OPERATOR BASED MULTISCALE METHOD
FOR COMPRESSIBLE FLOW

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I certify that I have read this report and that, in my opinion, it is fully adequate in scope and quality as a report for the degree of Master of Science.

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Approved for the University Committee on Graduate Studies.
Abstract

Accurate simulation of subsurface flow with detailed geologic description is of great academic and industrial interest. Fully fine-scale simulation is usually too expensive. The multiscale method is developed to capture fine-scale information without solving fine-scale equations. It is more efficient than fine-scale simulation methods and more accurate than traditional upscaling techniques.

Previous existing multiscale methods deal with the incompressible flow problems only. However, compressibility will be very significant if one of the fluid phases is gas. Gas has a large compressibility, and its compressibility is usually a strong function of pressure. Therefore, there can be a significant spatial variation of compressibility in the reservoir, and this is a challenge for multiscale modeling.

Motivated to construct a general multiscale framework that can deal with complicated physics, we develop an Operator Based Multiscale Method (OBMM). In this method, we first construct two multiscale operators — prolongation and restriction. Using the two operators, we construct the coarse-scale equations from fine-scale equations through simple algebraic operations. Then, we solve for the coarse-scale variables and reconstruct the fine-scale solution as in previous methods. The algorithm is succinct, and it is a general algebraic framework. As an example, compressibility can be naturally included into this framework. Numerical examples show the accuracy and efficiency of this method.
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I take this opportunity to thank my wife, Jing Peng, who always stands by me and shares my joy and sorrow. She gives me the motivation and confidence to do what I need to do. I am also indebted to my parents and younger brother for their support and love.
Nomenclature

\( \mathcal{P} \) prolongation Operator
\( \mathcal{R} \) restriction Operator
\( \mathcal{C}_f \) fine scale Compressibility matrix
\( \mathcal{C}_c \) coarse scale Compressibility matrix
\( \mathcal{T}_f \) fine scale transmissibility matrix
\( \mathcal{T}_c \) coarse scale transmissibility matrix
\( \mathbf{p}_f \) vector containing the pressure values on all fine nodes
\( \mathbf{p}_c \) vector containing the pressure values on all coarse nodes
\( p(x) \) fine scale pressure function
\( c(x) \) compressibility function
\( I \) x index of coarse scale nodes
\( J \) y index of coarse scale nodes
\( i \) x index of fine scale node
\( j \) y index of fine scale node
\( A \) node in coarse grid
\( a \) node in fine grid \( a \)
\(N\)  total number of coarse nodes

\(n\)  total number of fine nodes

\(\Omega\)  the whole physical domain

\(\tilde{\Omega}_A\)  coarse block centered on coarse nodes \(A\)

\(\Omega_A\)  dual block with upper left corner on coarse node \(A\)
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Chapter 1

Introduction

1.1 Background

The accuracy of simulating subsurface flow relies strongly on the detailed geologic description of the porous formation. Formation properties such as porosity and permeability typically vary over many scales. A detailed geologic description may require $O(10^8)$ grid cells. That is too expensive for calculation since state-of-the-art reservoir simulators deal efficiently with $O(10^6)$ grid cells.

Traditionally, this difficulty is tackled by upscaling techniques (refer to [5] and references therein). Upscaling is used to coarsen the fine-scale geologic models, while maintaining some kind of globally accurate information (i.e., give an accurate estimation of production history in the context of reservoir simulation). However, it is difficult for upscaling descriptions to model some complex flow scenarios, since the upscaled coarse models are usually constructed via simplified flow setting and strong localization assumptions. Thus, it is possible that an upscaled model works fine at first, but fails when some dramatic change happens during production. In addition, upscaling is a preprocess for simulation. The two processes are not automatically coupled.

Recently, there has been increasing interest in multiscale methods. Hou and Wu [6]
proposed a multiscale finite-element method (MsFEM) that captures the fine-scale information by constructing special finite element basis functions within each element. However, the reconstructed fine-scale velocity is not conservative. Later, Chen and Hou proposed a conservative mixed finite-element multiscale method that applied the same ideas for the basis functions. Another multiscale mixed finite-element method has been presented by Arbogast and Arbogast and Bryant. Numerical Green functions were used to resolve the fine-scale information and then were coupled with coarse-scale operators to obtain the global solution. These methods considered incompressible flow in heterogeneous porous media where the flow equation is elliptic.

The multiscale finite-volume method (MsFVM) was first proposed by Jenny, Lee and Tchelepi for heterogeneous elliptic problems. They employed two sets of basis functions — dual and primal. The dual basis functions are identical to those of Hou and Wu, while the primal basis functions are obtained by solving elliptic equations with Neumann boundary conditions, which are calculated from the dual basis functions.

Previous existing multiscale methods deal with the incompressible flow problem only. However, compressibility will be very significant if one of the fluid phases is gas. Gas has a large compressibility, and its compressibility is usually a strong function of pressure. Therefore, there can be a significant spatial variation of compressibility in the reservoir, and this is a challenge for multiscale modeling.

Very recently, Lunati and Jenny considered compressible multiphase flow, where the flow equation is parabolic. The key idea there is to homogenize the compressibility effects according to some assumptions within the framework of the MsFVM, and then obtain the coarse-scale pressure equation. They considered three models. The first model assumes piece-wise constant porosity and formation volume factor for each phase \( \alpha (B_\alpha) \) in a coarse block. Then the coarse-scale transmissibility and accumulation terms are easy to obtain. The second model is a modification of the first where the reconstructed fine-scale pressure is used to calculate the accumulation
term. They described a third method where the fine-scale porosity and formation volume factor are used to calculate both the accumulation and flux parts.

Motivated to construct a general multiscale framework that can deal with complicated physics, we develop an Operator Based Multiscale Method (OBMM). In this method, we need to construct two multiscale operators — prolongation and restriction. Using the two operators, we can construct the coarse-scale equations from fine-scale equations through simple algebraic operations. Then, we solve for the coarse-scale variables and reconstruct the fine-scale solution as in previous methods. The algorithm is succinct, and it is a general algebraic framework. As an example, compressibility can be naturally included into this framework quite easily. This algebraic framework can also be naturally extended from structured to unstructured grid. Moreover, the OBMM approach may be used to employ multiscale solution strategies with limited changes to existing fine-scale reservoir simulators. This is of great practical value.

1.2 Multiscale Finite-Element method (MsFEM)

Hou and Wu [6] first proposed the multiscale finite-element method (MsFEM) for the elliptic problem

\[-\nabla \cdot \lambda(x) \nabla u = f \quad \text{in } \Omega.\]  

For simplicity, let \( f = 0 \) and use rectangular mesh as shown in Fig. (1.1). The main idea of the MsFEM is to construct basis functions that represents the fine-scale information. Then the finite element space is represented by these basis functions, i.e.,

\[ V_h = \text{span}\{ \phi_K^i; i = 1, ..., N; K \in \mathcal{K}^h \} \subset H_0^1(\Omega), \]  

where \( \phi^i \) is the basis functions of node \( i \) in element \( K \), and \( \mathcal{K}^h \) denotes a partition of \( \Omega \).
The basis functions can be constructed locally in each coarse element. Take for example the coarse element $\bar{\Omega}_A$ shown in Fig. (1.1). Then, the basis function is the solution of

$$\nabla \cdot \lambda(x) \nabla \phi^i_A = 0 \quad \text{in} \quad \bar{\Omega}_A, \quad (1.3)$$

with reduced elliptic boundary conditions, which localize the basis function calculation. Specifically, assume that on the element boundary $\partial \bar{\Omega}_A$, we have

$$\frac{\partial}{\partial x_t} \left( \lambda(x) \frac{\partial \phi^i_A}{\partial x_j} \right)_t = 0, \quad (1.4)$$

where subscript $t$ denotes the component parallel to the boundary of element $\bar{\Omega}_A$. In addition, the basis function should possess the interpolation property. For this 2D example, one has

$$\sum_{i=1}^{4} \phi^i_A(x) = 1. \quad (1.5)$$

Then, we can set

$$\phi^i_A(x_j) = \delta_{ij} \quad i, j = 1, \ldots, 4, \quad (1.6)$$
where \( x_j \) denotes the coordinates of node \( j \). Eq. (1.4) is an imposed localization condition, which suggests a uniform flux along the boundary. Numerical experiments indicate that it is an excellent approximation [6].

After the calculation of the basis functions, the MsFEM follows the same procedure as standard finite element methods to solve for the nodal variables. The fine-scale information can then be naturally retrieved from the basis functions, i.e,

\[
\phi^j_i (x) = \sum_{i=1}^{4} \phi^j_i (x) u_i \quad \text{if} \quad x \in \Omega_A, \tag{1.7}
\]

where \( u_i \) is the finite element solution for coarse node \( i \) of element \( A \).

It was pointed out by Hou and Wu [3] that when the scale of oscillations in the coefficient (e.g., permeability) is close to the scale of the grid, there will be large errors due to “resonance”. Hou and Wu [6] found that the resonance error can be eliminated by improving the boundary conditions of the basis functions. They proposed an oversampling method. The idea is to impose the reduced boundary condition on a sampled domain that is larger than the coarse element and solve for the basis functions on that sampled domain. Since the basis functions use the interior information only, the boundary layer in the larger domain, where the resonance effects are pronounced, has much less influence on the basis functions.

1.3 Multiscale Finite-Volume method (MsFVM)

Jenny, Lee and Tchelepi [8] proposed the MsFVM, which employs two sets of basis functions — dual and primal. The dual basis functions are identical to those of Hou and Wu [6], while the primal basis functions are obtained by solving elliptic equations with Neumann boundary conditions, which are calculated from the dual basis functions. The main advantage of the MsFVM is that it is a cell-centered finite volume scheme and is locally conservative. The original MsFVM is used to solve the elliptic problem. Very recently, Lunati and Jenny extended the MsFVM to parabolic
problems [10].

1.3.1 MsFVM for elliptic problems

The dual grid is constructed by connecting the center point of each coarse (primal) control volume as shown in Fig. (1.2). Note that since the center points are vertices of the dual grid, we could then follow the same strategy as in MsFEM to calculate the dual basis functions for each dual block.

Consider the same elliptic governing equation as in Eq.(1.1). The flux is defined by

\[ \mathbf{v} = -\lambda(\mathbf{x}) \nabla p. \quad (1.8) \]

The finite volume scheme gives

\[ \int_{\Omega_A} \nabla \cdot \mathbf{v} \, d\Omega = \int_{\partial\Omega_A} \mathbf{v} \cdot \mathbf{n} \, d\Gamma = - \int_{\Omega_A} f \, d\Omega. \quad (1.9) \]

The challenge is to find a good approximation for the flux

\[ \mathbf{v} \cdot \mathbf{n} = \sum_{k=1}^{n} T^k \hat{v}^k, \quad (1.10) \]
where \( \bar{u}^k \) is the finite volume solution in control volume \( k \), \( n \) is the total number of volumes, and \( T^k \) denotes the transmissibility.

The effective coarse-scale transmissibility can be constructed from the basis functions. First, in each dual block, we compute the flux across a section of the coarse control-volume boundary (the solid line in Fig. (1.2)) from the contribution of basis function of each node (denoted as \( q^k \)). Then, the total flux across the coarse volume boundary can be obtained by assembling the flux contributions as follows

\[
q = \sum_{k=1}^{4} \bar{u}^k q^k. \tag{1.11}
\]

Here, \( q^k \), which is a normalized flux, is analogous to the transmissibility. Then, the effective transmissibility can be obtained by assembling the integral flow rate contributions, obtained from the basis functions, across the control volume interface.

The fine-scale information for \( u \) can be obtained from the basis functions as in Eq.(1.7). However, this information cannot be directly used to construct the fine-scale velocity field. A MsFVM, which employs the dual basis functions only, can guarantee conservation on the coarse grid only. This is because the fine-scale information is reconstructed using the dual grid locally and continuity of the fine-scale velocity is not guaranteed. However, a conservative fine-scale velocity field can be constructed in each primal coarse block locally. The approach is as follows.

First, since the flux across the interface of a primal coarse block is continuous, we can use it as a boundary condition for each coarse primal block (i.e, \( q \)). Thus, we can localize the problem. We define variable \( \tilde{u} \) as the reconstructed local fine-scale variable, which is used for the purpose of reconstructing the fine-scale velocity only. Now we have Neumann boundary conditions for \( \tilde{u} \); that is,

\[
-\lambda(x) \nabla \tilde{u} \cdot n = q \quad \text{on } \partial \bar{\Omega}. \tag{1.12}
\]
The governing elliptic problem in $\Omega_A$ is given by
\[ \nabla \cdot (\lambda(x)\nabla \tilde{u}) = f', \]  
where $f'$ is defined to ensure the solvability of the elliptic problem, namely,
\[ f' = \frac{\int_{\partial\Omega_A} q \, d\Gamma}{\int_{\Omega_A} d\Omega}. \]  

1.3.2 MsFVM for parabolic problems
Lunati and Jenny [10] extended the original MsFEM to study multiphase compressible flow, where the governing equation takes the form
\[ \frac{\partial}{\partial t} (\phi b_\alpha S_\alpha) = \nabla \cdot (b_\alpha \lambda_\alpha \nabla p), \]  
where $\phi$ is porosity, $b_\alpha$ is the inverse of formation volume factor of phase $\alpha$, $b_\alpha = 1/B_\alpha$. $\lambda_\alpha$ is the mobility of phase $\alpha$. They proposed to use the same dual basis functions as in the incompressible problem. The compressibility effect is taken into account in the solution of the coarse-scale equation and in the reconstruction of the fine-scale fluxes. First, to calculate the effective transmissibility that accounts for compressibility, they calculate the flux of phase $\alpha$ in dual block $\tilde{\Omega}_A$ by
\[ \mathbf{v}_\alpha^k = -b_\alpha \lambda_\alpha \nabla \phi_\alpha^k, \]  
where $\phi_\alpha^k$ is the basis function of node $k$ in dual block $\tilde{\Omega}_A$. Then, the effective transmissibility is used to construct the flow part of coarse-scale system. The accumulation part of the coarse-scale system is calculated from the average in the coarse volume $\hat{\Omega}_A$ as
\[ V_A \frac{\partial}{\partial t} (\langle \phi b_\alpha S_\alpha \rangle_A), \]
where \( \langle \cdot \rangle_A = \frac{1}{V_A} \int_{\Omega_A} \cdot \, dV \).

Three models are proposed in order to calculate the flux and accumulation terms in Equation (1.16) and (1.17), respectively. The first model assumes piece-wise constant \( \phi \) and \( b_\alpha \) in a coarse block. Thus, they assume in coarse volume \( \Omega_A \):

\[
\begin{align*}
\langle \cdot \rangle_A &= \frac{1}{V_A} \int_{\Omega_A} \cdot \, dV, \\
\langle \phi_A \rangle &= \phi_A(\bar{p}_A) = \text{const} \\
\langle b_\alpha \rangle &= b_\alpha(\bar{p}_A) = \text{const}.
\end{align*}
\]

The second model modifies the first model by using the reconstructed fine-scale pressure to calculate the accumulation term, i.e., in Eq. (1.16), \( b_\alpha \) is calculated from

\[
b_\alpha = b_\alpha(p^{\nu}),
\]

where \( p^{\nu} \) is the reconstructed fine-scale pressure from the previous iteration step \( \nu \).

The third model, which is the most accurate one, uses the fine-scale \( b_\alpha \) and \( \phi \) for flux and accumulation, i.e,

\[
\begin{align*}
b_\alpha &= b_\alpha(p^{\nu}), \\
\phi_A &= \phi_A(p^{\nu}),
\end{align*}
\]

with constraints \( \langle b_\alpha(p^{\nu}) \rangle_A = b_\alpha(\bar{p}_A) \) and \( \langle \phi(p^{\nu}) \rangle_A = \phi(\bar{p}_A) \).

### 1.4 Overview

The report is organized as follows. In Chapter 2, we introduce the OBMM framework. We describe how to construct the multiscale operators and coarse-scale systems in this framework. The correspondence of MsFEM [6] and MsFVM [8, 10, 9] with OBMM is discussed. We analyze a 1-dimensional problem in details to demonstrate the correspondence. We also give the physical interpretation of the constructed coarse-scale operators in parabolic problems. In Chapter 3, application of the OBMM for single-phase compressible flow problems is presented. Extensive numerical tests are given
for both 1-dimensional and 2-dimensional problems. The accuracy and efficiency of
the OBMM are also analyzed. In Chapter 4, we use the OBMM to study multiphase
problems. The algorithm of OBMM for multiphase problem is discussed and numeri-
cal studies on tracer transport and multiphase displacement problems are presented.
We conclude the report and discuss future research directions in Chapter 5.
Chapter 2

Operator Based Multiscale Method (OBMM)

In this chapter we present an Operator Based Multiscale Method (OBMM) for parabolic problems. The focus is on accurate modeling of compressible flow in heterogeneous porous media. We present the construction details of the multiscale operators based on the finite-volume method. The correspondence between OBMM and Ms-FVM [8], [10] is analyzed and the physical interpretation of the constructed coarse-scale operators is discussed. The basic advantages of the OBMM are presented at the end of the chapter. The OBMM is a general framework. The corresponding multiscale operators based on the finite-element method are shown in the Appendix.

2.1 Model Equation

The flow equation for compressible single-phase flow can be expressed as

\[ \frac{\partial}{\partial t} \left( \frac{\phi}{B} \right) + \nabla \cdot \left( \frac{k}{B \mu} \nabla p \right) = q, \]  

(2.1)

where \( \phi \) is the formation porosity, \( B \) is the so-called formation volume factor, which is defined by the density ratio as \( B = \rho_0/\rho(p) \), \( \mu \) is fluid viscosity, \( k \) is the absolute permeability, and \( q \) is the source term. For simplicity, we consider the case without
source terms \((q = 0)\). We assume the fluid is compressible, but the rock is incompressible \((\phi = \text{const})\). Then Eq.\((2.1)\) can be recast into

\[
\nabla \cdot (\lambda \nabla p) = c \frac{\partial p}{\partial t}, \tag{2.2}
\]

and the velocity (flux) is given by

\[
\mathbf{u} = -\lambda \nabla p, \tag{2.3}
\]

where \(\lambda = \frac{k}{B \mu}\) is the mobility, and \(c = \frac{\partial (\phi/B)}{\partial p}\) defined here is a characteristic parameter of compressibility, which is not the same as the compressibility defined in reservoir simulation literature. This definition offers us some notation simplicity. Therefore throughout this report, we refer to \(c\) as \textit{compressibility}.

Note that Eq.\((2.2)\) is parabolic. As we will show in Chapter 4, the flow equation for multiphase flow can also be expressed in this form. Therefore, Eq.\((2.2)\) is the model pressure equation for compressible flow.

Note that when the permeability, \(k\), is highly heterogeneous, the mobility will also be highly heterogeneous. In the context of reservoir simulation, \(c\) is not expected to be highly heterogeneous. However, to be safe in extreme cases, our proposed method should be able to handle problems with strong heterogeneity in both \(\lambda\) and \(c\). Therefore, in some test cases, we take \(c\) to be a highly heterogeneous spatial function.

## 2.2 General Multiscale Operators

The general multiscale operators can be introduced quite naturally in a manner analogous to multigrid methods [11]. Assume that the fine-scale discretization form (which can be obtained from any kind of numerical method) of the model equation \((2.2)\) is

\[
T_f \mathbf{p}_f = C_f \mathbf{p}_f + \mathbf{r}_f, \tag{2.4}
\]
where $C_f$ is the fine-scale compressibility matrix and $T_f$ the fine-scale transmissibility matrix. The multiscale prolongation operator, $\mathcal{P}$, relates the coarse-scale and fine-scale variables by

$$p_f = \mathcal{P} p_c. \quad (2.5)$$

Substituting Eq. (2.5) into Eq. (2.4) gives

$$T_f \mathcal{P} p_c = C_f \mathcal{P} p_c + r_f. \quad (2.6)$$

Obviously, Eq. (2.6) is a set of overdetermined equations. In order to solve for the coarse scale variable $p_c$, we need to restrict the $n$ fine-scale equations of Eq. (2.6) to $N$ coarse equations. This can be done by pre-multiplying Eq. (2.6) by a restriction operator $\mathcal{R}$, of order $N$ by $n$, which leads to

$$\mathcal{R} T_f \mathcal{P} p_c = \mathcal{R} C_f \mathcal{P} p_c + \mathcal{R} r_f. \quad (2.7)$$

We rewrite Eq. (2.7) as

$$T_c p_c = C_c p_c + r_c, \quad (2.8)$$

where the $T_c$ and $C_c$ are the coarse-scale transmissibility and compressibility matrices, respectively, and $r_c$ is the right hand side vector. Specifically,

$$T_c = \mathcal{R} T_f \mathcal{P},$$

$$C_c = \mathcal{R} C_f \mathcal{P},$$

$$r_c = \mathcal{R} r_f. \quad (2.9)$$

We now discuss how to construct the two operators algebraically. The prolongation operator, or interpolation, represents the weights applied to the coarse-scale variables in order to reconstruct the fine-scale variables. The more we know about the fine-scale structure of the variables, the better we can reconstruct them from coarse-scale information. Therefore, the accuracy of the prolongation operator is crucial for this multiscale method. The restriction operator, $\mathcal{R}$, projects the description from fine to coarse scales. As we will show in the following sections, $\mathcal{R}$ is related to the specific
numerical scheme used for the coarse-scale problem.

2.3 Multiscale Prolongation Operator

The multiscale prolongation operator $\mathcal{P}$ has the same interpolation properties as the basis functions shown in Chapter 1 for MsFEM and MsFVM. They are actually identical. The prolongation operator is simply the algebraic form of the basic functions but with a global numbering scheme.

The basis functions are usually expressed from a local point of view (i.e., interpolation can be done in each element (or dual block in the MsFVM) separately). However, here the prolongation operator is expressed from a global point of view. The local basis functions can simply be assembled and expressed globally as shown below.

Existing multiscale methods did not consider parabolic problems. Because compressible flow is our focus, we need to pay close attention to both the robustness and computational cost of the basis functions. To keep the efficiency of the multiscale method, we want to update the basis functions infrequently. Thus, we would like to avoid calculating time dependent basis functions. Consequently, we use the same basis functions as in elliptic problems. The philosophy is that the spatial distribution of the solution is dominated by the elliptic part. Later, we show using numerical examples that this is a good choice.

A two-dimensional multiscale grid is shown in Fig. (2.1). The notation is described in the nomenclature section. Now we describe the construction of the prolongation operator. First, the basis functions are constructed locally in each local dual block. Let $\phi^B_A$ to be the basis function that is associated with node $A$ and dual block $\bar{\Omega}_B$ as shown in Fig. (2.2). As we discussed in Chapter 1, the elliptic basis functions $\phi^B_A$
Figure 2.1: Two-dimensional multiscale grid. Thin dash-dot lines denote the fine grid, bold solid lines denote coarse grid, and bold dash lines denote dual grid; ◦ denotes fine-scale nodes, × denotes coarse-scale nodes.

Figure 2.2: A dual block with underlying fine cells.
satisfies
\[\nabla \cdot (\lambda \nabla \phi^B_A) = 0 \quad \text{in } \tilde{\Omega}_B,\]
\[\frac{\partial}{\partial x_t} \left( \lambda \frac{\partial \phi^B_A}{\partial x_j} \right) = 0 \quad \text{on } \partial \tilde{\Omega}_B,\]  
(2.10)
where \(C\) denotes a coarse node.

Assume \(\phi^B_A(x)\) is expressed from a global point of view, i.e., \(x\) is in global coordinates. Note that \(\phi^B_A(x)\) is zero in regions outside of \(\bar{\Omega}_A\). We can then assemble all the local functions to obtain a global representation
\[\phi_A = \sum_{B=1}^{N_d} \phi^B_A,\]
(2.11)
where \(N_d\) is the number of adjacent dual coarse blocks of a coarse node. For the rectangular grid shown in Fig. (2.2), the basis functions of each node (e.g., node A) has support only in four dual coarse blocks and thus \(N_d = 4\). Therefore, we need to solve four sets of equations (2.10) for the global basis function of every node. Finally, the prolongation operator can be expressed as
\[[P]_{a,A} = \phi_A(x_a),\]
(2.12)
where \(P\) is an \(n \times N\) matrix.

### 2.4 Multiscale Restriction Operator

The multiscale restriction operator is not unique. It depends on the numerical scheme adopted for the coarse-scale calculation, e.g., finite element method (FEM) or finite-volume method (FVM). We construct the multiscale restriction operators directly from the coarse-scale equations of the FVM. In the next section, we show that the multiscale method constructed algebraically by the two operators is equivalent to the
original MsFVM [8]. In the Appendix, we also show the FEM-based OBMM and its correspondence to the original MsFEM [6].

We use the finite-volume method as the fine-scale numerical scheme with backward difference in time and central difference in space, which is the most popular scheme in reservoir simulation. We note that any appropriate discretization scheme is feasible and fits equally well into the framework.

The finite-volume method is based on honoring the conservation law in each control volume. In the grid shown in Fig. (2.1), the fine cell and the coarse cell are taken to be the fine and coarse control volumes, respectively. The finite volume formulation starts with

$$\int_{\Omega_a} \nabla \cdot (\lambda \nabla p) \, d\Omega = \int_{\Omega_a} c(x) \frac{\partial p}{\partial t} \, d\Omega \quad (a = 1, \ldots, n).$$  

(2.13)

Also, the coarse-scale FVM formulation requires that

$$\int_{\Omega_A} \nabla \cdot (\lambda \nabla p) \, d\Omega = \int_{\Omega_A} c \frac{\partial p}{\partial t} \, d\Omega \quad (A = 1, \ldots, N).$$  

(2.14)

Comparing Eq.(2.13) and Eq.(2.14), it is clear that Eq.(2.14) can be obtained by summing Eq.(2.13) for all the fine blocks within coarse block A. The restriction operator, \( \mathcal{R} \), serves this purpose. Thus, \( \mathcal{R} \) simply takes the form

$$[\mathcal{R}]_{A,a} = \begin{cases} 
1 & \text{if } \Omega_a \subset \Omega_A \\
0 & \text{otherwise}
\end{cases} \quad (A = 1, \ldots, N; \ a = 1, \ldots, n).$$  

(2.15)

Assume the fine-scale discrete form is given by Eq.(2.4), then the coarse scale matrix system can be written as

$$T_c p_c = C_c p_c + r_c,$$  

(2.16)
with

\[
T_c = \mathcal{R} T_f \mathcal{P}, \\
C_c = \mathcal{R} C_f \mathcal{P}, \\
r_c = \mathcal{R} r_f.
\]  

(2.17)

### 2.5 Analysis of OBMM

So far, we presented the OBMM framework from a purely algebraic point of view. In this section, we analyze this algebraic formulation and illustrate the method with examples. We begin with 1D problems. Fig. (2.3) shows a section of the a scale grid with coarse blocks \( I - 1, \ I, \ I + 1 \), fine blocks \( i - 4, \ldots, i + 4 \) and two dual blocks \( D - 1 \) and \( D \). Fig. (2.4) shows a dual block. We analyze both elliptic and parabolic governing equations.

![Figure 2.3: 1D multiscale grid](image)

![Figure 2.4: a dual coarse block with underlying fine cells](image)
We denote the basis functions of the two dual blocks, $D-1$ and $D$, from a local point of view, where each dual block has two coarse nodes, $k = 1, 2$, associated with it. Thus for each dual block, we have two basis functions. Each basis function has four degrees of freedom, $n_{cf} = 4$, which coincide with the fine block values underlying the dual block. That is,

$$
\phi_n^k(J) \quad (k = 1, 2; \, n = D - 1, D; \, J = 1, \ldots, 4).
$$

(2.18)

Globally, the basis functions is denoted by

$$
\phi_C(i) \quad (C = I - 1, I, I; \, i = 1, \ldots, n_f).
$$

(2.19)

### 2.5.1 OBMM for elliptic equations

In this subsection, we analyze a 1-dimensional elliptic problem (i.e., Eq. (2.2) with $c = 0$). First consider the original MsFVM. Following the procedure of MsFVM as proposed in [8], the so-called effective transmissibility $T_c$, is calculated from the fluxes given by the dual basis functions. We choose central differences to calculate the fine-scale fluxes. Thus,

$$
T_{c_{I,I-1}} = -(flux)_{D-1}^{I} = -\frac{\lambda_i^{-2+1/2}}{\Delta x} (\phi_{D-1}^1(3) - \phi_{D-1}^1(2)),
$$

$$
T_{c_{I,I}} = -(flux)_{D-1}^{2} + (flux)_{D}^{I} = -\frac{\lambda_i^{-2+1/2}}{\Delta x} (\phi_{D-1}^2(3) - \phi_{D-1}^2(2)) + \frac{\lambda_i^{1+1/2}}{\Delta x} (\phi_{D}^1(3) - \phi_{D}^1(2)),
$$

(2.20)

$$
T_{c_{I,I+1}} = (flux)_{D}^{2} = \frac{\lambda_i^{1+1/2}}{\Delta x} (\phi_{D}^2(3) - \phi_{D}^2(2)).
$$

Now, consider the FVM-based OBMM. We only need to consider the $I^{th}$ row of the coarse-scale transmissibility matrix. The $I^{th}$ row of the multiscale restriction operator is

$$
\mathcal{R}_I = (\ldots 1 \ 1 \ 1 \ldots),
$$

(2.21)
where 1 lies in fine-scale position $i - 1$, $i$, and $i + 1$; 0 lies in all other positions. Therefore, we only need to consider the $i - 1$, $i$, and $i + 1$ rows of the fine-scale transmissibility matrix. To be consistent with the MsFVM, we also choose central differences as the fine-scale numerical scheme, so that

\[
\frac{1}{\Delta x} \begin{pmatrix}
\ldots & i - 2 & i - 1 & i & i + 1 & i + 2 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
i - 1 & \lambda_{i-3/2} & -(\lambda_{i-3/2} + \lambda_{i-1/2}) & \lambda_{i-1/2} & 0 & 0 \\
i & \ldots & \lambda_{i-1/2} & -(\lambda_{i-1/2} + \lambda_{i+1/2}) & \lambda_{i+1/2} & 0 \\
i + 1 & \ldots & 0 & \lambda_{i+1/2} & -(\lambda_{i+1/2} + \lambda_{i+3/2}) & \lambda_{i+3/2} \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix}.
\]

We have

\[
\mathcal{R}_f T_f = \frac{1}{\Delta x} \begin{pmatrix}
\ldots & \lambda_{i-2+1/2} & -\lambda_{i-2+1/2} & 0 & -\lambda_{i+1+1/2} & \lambda_{i+1+1/2} & \ldots 
\end{pmatrix},
\]

where the five elements lie at location $i - 2$, $i - 1$, $i$, $i + 1$, and $i + 2$. For the prolongation operator, we only need to consider columns $I - 1$, $I$, and $I + 1$ and row $i - 2$, $i - 1$, $i$, $i + 1$, and $i + 2$, so we write

\[
P = \begin{pmatrix}
\ldots & \ldots & I - 1 & I & I + 1 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
i - 2 & \ldots & \phi_D^{1}(2) & \phi_D^{2}(2) & 0 & \ldots \\
i - 1 & \ldots & \phi_D^{1}(3) & \phi_D^{2}(3) & 0 & \ldots \\
i & \ldots & 0 & 1 & 0 & \ldots \\
i + 1 & \ldots & 0 & \phi_D^{1}(2) & \phi_D^{2}(2) & \ldots \\
i + 2 & \ldots & 0 & \phi_D^{1}(3) & \phi_D^{2}(3) & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots 
\end{pmatrix}.
\]
CHAPTER 2. OPERATOR BASED MULTISCALE METHOD (OBMM)

We apply the multiscale operators to $T_f$ and obtain

\[
T_{c_{l,I-1}} = -\frac{\lambda_i-2+1/2}{\Delta x} (\phi_{D-1}^I(3) - \phi_{D-1}^I(2)),
\]

\[
T_{c_{l,I}} = -\frac{\lambda_i-2+1/2}{\Delta x} (\phi_{D-1}^I(3) - \phi_{D-1}^I(2)) + \frac{\lambda_i+1+1/2}{\Delta x} (\phi_D^I(3) - \phi_D^I(2)),
\]

\[
T_{c_{l,I+1}} = \frac{\lambda_i+1+1/2}{\Delta x} (\phi_D^I(3) - \phi_D^I(2)).
\]

Eq. (2.25) is exactly the same as Eq. (2.20), which demonstrates the equivalence of the two methods. That is, the coarse-scale transmissibility matrix constructed from the FVM-based OBMM is identical to the effective coarse-scale transmissibility matrix of MsFVM.

Therefore, we have clearly shown that for the 1D incompressible flow (elliptic) equation, the OBMM constructs coarse systems that are identical to the corresponding MsFVM. A similar analysis of the MsFEM is shown in the Appendix. This conclusion can also be extended to 2D elliptic equations, which we demonstrate by numerical examples.

### 2.5.2 OBMM for parabolic problems

The parabolic equation consists of two parts, an elliptic part and a time derivative part (accumulation). Construction of the coarse-scale system for the elliptic part has been discussed in the above subsection. Here, we focus on the accumulation term and analyze the coarse-scale compressibility matrix constructed by the OBMM.

In the FVM-based OBMM, we have the approximation

\[
[\mathcal{RC}_f \mathcal{P}_c]_I \simeq \int_{\Omega} c(x)p(x) \, dx.
\]

Using the grid shown in Fig. (2.3), again for the $I^{th}$ row of the compressibility matrix,
there are only three nonzero elements, we obtain

\[
[ \mathcal{R}_f \mathcal{P}_{p_{e+1}} ]_I = (C_{c_{i-1}} p_{c_{i-1}} + C_{c_i} p_{c_i} + C_{c_{i+1}} p_{c_{i+1}}) \Delta x_f, \tag{2.27}
\]

with

\[
C_{c_{i-1},I} = \sum_i \mathcal{R}_{f,i} C_{f_{i,i}} \mathcal{P}_{i,I-1} = \int_{\Omega_I} \phi_{i-1}(x) c(x) \, dx,
\]

\[
C_{c_i,I} = \sum_i \mathcal{R}_{f,i} C_{f_{i,i}} \mathcal{P}_{i,I} = \int_{\Omega_I} \phi_i(x) c(x) \, dx, \tag{2.28}
\]

\[
C_{c_{i+1},I} = \sum_i \mathcal{R}_{f,i} C_{f_{i,i}} \mathcal{P}_{i,I+1} = \int_{\Omega_I} \phi_{i+1}(x) c(x) \, dx.
\]

Eq. (2.28) shows that the \(I^{th}\) row of the compressibility matrix is the weighted sum of compressibility in coarse block \(I\), where the weights are the basis function values of the corresponding coarse node. The idea here is analogous to that of Lunati and Jenny’s third model \([10]\), where \(b\) is calculated from the fine-scale pressure given by the basis functions and the coarse-scale pressure. However, the construction method using OBMM is natural and straightforward.

For the special case where \(\lambda = 1, c = 1\), using the FVM-based OBMM, we get

\[
C_{c_{i-1},I} = \frac{n_{cf}^2 - 1}{8n_{cf}^2} \Rightarrow \frac{1 - 1/n_{cf}^2}{8} \to \frac{1}{8} \text{ when } n_{cf} \to \infty, \tag{2.29}
\]

\[
C_{c_i,I} = \frac{6n_{cf}^2 + 1}{8n_{cf}^2} \Rightarrow \frac{6 + 1/n_{cf}^2}{8} \to \frac{3}{4} \text{ when } n_{cf} \to \infty, \tag{2.30}
\]
\[
C_{cI,I+1} = \frac{n_{cf}^2 - 1}{8n_{cf}^2} \\
\Rightarrow \frac{1 - 1/n_{cf}^2}{8} \to \frac{1}{8} \text{ when } n_{cf} \to \infty.
\]

That is the coarse scale compressibility stencil can be represented in compact form as

\[
\left[ \frac{1}{8}, \frac{3}{4}, \frac{1}{8} \right].
\]

\section*{2.5.3 Advantages of the OBMM}

The OBMM extends the original MsFEM and MsFVM naturally to parabolic problem. The equivalence for elliptic problem was clearly demonstrated. The OBMM has three advantages over existing multiscale methods.

1. The OBMM is a general algebraic framework. Thus, it does not depend on the geometry of the discretization grid and can be directly applied to unstructured grid.

2. The implementation of the OBMM is straightforward. Existing fine-scale computational programs can make immediate use of the OBMM to develop a multiscale program. Therefore, it is possible to build a multiscale reservoir simulator from existing fine-scale simulators. Compared with building a multiscale simulator from scratch, that will save a great deal of effort.

3. The algorithm is flexible and allows for including complex physics. In the compressible flow problem, compressibility can be naturally included in this framework without resorting to special treatments. In contrast, the method proposed by Lunati and Jenny \cite{10} modifies the form of the coarse-scale equations to account for the influence of compressibility. Moreover, the OBMM is analogous to their method using the most accurate model.
Chapter 3

Single-Phase Compressible Flow

In this chapter, we focus on the numerical performance of the OBMM for single-phase compressible flow in heterogeneous porous media. We first study 1D problems. Since in 1D, the boundary conditions used for the basis functions are exact for the elliptic problem, we can separate the error introduced by compressibility from that due to the imposed boundary condition. In the second part, we investigate the 2D problems. In the end, the computational efficiency of the OBMM is analyzed.

3.1 Numerical study for 1D Compressible Flow

We present a detailed numerical study to demonstrate the numerical performance of the OBMM. We focus on two factors: permeability and compressibility. We study a broad range of permeability fields, from homogeneous to extremely heterogeneous. Both small and large spatial changes of compressibility are considered. As discussed in Chapter 2, the governing equation for single-phase flow is

\[ \nabla \cdot (\lambda \nabla p) = \frac{c}{\tilde{t}} \frac{\partial p}{\partial \tilde{t}}, \]

(3.1)

where all the parameters used in the 1D study are dimensionless as

\[ p = \frac{\tilde{p}}{\bar{p}}, \quad c = \frac{\tilde{c}}{\bar{c}}, \quad \lambda = \frac{\tilde{\lambda}}{\bar{\lambda}}, \quad x = \frac{\tilde{x}}{L}, \quad t = \frac{\tilde{t}}{\tau} \]

(3.2)
where \( \tilde{\cdot} \) denotes the original parameter with dimension, \( \bar{\cdot} \) denotes the corresponding characteristic parameter. Then the characteristic time is defined as \( \tau = \bar{c}L^2/\bar{\lambda} \), where \( \bar{c} \) and \( \bar{\lambda} \) are the characteristic parameters.

### 3.1.1 Test case

In all the following 1D examples, the physical domain is \([0, 1]\). The field is initially at a dimensionless pressure of 100. The left side boundary is kept at constant pressure of 100, while the right side boundary is kept at a constant pressure of 1. There is no source term. There are 65 fine cells and 7 coarse cells. There are 9 fine cells in each coarse block (an upscaling factor of 9). The fluid property is taken to be the same as ideal gas \( (B(p) = B_0/p) \).

The relative error is expressed as

\[
e = \frac{\| p^{\text{fine}} - p^{\text{ms}} \|_2}{\| p^{\text{fine}} \|_2},
\]

where \( \| \cdot \|_2 \) denotes the \( L^2 \) norm, \( p^{\text{fine}} \) is the fine-scale pressure solution obtained from fine-scale calculation, which is taken to be the reference ”true” solution, and \( p^{\text{ms}} \) is the fine-scale pressure solution obtained from multiscale calculations using the OBMM.

### 3.1.2 Homogeneous permeability field

We first examine ideal gas flow in a homogeneous permeability field, i.e. let

\[
B = \frac{1}{p}, \quad \text{and} \quad k = 1.
\]

The problem is nonlinear, and the Newton iterative method is used to solve it. We compare the pressure profiles calculated from the fine-scale and multiscale methods.
Fig. (3.1) shows the pressure profiles at three time steps and also the pressure profile at steady state (or incompressible problem). In this figure, $ms$-$fine$ refers to the fine-scale pressure calculated from the multiscale method. At early time, there is noticeable error in the block around the pressure diffusion front; however, the overall results given by OBMM are quite accurate. The relative error in the three time is 2.1%, 0.4% and 0.01% respectively. For the incompressible problem, the result is exact (i.e., $e < 10^{-10}$).

One reason for this numerical performance is that the basis functions are based on elliptic equations, while the actual pressure distribution is parabolic. This introduces some error, but the error is limited to only one coarse block as can be seen in Fig. (3.1). At later time, but before steady state, the solution become nearly exact everywhere. The solution for steady state is exact.

![Figure 3.1: 1d pressure profile for a homogeneous permeability field](image)
3.1.3 Heterogeneous permeability field

We continue to study ideal gas flow, but now for a heterogeneous permeability field. Take $B = 1/p$ and consider a highly heterogeneous 1D permeability field given by

$$k = 1 + 10^4 \sin(10^3 \pi x) \times \text{rand}(0,1),$$

where $\text{rand}(0,1)$ is a random variable equally distributed between 0 and 1. This permeability is highly oscillatory and heterogeneous. Fig. (3.2) shows that the OBMM also gives good results for this heterogeneous case. The numerical performance is similar to the homogeneous case: noticeable error at early time and nearly exact results for later time. The relative error in the three time steps is 3.1%, 1.1% and 0.05%. Recall that, the result for steady state is exact.

![Figure 3.2: 1d pressure profile for a heterogeneous permeability field](image-url)
3.1.4 Heterogenous permeability and compressibility fields

We now consider a model case with highly spatial oscillations in both permeability and compressibility fields, i.e., we take

\[
  k = 10^{-4}(1 + 10^4 \sin(10^3 \pi x) \times \text{rand}), \quad \text{and} \\
  c = 1 + 10^4 \sin(10^3 \pi x) \times \text{rand}).
\]  

(3.6)

Note that \( c \) is highly heterogenous and several orders larger than \( \lambda \). We use these parameters to model extreme cases where compressibility is dominant and oscillatory. Fig. (3.3) shows that the results are quite good. The numerical performance is similar in the three cases.

Figure 3.3: 1D pressure profile for a problem with heterogeneous permeability and compressibility fields

3.1.5 Comments

Based on all the previous results, we can conclude that
1. For the elliptic problem (incompressible flow) in 1D, the OBMM gives exact results.

2. For the parabolic problem (compressible flow) in 1D, the OBMM gives very good results, with the error decreasing with time.

3. The accuracy of the OBMM strongly depends on the quality of the basis functions. For the elliptic problem in 1D, the basis functions represent the pressure distribution exactly, and thus we can obtain the exact solution. For the parabolic problem in 1D, the basis functions can only represent the pressure distribution approximately. Fortunately, this approximation is quite reasonable, and we get excellent results even for extremely difficult cases.

3.2 Numerical study for 2D Compressible Flow

3.2.1 Test case

For the 2D studies, the physical domain is the unit square. The field is initially at a dimensionless pressure of 100. We have constant pressure boundary conditions on the west and east sides of 100 and 1, respectively. The north and south sides are no-flow boundaries. There is no source term. The characteristic time is \( \tau = \bar{c}L_xL_y/\bar{\lambda} \).

3.2.2 Homogeneous permeability field

We study ideal gas flow, \( B = 1/p \), in a homogeneous permeability field, \( k = 1 \). We compare the fine-scale pressure profiles of the reference solution with the solution from the OBMM. Fig. (3.4) shows the results at three time steps. The relative error for the three time steps is 3.1%, 1.9%, and 0.98%. The pressure history of the middle cell is shown in Fig. (3.5). The steep pressure drop is captured accurately and the match is excellent.

For early time, we can also notice some error for some fine cells near the boundary. The reason is that these cells are the boundary of dual blocks, and thus the pressure
is fully controlled by the reduced boundary condition.

Figure 3.4: 2d pressure field comparison for homogeneous permeability field

The coarse-scale solution at early time has relatively larger error in regions with a large pressure drop. Pressure on the dual block boundary will not be corrected by the dual basis functions, while other cells can have a much better correction from the basis functions. Therefore, there is relatively large error for the dual boundary cells. Note that the error is still small and dissipates as time increases. We can also appreciate the advantage of the multiscale method over the coarse-scale method (i.e., upscaled method) — the fine-scale information can correct the coarse-scale pressure for most fine cells. Moreover, although the reduced elliptic boundary condition is an imposed approximate condition, it results in very good overall numerical performance and only on a few cells suffer from these imposed conditions.

### 3.2.3 Heterogeneous permeability field

Now, we use the OBMM to study 2D single-phase flow. To challenge the proposed method, we use the permeability model from the 10th SPE Comparative Solution Project (SPE 10 [1]). We take the top layer of the model as shown in Fig. (3.6). The permeability varies by six orders of magnitude, and the variance of the logarithm
Figure 3.5: Pressure history of the middle cell for homogeneous permeability field of permeability is 5.49. This is a difficult problem due to the high oscillations in permeability. The fluid is assumed to be an ideal gas.

Fig. (3.7) shows a comparison of the pressure profiles at three time steps. All the steps show excellent agreement between the multiscale and fine-scale solutions with relative error of 3.84%, 2.52%, and 2.10%, respectively. Fig. (3.8) shows the pressure history of the middle cell, the match is nearly exact.

3.2.4 Comments

Through all the previous results, we have shown that

1. For 2D problems, we have to use the imposed local boundary condition to calculate the basis functions. It is an excellent approximation, and it results in very good overall performance.

2. For 2D parabolic problems, the numerical performance is similar to the 1D
CHAPTER 3. SINGLE-PHASE COMPRESSIBLE FLOW

Figure 3.6: SPE 10 logarithm permeability field

Figure 3.7: 2d pressure field comparison for heterogeneous permeability field
problem in that the error is noticeable at early time due to using elliptic basis functions. The error decreases quickly with time. Therefore, it is still reasonable to use the elliptic basis functions for the parabolic problem.

3.3 Computational Efficiency of the OBMM

We point out that the efficiency of the OBMM strongly depends on the frequency of the basis function calculation. In the previous results, to ensure maximum accuracy, we update the basis functions in each iteration since the mobility, $\lambda$, is iteration dependent. In most cases, this is not necessary. Usually, we only need to update the basis functions in each time step. To maximize the efficiency, we need to have some adaptive criterion to update the basis function. This has been discussed in [9].

Here we first check the accuracy of OBMM when updating the basis function only once at each time step. We run the 2D single-phase compressible flow in the SPE 10 permeability field. The settings are the same as the 2D heterogeneous case in the previous section. The relative errors are compared in Table (3.1). We can see that
the accuracy of the two methods is comparable, but of course it is desirable to update
the basis function only every time step.

<table>
<thead>
<tr>
<th>time ($\times \tau$)</th>
<th>update per iteration</th>
<th>update per time step</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>3.65%</td>
<td>3.84%</td>
</tr>
<tr>
<td>0.1</td>
<td>2.68%</td>
<td>2.52%</td>
</tr>
<tr>
<td>1</td>
<td>2.22%</td>
<td>2.10%</td>
</tr>
</tbody>
</table>

Table 3.1: Relative error of the pressure field by updating basis functions every
iteration versus every time step.

Let us analyze the computational efficiency following the strategy in [8]. We do
not use any adaptive strategy. First we introduce the definitions in Table (3.2).

<table>
<thead>
<tr>
<th>$n_v$</th>
<th>number of cells of the fine grid</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_V$</td>
<td>number of cells of the coarse grid</td>
</tr>
<tr>
<td>$n_N$</td>
<td>number of nodes of the coarse grid</td>
</tr>
<tr>
<td>$n_t$</td>
<td>total number of time steps</td>
</tr>
<tr>
<td>$a_N$</td>
<td>average number of adjacent coarse volumes to a coarse node</td>
</tr>
<tr>
<td>$a_V$</td>
<td>average number of adjacent coarse volumes to a coarse volume</td>
</tr>
<tr>
<td>$a_n$</td>
<td>average number of adjacent coarse volumes to a fine node</td>
</tr>
<tr>
<td>$a_v$</td>
<td>average number of adjacent coarse volumes to a fine volume</td>
</tr>
<tr>
<td>$f(n)$</td>
<td>CPU time to solve a linear system with $n$ unknowns</td>
</tr>
<tr>
<td>$t_m$</td>
<td>CPU time for one multiplication</td>
</tr>
<tr>
<td>$\nu$</td>
<td>average iterations per time step</td>
</tr>
</tbody>
</table>

Table 3.2: Definitions for computational efficiency analysis

Also assume that $f(n) = ct_m n^\alpha$ where $c$ and $\alpha$ are constants.

1. Time for updating the basis functions every time step is

$$t_{b0} = n_N a_N f(n_v/n_V) \times n_t. \tag{3.7}$$

If we update the basis functions per iteration, the time is

$$t_b = t_{b0} \times \nu. \tag{3.8}$$
2. Time for constructing the coarse-scale operators

\[ t_o = n_N t_m \frac{n_v}{n_V} (a_n + a_N) \times n_t \nu. \]  

(3.9)

3. Time for solving the coarse-scale equation

\[ t_c = f(n_V) \times n_t \nu. \]  

(3.10)

4. Time for reconstruction of pressure

\[ t_p = t_m n_N a_N \frac{n_v}{n_V} \times n_t. \]  

(3.11)

5. Time for reconstruction of velocity

\[ t_v = f \left( \frac{n_v}{n_V} \right) n_v \times n_t. \]  

(3.12)

The fine-scale calculation time will be

\[ t_f = n_t \nu f(n_v). \]  

(3.13)

Now, considering a structured 1000 × 1000 fine grid and 50 × 50 coarse grid. Then
\[ n_v = 1 \times 10^6, n_V = 2.5 \times 10^3, a_N = a_V = a_n = a_N = 4. \]  

We take \( c = 10, \alpha = 1.2, \nu = 8. \) Then, when we update basis functions per time step the efficiency factor is \( t_f/t_{ms} = 65. \) If we update basis functions per iteration, it is only 12. The gain due to using OBMM increases as the size of the fine-scale problem increases.

Note that the basis function computation and the velocity reconstruction procedures are perfectly suitable for parallel processing. Also, adaptivity can greatly improve the efficiency. [9]
Chapter 4

Multiphase Compressible Flow

In this chapter, we use OBMM to study multiphase compressible flow in heterogeneous porous media. The algorithm of OBMM for multiphase problem is discussed. Numerical studies on tracer transport and multiphase displacement problems are presented.

4.1 OBMM for Multiphase Problems

The OBMM algorithm for multiphase flow problems includes three major procedures: solving the flow equation for pressure, reconstructing the fine-scale velocity field and updating the saturation. Details of the algorithm are presented.

4.1.1 Solve flow equation for pressure

The governing equations of multiphase immiscible flow can be written into the form

$$\frac{\partial}{\partial t} (\phi b_\alpha S_\alpha) = \nabla \cdot (b_\alpha \lambda_\alpha \nabla p), \quad (4.1)$$

where $\alpha$ denotes a phase. We focus on two-phase flow problem. Subscript $\alpha$ and $\beta$ is used to denote the two phases, respectively. We have to solve for the pressure and saturation separately. This is because OBMM is used to obtain the flow field (pressure and total velocity). The transport problem is solved using conventional methods.
CHAPTER 4. MULTIPHASE COMPRESSIBLE FLOW

in the fine grid. Therefore, a natural choice would be the IMPES method (implicit pressure, explicit saturation). The sequential fully implicit method, or SFIM, is also suitable to use OBMM for the multiphase problems. We focus on the IMPES method here.

Following the IMPES scheme, the discrete pressure equation for two-phase flow can be written as

\[
\frac{\phi}{\Delta t} (S_{\alpha}^{n} b_{\alpha}^{n+1} \Delta t b_{\alpha} + S_{\beta}^{m} b_{\alpha}^{n+1} \Delta t b_{\beta}) = [b_{\beta}^{n+1} \nabla \cdot (b_{\alpha}^{n} \lambda^{n} \nabla p^{n+1}) + b_{\alpha}^{n+1} \nabla \cdot (b_{\beta}^{n} \lambda^{n} \nabla p^{n+1})],
\]

(4.2)

using \(\Delta t b_{\alpha} = (db_{\alpha}/dp) \Delta t p = b'_{\alpha}\), we have

\[
\frac{\phi}{\Delta t} (S_{\alpha}^{m} b_{\beta}^{n+1} b'_{\alpha} + S_{\beta}^{m} b_{\alpha}^{n+1} b'_{\beta}) \Delta t p = [b_{\beta}^{n+1} \nabla \cdot (b_{\alpha}^{n} \lambda^{n} \nabla p^{n+1}) + b_{\alpha}^{n+1} \nabla \cdot (b_{\beta}^{n} \lambda^{n} \nabla p^{n+1})].
\]

(4.3)

We define

\[
C^{n+1} = \phi (S_{\alpha}^{m} b_{\beta}^{n+1} b'_{\alpha} + S_{\beta}^{m} b_{\alpha}^{n+1} b'_{\beta}),
\]

(4.4)

where \(C^{n+1}\) refers to the characteristic compressibility coefficient in multiphase problems. Eq.(4.3) is a parabolic equation with only pressure as the unknown, which can be solved using the OBMM approach as described previously.

4.1.2 Fine-scale velocity reconstruction and saturation

Note that \(p^{n+1}\) is constructed on dual coarse blocks using the elliptic basis function. Therefore, the fine-scale velocity calculated directly from \(p^{n+1}\) is generally discontinuous across dual coarse boundaries. This velocity cannot be directly used in the transport equation, since it leads to large errors due to the violation of mass conservation. However, this velocity is continuous at the primal coarse block interfaces, which lie in the center of the dual coarse block. Therefore, we can use the flux across the primal coarse block interfaces as a boundary condition to reconstruct a conservative velocity field locally in each primal coarse block. Since this reconstructed velocity is also continuous on primal coarse block interfaces, it is thus locally conservative everywhere.
CHAPTER 4. MULTIPHASE COMPRESSIBLE FLOW

From Eq. (4.2) and with $p^{n+1}$ known, we have

$$[b_{\beta}^{n+1}\nabla \cdot (b_{\alpha}^{n+1} \lambda_{\alpha}^{n} \nabla \tilde{p}^{n+1}) + b_{\alpha}^{n+1}\nabla \cdot (b_{\beta}^{n+1} \lambda_{\beta}^{n} \nabla \tilde{p}^{n+1})] = q_{c}^{n+1} \quad \text{in } \tilde{\Omega}_{I},$$

(4.5)

where $\tilde{\Omega}_{I}$ is a primal coarse block, $\tilde{p}$ is the local fine scale pressure in $\tilde{\Omega}_{I}$ which is calculated only for the purpose of reconstructing the fine scale velocity; $q_{c}$ is the accumulation term due to compressibility, and it is given by

$$q_{c}^{n+1} = \frac{\phi}{\Delta t}(S_{\alpha}^{n} b_{\beta}^{n+1} \Delta t b_{\alpha} + S_{\beta}^{n} b_{\alpha}^{n+1} \Delta t b_{\beta}).$$

(4.6)

We also have the flux boundary condition for each primal coarse block $\tilde{\Omega}_{I}$,

$$\nabla \tilde{p} = \nabla p^{n+1} \quad \text{(on } \partial \tilde{\Omega}_{I}).$$

(4.7)

Note that we only care about the gradient of $\tilde{p}$ since our purpose is to calculate the fine scale velocity from $\tilde{p}$. Eq. (4.5) and Eq. (4.7) are adequate to solve for the gradient of $\tilde{p}$ in each primal coarse block locally. Also note that the linear matrix given by Eq. (4.5) and Eq. (4.7) is singular, which can be scaled by simply modifying one dialog element. Then we obtain the fine scale velocity of phase $\alpha$, that is,

$$u_{\alpha} = -\lambda_{\alpha} \nabla \tilde{p}.$$  

(4.8)

Now, we update the saturation either explicitly (IMPES) or implicitly (SFIM) using $u_{\alpha}$ and $p^{n+1}$. Using IMPES, we have

$$S_{\alpha}^{n+1} = S_{\alpha}^{n} \frac{b_{\alpha}^{n}}{b_{\alpha}^{n+1}} - \frac{\Delta t}{\phi b_{\alpha}^{n+1}} \nabla \cdot (b_{\alpha}^{n+1} u_{\alpha}).$$

(4.9)

4.2 Numerical Examples

4.2.1 Tracer transport

We first study the transport of an ideal tracer in single-phase compressible flow. The ideal tracer is assumed not to affect the flow and thus can be taken as a good
indicator of the quality of the fine-scale velocity field. The governing equation for tracer transport is an advection equation, that is,

$$\frac{\partial (\phi s)}{\partial t} + \nabla \cdot (su) = q_c,$$

(4.10)

where $s$ is the tracer concentration, which is equivalent to the saturation, $q_c$ is the tracer source term and $u$ is the flow velocity.

Our numerical example is taken to be the tracer transport in a 2D heterogeneous reservoir. The permeability is extracted from the SPE 10 top layer permeability model as shown in Fig. (4.1). The fine grid is $50 \times 50$ and coarse grid is $10 \times 10$. The field is initially saturated with ideal gas at a dimensionless pressure of 1. We have constant pressure boundary conditions on the west and east sides of 100 and 1, respectively. The north and south sides are no-flow boundaries. At the left boundary an ideal tracer is injected at constant concentration, $s_{inj} = 1$.

Figure 4.1: Logarithm permeability field for tracer transport problem
For the transport equation, the characteristic time is defined as

$$
\tau = \frac{\bar{\phi} \bar{\mu} L^2}{k \Delta p}.
$$

(4.11)

Fig. (4.2) shows a comparison of the tracer concentration profiles at three time steps. Very good agreement between the fine-scale and multiscale concentration solutions are obtained. The relative error of concentration is defined by

$$
\epsilon_s = \frac{\|s^{ms} - s^f\|_2}{\|s^f\|_2},
$$

(4.12)

where $s^{ms}$ is the concentration solution from OBMM and $s^f$ is the reference fine scale solution. In this case, $\epsilon_s$ in the three time steps is 3.6%, 3.2%, and 1.1%, respectively.

Figure 4.2: 2D tracer concentration profiles at different time
4.2.2 Two-phase flow problem

We then study the displacement of one phase by another phase with different compressibility. We consider injecting pure water into a 2D heterogeneous reservoir that is initially saturated with 50% water and 50% gas. Take the same permeability field, grid and boundary condition as in the previous section. The relative permeabilities are modeled as quadratic functions of saturation. Assume water is incompressible, $b_w = 1$, and gas is ideal, $b_g = p/b_g0$.

The pressure and saturation fields are shown in Fig. (4.3) and Fig. (4.4), respectively. The overall match for pressure and saturation is quite good. The $L^2$ relative error for pressure and saturation, denoted by $\epsilon_p$ and $\epsilon_s$, in the three time steps are listed in Table (4.2.2). The pressure error, $\epsilon_p$, decreases with time, which is consistent with the results in Chapter 3. The saturation error, $\epsilon_s$, increases with time as the velocity front sweeps more cells. However, the error is still small at later time, which demonstrates the accuracy of the OBMM for multiphase problems.

<table>
<thead>
<tr>
<th>time ($\times \tau$)</th>
<th>$\epsilon_p$</th>
<th>$\epsilon_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>1.1%</td>
<td>0.82%</td>
</tr>
<tr>
<td>0.1</td>
<td>0.53%</td>
<td>0.92%</td>
</tr>
<tr>
<td>1</td>
<td>0.50%</td>
<td>1.7%</td>
</tr>
</tbody>
</table>

Table 4.1: $L^2$ norm relative error of pressure and saturation in imbibition problems
Figure 4.3: Pressure field of two-phase problem

Figure 4.4: Saturation field of two-phase problem
Chapter 5

Conclusions

We developed an operator based multiscale method (OBMM) that serves as a general algebraic multiscale framework. We analyzed the OBMM for both elliptic and parabolic problem. For elliptic problems, we construct an algebraic framework that is equivalent to multiscale methods to existing MsFEM \[6\] and MsFVM \[8\] approach by choosing appropriate prolongation and restriction operators. For parabolic problems, the constructed compressibility matrices represent the homogenized compressibility by taking the basis functions as weight functions.

We have run extensive test cases for single phase flow problems. The OBMM method produces accurate results for flow in both homogeneous and heterogeneous permeability fields with highly compressible fluid properties. The efficiency of the OBMM is also analyzed, which shows that it is computationally efficient for large-scale problems. Moreover, the OBMM is very suitable for parallel computation, and using an adaptive criteria for updating the basis functions is expected to greatly improve the efficiency.

The OBMM is readily extendible to multiphase flow problems. An IMPES scheme using the OBMM was constructed. Numerical studies for tracer transport and multiphase displacement show the accuracy of the OBMM.

Future research on the OBMM includes more intensive tests for multiphase flow
problems. It is also of strong interest to include more physics, such as blackoil, well models, gravity and capillarity. Another challenging aspect is the development of multiscale methods for nonlinear transport equations.
Appendix A

FEM-based OBMM

A.1 Construction of OBMM based finite element method

We develop the FEM-based multiscale method. The prolongation operator is the same as the FVM-based multiscale method. The difference lies in the restriction operator.

First, define the finite element space using the basis function expressed from a global point of view,

\[ V_h = \text{span}\{\phi_A : A = 1, \ldots, N\}. \]  

(A.1)

The Galerkin FEM requires

\[ \int_{\Omega} \phi_A \nabla (\lambda \nabla p) = \int_{\Omega} \phi_A c(x) \frac{\partial p}{\partial t} \quad A = 1, \ldots, N. \]  

(A.2)

Recall that the basis function \( \phi_A \) is defined on underlying fine cells, and we have

\[ \phi_A(x_a) = R_{a,A}. \]  

(A.3)

Then, assume the fine-scale discretization matrix is of the form of Eq. (2.4), then for
the left hand side of Eq. (A.2), we obtain

\[ \int_{\Omega} \phi A \nabla \cdot (\lambda \nabla p) \, dx = \sum_{a=1}^{n} \phi_A(x_a) \sum_{b=1}^{n} T_{f_{a,b}} p_b \]

\[ = \sum_{a=1}^{n} \phi_A(x_a) \sum_{b=1}^{n} T_{f_{a,b}} \sum_{B=1}^{N} \phi_B(x_b) p_{c_B} \]

\[ = \sum_{a=1}^{n} \phi_A(x_a) \sum_{b=1}^{n} T_{f_{a,b}} \sum_{B=1}^{N} P_{b,B} p_{c_B} \]

\[ = \sum_{a=1}^{n} \phi_A(x_a) \left[ T_f \mathbf{P} p_c \right]_a. \]  

(A.4)

Eq. (A.4) suggests taking the restriction operator as

\[ \mathbf{R} = \mathbf{P}^T = \mathbf{R}_{A,a} = \phi_A(x_a). \]  

(A.5)

Thus we have a succinct form for the left hand side of Eq. (A.2),

\[ \int_{\Omega} \phi_A \nabla \cdot (\lambda \nabla p) = \left[ \mathbf{R} T_f \mathbf{P} p_c \right]_A. \]  

(A.6)

Now, we consider the time dependent term on the right hand side of Eq. (A.2). As mentioned before, we use the first-order backward Euler scheme for time discretization. Then, the \( p(x) \) in Eq. (A.7) is at time step \( n + 1 \). We neglect the time step in the notation for convenience, and we have

\[ \int_{\Omega} \phi_A c(x) p(x) \, dx = \sum_{a=1}^{n} \phi_A(x_a) \sum_{b=1}^{n} C_{f_{a,b}} p_b \]

\[ = \sum_{a=1}^{n} \phi_A(x_a) \sum_{b=1}^{n} C_{f_{a,b}} \sum_{B=1}^{N} \phi_B(x_b) p_{c_B} \]

\[ = \sum_{a=1}^{n} \sum_{b=1}^{n} C_{f_{a,b}} \sum_{B=1}^{N} P_{b,B} p_{c_B} \]

\[ = \left[ \mathbf{R} C_f \mathbf{P} p_c \right]_A. \]  

(A.7)
Therefore, if we take the FEM as the coarse-scale scheme and take the multiscale operator in the form of Eq. (A.5), the coarse-scale system can be written as

\[ T_c p_c = C_c p_c + r_c \]  

(A.8)

with

\[ T_c = \mathcal{R} T_f \mathcal{P}, \]
\[ C_c = \mathcal{R} C_f \mathcal{P}, \]  

(A.9)
\[ r_c = \mathcal{R} r_f. \]
A.2 Analysis of FEM-based OBMM

A.2.1 OBMM for elliptic equations

In this subsection, we analyze a 1-dimensional elliptic problem (i.e., Eq. (2.2) with $c = 0$). First use the original MsFEM. Considering only the $I$th row of the coarse-scale matrix (i.e, stiffness matrix in the context of finite element method [7]). Using the FEM, one can write

$$T_{I,J} = \int_{\Omega} \lambda \frac{\partial \phi_I}{\partial x} \frac{\partial \phi_J}{\partial x} \, dx.$$

(A.10)

Obviously, $T_{I,J}$ is nonzero only if $J = I - 1, I, I + 1$.

To be consistent with the fine-scale discretization scheme, we calculate the derivatives in Eq. (A.10) by central differences. Since in each fine cell we have only one unknown, we use a piecewise constant quadrature rule. Thus, using MsFEM, one can obtain the stiffness matrix with elements as

$$T_{cI,I-1} = \sum_{k=i-3}^{i-1} \lambda_{k+1/2} [\phi_{D-1}^1(l+1) - \phi_{D-1}^1(l)] [\phi_{D-1}^2(l+1) - \phi_{D-1}^2(l)], \quad (l = k - (i - 4))$$

(A.11)

where $l$ denotes the local index of a fine cell in element $D - 1$.

$$T_{cI,I} = \sum_{k=i-3}^{i-1} \lambda_{k+1/2} [\phi_{D-1}^1(l+1) - \phi_{D-1}^1(l)] [\phi_{D-1}^2(l+1) - \phi_{D-1}^2(l)] + \sum_{k=i}^{i+2} \lambda_{k+1/2} [\phi_{D-1}^1(m+1) - \phi_{D-1}^1(m)] [\phi_{D-1}^2(m+1) - \phi_{D-1}^2(m)], \quad (m = k - (i - 1))$$

(A.12)

where $m$ denotes the local index of a fine cell in element $D$.

$$T_{cI,I+1} = \sum_{k=i}^{i+2} \lambda_{k+1/2} [\phi_{D-1}^1(m+1) - \phi_{D-1}^1(m)] [\phi_{D-1}^2(m+1) - \phi_{D-1}^2(m)], \quad (m = k - (i - 1))$$

(A.13)

Now, consider the FVM-based OBMM. Again, we only care about the $I$th row of the coarse-scale transmissibility matrix. The $I$th row of the multiscale restriction
operator is

\[
\mathcal{R}_i = (\ldots \phi_{D-1}^2(2) \phi_{D-1}^2(3) 1 \phi_D^1(2) \phi_D^1(3) \ldots),
\]

(A.14)

where the first nonzero element lies in position \(i - 2\). Therefore we only need to consider the \(i - 2, i - 1, \ldots i + 2\) rows of the fine-scale transmissibility matrix. To be consistent, we also choose central differences as the fine-scale numerical scheme. The sub-matrix is

\[
\begin{pmatrix}
\lambda_{i-\frac{5}{2}} & -(\lambda_{i-\frac{5}{2}} + \lambda_{i-\frac{3}{2}}) & \lambda_{i-\frac{3}{2}} & 0 & 0 & 0 & 0 \\
0 & \lambda_{i-\frac{3}{2}} & -(\lambda_{i-\frac{3}{2}} + \lambda_{i+\frac{1}{2}}) & \lambda_{i+\frac{1}{2}} & 0 & 0 & 0 \\
0 & 0 & \lambda_{i+\frac{1}{2}} & -(\lambda_{i+\frac{1}{2}} + \lambda_{i+\frac{3}{2}}) & \lambda_{i+\frac{3}{2}} & 0 & 0 \\
0 & 0 & 0 & \lambda_{i+\frac{3}{2}} & -(\lambda_{i+\frac{3}{2}} + \lambda_{i+\frac{5}{2}}) & \lambda_{i+\frac{5}{2}} & 0 \\
0 & 0 & 0 & 0 & \lambda_{i+\frac{5}{2}} & -(\lambda_{i+\frac{5}{2}} + \lambda_{i+\frac{7}{2}}) & \lambda_{i+\frac{7}{2}} \\
\end{pmatrix}
\]

(A.15)

For \(\mathcal{P}\), we only need to consider rows \(I - 1, I, I + 1\) and columns \(i - 3, i - 2, \ldots, i + 3\). The sub-matrix is

\[
\begin{pmatrix}
\ldots & \ldots & \ldots & \ldots & \ldots \\
i - 3 & \ldots & 1 & 0 & 0 \\
i - 2 & \ldots & \phi_{D-1}^1(2) & \phi_{D-1}^2(2) & 0 \\
i - 1 & \ldots & \phi_{D-1}^1(3) & \phi_{D-1}^2(3) & 0 \\
i & \ldots & 0 & 1 & 0 \\
i + 1 & \ldots & 0 & \phi_D^1(2) & \phi_D^2(2) \\
i + 2 & \ldots & 0 & \phi_D^1(3) & \phi_D^2(3) \\
i + 3 & \ldots & 0 & 0 & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]

(A.16)

It can be shown by basic algebraic manipulation that the resulting coarse-scale system has only three nonzero elements in the \(I^{th}\) row that are identical to the obtained using MsFEM. Thus, we have shown that the coarse-scale compressibility matrix from FEM-based OBMM is actually the stiffness matrix in MsFEM.
A.2.2 OBMM for parabolic problems

Construction of the elliptic part of the coarse-scale system has been discussed in the previous subsection. We focus on the accumulation part and analyze the coarse-scale compressibility matrix constructed by the OBMM. We have shown that

\[
[R_C f P_c]_I = \int_\Omega N_I(x)c(x)p(x) \, dx. \tag{A.17}
\]

Again, considering the 1D multiscale grid as shown in Fig. (2.3), we know \( R_{I,i} \) is nonzero if and only if the fine block \( i \) is within dual blocks \( D - 1 \) and \( D \). Moreover \( P_{t,J} \) is only nonzero only when \( J \) is \( I - 1 \), \( I \), or \( I + 1 \). Thus, Eq. (A.17) can be written as

\[
[R_C f P_c^n]_I = (C_{c_{I-I-1}} p_{c_{I-1}} + C_{c_{I}} p_{c_{I}} + C_{c_{I+1}} p_{c_{I+1}}) \Delta x_f \tag{A.18}
\]

with,

\[
C_{c_{I-I-1}} = \sum_i R_{I,i} C_{f_{I,i}} P_{i,I-1} \\
\approx \int_\Omega \phi_I(x)\phi_{I-1}(x)c(x) \, dx,
\]

\[
C_{c_{I}} = \sum_i R_{I,i} C_{f_{I,i}} P_{i,I} \\
\approx \int_\Omega \phi^2_I(x)c(x) \, dx, \tag{A.19}
\]

\[
C_{c_{I+1}} = \sum_i R_{I,i} C_{f_{I,i}} P_{i,I+1} \\
\approx \int_\Omega \phi_I(x)\phi_{I+1}(x)c(x) \, dx.
\]

Clearly, the coarse-scale compressibility matrix constructed by the FEM-based OBMM is identical to the mass matrix in the context of the finite element method \cite{7}.

Now consider the special case where \( \lambda = 1, c = 1 \), we know that when using
piecewise linear element, the mass matrix in standard FEM should be

\[
\begin{align*}
C_{cI,I-1} &= 1/6, \\
C_{cI,I-1} &= 2/3, \\
C_{cI,I-1} &= 1/6. 
\end{align*}
\]  
\(\text{(A.20)}\)

When we use FEM-based OBMM, we can get

\[
\begin{align*}
C_{cI,I-1} &= \frac{1}{n_{cf}} \sum_{i} R_{I,i} C_{f_{i,i}} P_{i,I} \left( C_{cI,I-1} - 1 \right) \\
&= \frac{1}{n_{cf}} \sum_{i=1}^{n_{cf}} \frac{i}{n_{cf}} \frac{n_{cf} - i}{n_{cf}} \\
&= \frac{n_{cf}^2 - 1}{6n_{cf}^2} \\
&\Rightarrow \frac{1 - 1/n_{cf}^2}{6} \to \frac{1}{6} \text{ when } n_{cf} \to \infty. 
\end{align*}
\]  
\(\text{(A.21)}\)

Also,

\[
\begin{align*}
C_{cI,I} &= \frac{1}{n_{cf}} \sum_{i} R_{I,i} C_{f_{i,i}} P_{i,I} \\
&= \frac{1}{n_{cf}} \sum_{i=1}^{n_{cf}} \frac{i^2}{n_{cf}} - 1 \\
&= \frac{2n_{cf}^2 + 1}{3n_{cf}^2} \\
&\Rightarrow \frac{2 + 1/n_{cf}^2}{3} \to \frac{2}{3} \text{ when } n_{cf} \to \infty. 
\end{align*}
\]  
\(\text{(A.22)}\)

By symmetry, we get

\[
\begin{align*}
C_{cI,I+1} &= C_{I,I-1} \\
&= \frac{n_{cf}^2 - 1}{6n_{cf}^2} \\
&\Rightarrow \frac{1 - 1/n_{cf}^2}{6} \to \frac{1}{6} \text{ when } n_{cf} \to \infty. 
\end{align*}
\]  
\(\text{(A.23)}\)
Therefore, we have verified that the coarse-scale compressibility matrix constructed by FEM-based OBMM is identical to the mass matrix in FEM. The approximation error is due to the fact that we solve the basis functions numerically (central difference in this example) and the value is piecewise constant in each fine cell. Therefore, the integration is only first order accuracy approximation in this case.
Bibliography


