
64. C
IF(I8.EQ.1.AND.VX.LT.1.0)K=KK
IF(I8.EQ.2.AND.VX.GT.1.0)K=KK
U=4.0*K**2
C
UPT=1.0
FACJ=1
SUMT=1.0
C
CO 40 J=1,10
FACJ=FACJ*J
UPR=U-(2.0*J-1.0)**2.0
UPT=-UPR*UPT
TRM=UPT/(FACJ*(3.0*Z)**J)
SUMT=SUMT+TRM
40 CONTINUE
KS=K
IF(VX.GE.1)KS=-K
VY=VX*K
BEI=VY*SUNT
ADD=ADD+BEI
80 CONTINUE
C
CAN=Z-(TD+RD)
IF(CAN.LT.-120.0)GO TO 82
TE=(1.0/(2.*3.1415927*Z))**.5*DEXP(CAN)*ADD
82 IF(CAN.LT.-120.0)TE=0.0
80 CONTINUE
RETURN
END
$DATA
STEINFEST ROUTINE

1. // JOB
2. // EXEC WATFIV
3. IMPLICIT REAL*8 (A-H,O-Z)
4. EXTERNAL P
5. COMMON G(50),V(50),H(25),GZ(1)
6. COMMON TINJ,THI,D,RW,PD,B,SIGM,SIGW
7. DOUBLE PRECISION P
8. M=1
9. N=16
10. QF=0.0
11. DENW=60.0
12. CVW=1.00
13. HT=25.0
14. DENR=170.00
15. CVR=0.2280
16. DELT=0.004
17. HC=0.00
18. PHI2=0.20
19. RW=0.750
20. THI=400.00
21. TINJ=60.00
22. C
23. ASSIGNS CONSTANTS
24. C
25. B=DENW*QF*CVW/(2.0*3.1415927*HT)
26. D=2.0*HC*PHI2/DELT
27. SIGM=DENR*CVR*(1.0-PHI2)
28. SIGW=DENW*CVW*PHI2
29. C
30. 250 FORMAT(5X,3(3X,E19.10))
31. C
32. C START LAPLACE INVERSION
33. C
34. TD=15.0
35. C
36. PRINT,' '
37. PRINT,' TD= ',TD
38. PRINT,' '
39. C
40. DO 20 LM=1,60
41. RD=0.0+.50*(LM-1)
42. PRINT,' '
43. CALL LINV(P,TD,1,N,M,TAU1)
44. CALL LINV(P,TD,2,N,M,TAU2)
45. WRITE(6,250)RD,TAU1
46. 20 CONTINUE
47. STOP
48. END
49. FUNCTION P(S,I7)
50. IMPLICIT REAL*8 (A-H,O-Z)
51. COMMON G(50),V(50),H(25),GZ(1)
52. COMMON TINJ,THI,D,RW,PD,B,SIGM,SIGW
53. YY=1.0+SIGW*S/SIGM-1.0/(S+1.0)
54. XY=S+1.0
55. IF(RD*YY.GT.100.0)GO TO 91
56. IF(17.EQ.2)GO TO 90
57. F=DEXP(-RD*YY)/S
58. GO TO 92
59. 90 F=DEXP(-RD*YY)/(S*XY)
60. GO TO 92