TEMPERATURE TRANSIENT ANALYSIS IN NATURALLY FRACTURED RESERVOIRS

A THESIS
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Jose Eduardo Ramirez Lopez Miro
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I certify that I have read this thesis and that, in my opinion, it is fully adequate in scope and quality as partial fulfillment of the degree of Master of Science in Petroleum Engineering.

(Roland N. Horne) Principal Adviser
Abstract

Naturally fractured reservoirs are very important for the oil industry and have been thoroughly studied in the literature. Historically in the well testing community, pressure transient analysis has been the main method to characterize the rock and fluid properties of a naturally fractured reservoir.

In this study we investigated the temperature behavior in a well producing from a naturally fractured reservoir. We derived early- and late-time solutions that describe the temperature behavior under the assumption that the diffusion term can be neglected. To determine the range of applicability of these approximations we developed a numerical model that takes into account both diffusion and convection heat transport mechanisms as well as heat flow between fracture and matrix. It was found that only four parameters are needed to describe the early and late-time temperature behavior if the pressure behavior is already known. Additionally it was found that the normalized dimensionless temperature behaves remarkably similar to the dimensionless pressure, showing two parallel straight lines in a semilog plot with slope equal to one half and with separation related to the ratio of heat storage between the fracture medium and the reservoir. Furthermore it seems that the transition period that connects the two straight lines is given by another straight line with slope equal to one quarter.

In conclusion, the temperature measurements were found to be a robust and reliable source of information to characterize a double porosity reservoir and the transition period that appears in the temperature transient seems to be a promising area of research.
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Chapter 1

Introduction

According to the 2016 BP Energy Outlook [11] significantly more energy will be required over the next twenty years to enable the world economy to grow and prosper. The Energy Outlook considers a base case where fossil fuels will provide almost 80% of total energy supplies in 2035. Furthermore the oil industry is facing a very difficult moment with very unstable oil prices; for example, the West Texas Intermediate reached a 12 year low on February 2016 with $26.14 USD per barrel (Figure 1.1). This difficult scenario prevents oil companies from investing money in high risk projects such as deep water oil fields or the so-called unconventional resources. A more viable solution is to sustain production of existing fields; as can be seen from Figure 1.2 most of the proved oil reserves are carbonate reservoirs and due to their geologic characteristics, carbonate reservoirs are highly heterogeneous, usually containing natural fractures and other geologic features such as vugs and caverns (see [33]). Therefore is of the utmost importance to properly characterize naturally fractured reservoirs for an efficient and sustainable oil production in the present and future of the oil industry.

Naturally fractured reservoirs have been a popular research topic since the last century. In the well testing community the typical approach is to characterize naturally fractured reservoirs using pressure measurements, the pressure data are matched with a suitable reservoir model (a mathematical representation of the underlying physics) and once the match is achieved, the reservoir engineer can infer the reservoir rock
properties such as porosity, compressibility, permeability, drainage area, etc. One of the main problems is that the rock properties obtained in this type of analysis are highly uncertain due to the fact that pressure measurements always contain some degree of noise and the mathematical model is often ill-posed, in the sense that there is a large spectrum of possible values that can lead to the same result. The simplest way to reduce this uncertainty is to match the model subject to some constraints such as the information obtained from well logs, core experiments, seismic measurements, etc.

1.1 Statement of the Problem

Recent studies have shown the huge potential that temperature measurements contain to characterize rock properties (see [16, 26]). Inspired by these new discoveries, the objective of this work was to determine how the temperature information can be used to characterize the heterogeneous properties of naturally fractured reservoirs. To accomplish this goal, we represented the naturally fractured reservoir using a double porosity model, and solved the energy and mass conservation equations to study the pressure and temperature transients as function of the rock properties.
Figure 1.2: Carbonate reservoirs around the world [27].
1.2 Report Outline

This report is organized in seven chapters including the Introduction and Concluding Remarks.

Chapter 2 reviews the history and development of analytical solutions and methodologies to analyze naturally fractured reservoirs using pressure transient analysis and gives an overview of the importance of temperature information to determine reservoir and fluid properties in the context of well testing.

Chapter 3 shows the general conservation laws: mass conservation and energy conservation equations for a unit of rock volume. And derives the equations that govern the evolution of temperature and pressure of a fluid flowing in porous media.

Chapter 4 derives the pressure and temperature equations for the fracture and matrix media. Then shows the analysis of the equations and gives some analytical solutions to the fracture problem.

Chapter 5 shows the energy and pressure equations in dimensionless form and describes a numerical model that considers both convection and diffusion heat transport mechanism.

Chapter 6 compares the analytical solutions derived in Chapter 4 with the numerical solution obtained from the numerical model described in Chapter 5. Moreover shows some sensitivity analysis, range of applicability of the analytical solutions and some results concerning the analysis of the semilog plot and the double logarithmic plot of the dimensionless pressure and temperature.

Chapter 7 gives a summary of the report, final conclusions and direction for future work to expand the theory of temperature transients in double porosity reservoirs.
Chapter 2

Literature Review

2.1 Pressure Transient Analysis

Naturally fractured reservoirs (NFRs) have been studied thoroughly using pressure transient analysis. The first mathematical model was developed in 1960 by Barenblatt and Zheltov [4]. They recognized that trying to model a NFR by representing fractures explicitly in a mathematical model is a highly nontrivial task; furthermore, is virtually impossible to determine the fracture network topology accurately even with current state of the art knowledge (see [1]); therefore, to honor the heterogeneous behavior of a NFR, Barenblatt and Zheltov represented the NFR as the superposition of two distinct media, a high flow capacity but low fluid storage medium that represents the fracture, and a high fluid storage but low flow capacity medium that represents the matrix. Thus at each point in space they considered two distinct pressures, the fracture pressure and the matrix pressure. In addition to this, the model considered that the interporosity flow from the matrix to the fracture ($q_{MF}$) occurred in a pseudosteady state, i.e. the fluid velocity is proportional to the pressure drop between the two media as shown in Equation (2.3) and that all the production comes from the fracture medium. This model is known in the literature as the double porosity model, the fracture and matrix media are referred as secondary and primary porosity
respectively. The complete set of equations of this model is:

\[
\phi_{FcF} \frac{\partial P_F}{\partial t} = \nabla \cdot \left( \frac{k_F}{\mu} \nabla P_F \right) + q_{MF} \tag{2.1}
\]

\[
\phi_{FcM} \frac{\partial P_M}{\partial t} = \nabla \cdot \left( \frac{k_M}{\mu} \nabla P_M \right) - q_{MF} \tag{2.2}
\]

\[
q_{MF} = -\frac{\alpha}{\mu} (P_F - P_M) \tag{2.3}
\]

where the subindices \( F, M \) stand for fracture and matrix properties respectively and \( \alpha \) is a dimensionless variable that controls the interporosity flow between fracture and matrix. Later in the same year, Barenblatt, Zheltov and Kochina [5] simplified and solved Equations (2.1) and (2.2) by assuming that the fluid storage capacity in the fracture medium is negligible compared to the storage capacity within the matrix, thus neglecting the time derivative in the fracture Equation (2.1), and because the flow velocity within the matrix medium is expected to be very small due to the low matrix permeability, they discarded the divergence term of the matrix Equation (2.2).

In 1963, J. E. Warren and P. J. Root [32] considered the time derivative in the fracture Equation (2.1), enabling the formal study of pressure transients in double porosity reservoirs. In their work Warren and Root [32] developed a methodology to analyze the pressure transient of double porosity reservoirs under the assumptions that the matrix medium has a sufficiently low permeability to neglect the divergence term in the matrix equation (2.2), all the oil production comes from the fracture medium and the flow between fracture and matrix occurs under pseudosteady state conditions. In their analysis, Warren and Root introduced two dimensionless variables that, under the given assumptions, govern the fluid flow of fluids and distinguish a double porosity reservoir from a homogeneous reservoir model. These variables are the so-called transmissivity ratio \( \lambda \) and storativity ratio \( \omega \) defined as:

\[
\lambda = \alpha^* \frac{k_M}{k_F} \frac{r_w^2}{r_w} \tag{2.4}
\]

\[
\omega = \frac{\phi_{FcF}}{\phi_{FcF} + \phi_{McM}} \tag{2.5}
\]
where $\alpha^*$ is a dimensional parameter (dimension length squared) that represents the geometry of the matrix-fracture boundary and controls the flow between the two media. $\alpha^*$ is related to the dimensionless parameter $\alpha$ introduced by Barenblatt and Zheltov in 1960 by the following equation:

$$\alpha = \alpha^* k_m. \hspace{1cm} (2.6)$$

An important characteristic is that for the value of $\omega = 1$ the solution given by Warren and Root reduces to the solution obtained by Van Everdingen and Hurst [30] for a homogeneous reservoir and when $\omega = 0$ they recovered the solution obtained by Barenblatt, Zheltov and Kochina in [5]. The main contribution of Warren and Root is that in a semilog plot, the pressure transient shows two parallel lines, whose slopes are function of the fracture permeability, and the separation is equal to $\log(1/\sqrt{\omega})$; moreover, for values of $\lambda$ larger than $10^{-3}$ the level of heterogeneity or contrast in transmissivities is so small that the reservoir appears to have only single porosity [19]. Using this results, Warren and Root proposed a method to determine the fracture permeability, storativity ratio and transmissivity ratio.

In 1969 Kazemi [20] developed a mathematical model to describe the interporosity flow between matrix and fracture media under transient flow conditions. He realized that the double porosity reservoir behavior could be replicated by considering a special case of the idealized fractured reservoir of Warren and Root [32], where the reservoir consists of a set of horizontal fractures, separated by matrix layers as shown in Figure 2.1. Kazemi used a numerical approach to solve the pressure equation in his idealized double porosity model, he concluded that Warren and Root methodology [32] is only valid when the contrast between fracture and matrix flow capacities is large, on the contrary when the flow capacities are similar the double porosity reservoir behaves as an equivalent homogeneous reservoir and only one straight line is observed in the semilog plot, and for long times the fractured reservoir pressure transient approaches the pressure response of an equivalent homogeneous reservoir system.

In 1976 the transient interporosity flow was studied by de Swaan [14] from an analytical point of view. De Swaan [14] visualized the matrix medium, as small blocks
Figure 2.1: Visualization of a double porosity model of a cylindrical naturally fractured reservoir with a vertical well producing at its center.
CHAPTER 2. LITERATURE REVIEW

with different geometries (slabs and spheres), surrounded by the fracture medium. For early time, he considered that due to the “tight” nature of the matrix blocks (low matrix conductivity), the pressure transient is being influenced only by the fracture medium, and used the Van Everdingen and Hurst solution \[30\] to model the pressure transient for early times. For late times the matrix blocks’s influence becomes important, to model this phenomenon de Swaan used the already well known transfer function \((\psi)\) that describe the flow through the surface of a block with a given geometry from the heat conduction theory (see \[12\]) to express the interporosity flow \((q_{MF})\) as a convolution of the transfer function \((\psi)\) and the fracture pressure \((P_F)\) multiplying the inverse of the matrix block surface area \((A_M)\) and inverse of the fracture thickness \((h_F)\) as shown in Equation (2.7).

\[
q_{MF}(P_F, t) = -\frac{2}{A_M h_F} \int_0^t \frac{dP_F}{d\tau} \psi(t - \tau) d\tau. \tag{2.7}
\]

De Swaan’s results are in perfect agreement with what was found by Kazemi \[20\], nevertheless he did not give an equation to describe the transient period between the two semilog parallel lines. Based on de Swan’s work, Najurieta \[21\] expanded the analysis of the transient interporosity flow, by giving a unique equation that describes the pressure behavior in a double porosity reservoir for most practical ranges of time (early, transition and late times) and position within the reservoir. He concluded that the double porosity reservoir can be described using four distinct parameters, and gave a line source type equation with a composite hydraulic diffusivity that is a function of fracture transmissivity, fracture storage, matrix storage and an extra parameter that takes into account the size and diffusivity of the matrix. Using this line source type equation Najurieta \[21\] gave a methodology to do the semilog analysis of double porosity reservoirs under drawdown and buildup conditions.

After this, in 1982 Cinco and Samaniego \[13\] gave a thorough analysis of the behavior of a double porosity reservoir. They determined that the transient pressure behavior can be analyzed using the fracture storage parameter \(\omega\), a dimensionless
matrix hydraulic diffusivity $\eta_{mD}$ and the dimensionless fracture area $A_{FD}$:

$$\eta_{mD} = \left(\frac{k}{\phi c_t} \right)_M \left(\frac{\phi c_t}{k} \right)_F \frac{\rho^2}{h_M^2},$$  

$$A_{FD} = \frac{A_{Fb} h_M V_F}{V_M},$$

(2.9)

where $h_M$ is the matrix slab thickness, $A_{Fb}$ is the fracture area per unit bulk volume, and $V_F, V_M$ are the bulk volumes of fracture and matrix respectively. Furthermore, there are three distinct periods in the pressure transient, fracture storage dominated period, transient matrix linear flow period and pseudosteady state matrix dominated period. Perhaps their main contribution is that the matrix linear flow period appears as a semilog straight line with slope equal to half the slope of the early and late time semilog lines that have been found since the work done by Warren and Root [32].

In the analysis of double porosity reservoirs, reservoir engineers usually rely on the identification of the first semilog line to identify important information of the fracture medium and confirm that the reservoir has double porosity behavior. Nevertheless, due to wellbore storage and skin effects this portion of the pressure transient is usually distorted and the first semilog line does not appear, to solve this complication in 1980 Bourdet and Gringarten [10] gave a useful method that relies in a double logarithmic plot to create a new set of type curves and aid in the analysis of NFRs whose behavior was difficult to match using semilog analysis. Later Bourdet et al. ([9], [8]) applied the newly discovered pressure derivative as a robust tool to estimate the double porosity parameters $\omega$ and $\lambda$ using the position of the minimum (dip) in the derivative plot.

### 2.2 Temperature Transient Analysis

Historically, the application of temperature data in reservoir engineering used to be restricted to wellbore models, with perhaps the first model given by Ramey [25]. Ramey gave an algebraic expression that predicts the temperature of a fluid flowing from the well head to the bottom of the well as function of position and time. In his work, he took into account the heat losses due to conduction between the fluid and
the wellbore and then between the wellbore and the surrounding rock.

In the context of geothermal reservoirs Gringarten et al. [18] used the temperature information and a simple model of a geothermal reservoir with infinite fractures, to derive an analytical solution and predict heat production with different scenarios.

More recently due to advances in the technology of temperature sensors, some studies have investigated new methods to use temperature information in reservoir characterization. In 2009 Sui et al. [29] studied the use of transient temperature data at multiple locations combined with the pressure transient from a single location, to determine the properties of a layered reservoir, in particular the layer permeability and skin. Later in 2010 Duru and Horne [17] developed a temperature model that included the convection, diffusion and fluid expansion effects for single and multiphase flow. With this model they matched the temperature signal recorded from a permanent downhole gauge and determine the reservoir properties with success. The next year App and Yoshioka [2] performed a theoretical study of the influence of permeability in the temperature behavior. Their analysis is based on dimensionless variables (most notably a dimensionless Joule-Thomson coefficient and Peclet number) and developed an analytical theory for a steady state scenario. App and Yoshioka also studied the transient behavior of the temperature but only in the context of numerical simulation.

In 2012 Wang [31] did a thorough study on the uses of distributed temperature surveys. For the analysis, he developed a thermal reservoir simulator coupled with a wellbore model. Wang showed that the temperature information can be used to determine the flowrate of a production well. Furthermore Wang showed that the temperature signal is a better source of information than the pressure signal to characterize the near wellbore region. In the same year K.M. Muradov and D.R. Davies [24] developed a methodology to analyze the temperature transients for a horizontal, multi-zone, intelligent well. They concluded that the temperature information does not need to be considered only as a complement to the pressure information, but is sufficient for the analysis if the thermal reservoir properties are known. Moreover, Muradov and Davies concluded that because the temperature information is less affected by downstream process due its convective nature, the temperature signal is better than the pressure signal to characterize each individual zone in a multizone
well completion.

In recent years P. Ribeiro [26] studied the behavior of temperature during a hydraulic fracturing process. For her analysis she developed a numerical model that reproduces the pressure and temperature behavior during and after the hydraulic fracture process. The model also considers fracture growth and contraction depending on the effective stress. P. Ribeiro found that the temperature signal can help determine the fracture half length as well as the reservoir permeability.
Chapter 3

Mathematical Preliminaries

To study the relationship between pressure and temperature with the fluid and rock properties, we considered a mechanistic model that is deduced from the mass and energy conservation laws. It is important to recall that under this analysis we are always working on a Darcy scale, in the sense that all the rock and fluid properties are average values over a representative elemental volume such that the continuum assumption is valid (see Bear [6] Chapter 1). In this analysis we restrict the analysis to a normal Euclidean space.

3.1 Mass conservation

Consider a control volume Ω with closed boundary ∂Ω that moves with the fluid such that contains a constant amount of fluid mass. The amount of fluid mass per unit rock volume is expressed as ϕρ_f where ρ_f is the fluid’s density and ϕ is the rock porosity; therefore, the total amount of fluid mass contained inside Ω is given by:

\[ m = \int_{\Omega} \rho_f \phi d\Omega. \] (3.1)

We have specified as constraint that the mass inside Ω is constant; therefore, we can differentiate Equation (3.1) with respect to time and use the generalized Leibniz rule
to obtain:

\[ 0 = \frac{dm}{dt} = \frac{d}{dt} \int_{\Omega} \rho_f \phi d\Omega = \int_{\Omega} \left( \frac{\partial (\phi \rho_f)}{\partial t} \right) dV + \int_{\partial \Omega} \phi \rho_f (\vec{v} \cdot \vec{n}) dS \]  

(3.2)

where \( \vec{v} \) is the velocity of the boundary \( \partial \Omega \) (by constraint is also the fluid interstitial velocity), and \( \vec{n} \) is the unit outward normal. Recalling that the Darcy’s velocity (\( \vec{u} \)) is related to the interstitial velocity (\( \vec{v} \)) by:

\[ \vec{u} = \phi \vec{v}, \]  

(3.3)

we obtain the mass conservation equation in integral form:

\[ \int_{\Omega} \frac{\partial (\phi \rho_f)}{\partial t} dV = - \int_{\partial \Omega} \rho_f (\vec{u} \cdot \vec{n}) dS. \]  

(3.4)

Using the divergence theorem to transform the surface integral into a volume integral we arrive to the following equation:

\[ \int_{\Omega} \left( \frac{\partial (\phi \rho_f)}{\partial t} + \nabla \cdot (\phi \rho_f \vec{v}) \right) dV = 0. \]  

(3.5)

Assuming that the argument inside the parenthesis is a continuous function, and because the integral is zero for any arbitrary control volume \( \Omega \), we conclude that the argument inside the parenthesis must be identically zero and the mass conservation equation in differential form follows:

\[ \frac{\partial (\phi \rho_f)}{\partial t} + \nabla \cdot (\rho_f \vec{u}) = 0. \]  

(3.6)

For convenience we define the new operator \( \frac{D}{Dt} \) that is defined as:

\[ \frac{D(\cdot)}{Dt} = \frac{\partial (\cdot)}{\partial t} + \vec{v} \cdot \nabla (\cdot), \]  

(3.7)

this operator is called the material derivative because it tells us the rate of change of a physical quantity experienced by a particle when the particle is moving with
velocity \vec{v}; notice we are using the interstitial velocity to define this operator. Using the material derivative in Equation (3.6) and considering that the porous medium is incompressible (constant porosity) we obtain:

\[
\frac{1}{\rho_f \phi} \frac{D(\phi \rho_f)}{Dt} = -\nabla \cdot \vec{v}. \tag{3.8}
\]

The term \( \frac{D(\phi \rho_f)}{Dt} \) expresses the rate of change of fluid mass per unit bulk volume experienced by a particle advected by the velocity field \( \vec{v} \), if \( \frac{D(\phi \rho_f)}{Dt} > 0 \) it means that there is more mass inside the same rock volume; that is, the fluid is being compressed, thus the divergence term in the right hand side \( \nabla \cdot \vec{v} \) can be interpreted as the relative expansion experienced by the fluid inside the reservoir rock. Is important to notice that from a pore scale point of view, when the fluid density is constant we have that \( \nabla \cdot \vec{v} = 0 \), nevertheless if the fluid density is assumed constant in Equation (3.8) the term \( \nabla \cdot \vec{v} \) can be nonzero if the porosity is changing; therefore, the interstitial velocity \( \vec{v} \) should not be interpreted as the actual fluid velocity inside the porous medium, but as an average apparent fluid velocity inside the porous medium.

### 3.1.1 Pressure Equation

To transform the mass conservation equation into something more meaningful in terms of measurable quantities we need to introduce a constitutive equation that relates the fluid density with the pressure and temperature. Here we consider one of the simplest models, let the density be described as a total differential of both pressure \( P \) and fluid temperature \( T_f \):

\[
d\rho_f = \left( \frac{\partial \rho_f}{\partial P} \right)_{T_f} dP + \left( \frac{\partial \rho_f}{\partial T_f} \right)_P dT_f, \tag{3.9}
\]

recalling the definitions of the fluid volumetric compressibility \( c_f \) and volumetric thermal expansion coefficient \( \beta_f \):

\[
c_f = \frac{1}{\rho_f} \left( \frac{\partial \rho_f}{\partial P} \right)_{T_f}, \quad \beta_f = -\frac{1}{\rho_f} \left( \frac{\partial \rho_f}{\partial T_f} \right)_P, \tag{3.10}
\]
we obtain:

\[ d\rho_f = c_f \rho dP - \beta_f \rho dT_f. \] (3.11)

Furthermore we need a relationship between the velocity and the measurable quantities pressure and temperature, this relationship is given by Darcy’s law, it says that the fluid volumetric flow rate per unit of rock area is proportional to the gradient of a potential function, and in the absence of body forces reduces to:

\[ \vec{u} = -\frac{1}{\mu} k \nabla P, \] (3.12)

where \( \mu \) is the fluid’s viscosity, \( k \) is the second-order symmetric permeability tensor and \( \nabla \) is the gradient operator. If we assume that the reservoir rock is incompressible, using the chain rule to transform the derivatives of fluid density in terms of derivatives with respect to temperature and pressure by using Equation (3.11) and combining Equations (3.12) with (3.6) we arrive to the mass conservation equation in terms of pressure and temperature:

\[ \phi \left( c_f \frac{\partial P}{\partial t} - \beta_f \frac{\partial T_f}{\partial t} \right) = \nabla \cdot \left( \frac{k}{\mu} \nabla P \right) - (c_f \nabla P - \beta_f \nabla T_f) \cdot \vec{u}. \] (3.13)

Using the material derivative (Equation (3.7)) the Equation (3.13) becomes:

\[ \phi c_f \frac{DP}{Dt} - \phi \beta_f \frac{DT_f}{Dt} = \nabla \cdot \left( \frac{k}{\mu} \nabla P \right) = -\nabla \cdot \vec{u}. \] (3.14)

A typical assumption in well test analysis, is to consider isothermal flow of a slightly compressible fluid such that we can neglect temperature dependence and the term that multiplies the velocity vector in the right hand side of Equation (3.13) to obtain the pressure equation. This assumption converts the material derivative of pressure into a partial derivative of pressure, linearizing the mass conservation equation and making it more suitable for engineering analysis, such that Equation (3.15) is perhaps
the most important equation in well testing for reservoir engineers.

\[ \phi c_f \frac{\partial P}{\partial t} = \nabla \cdot \left( \frac{k}{\mu} \nabla P \right). \tag{3.15} \]

### 3.2 Momentum conservation

The conservation of linear momentum, takes into account how the forces are acting on a fluid. Consider again a control volume \( \Omega \) that contains a constant amount of fluid particles. The forces we are going to consider are of two types:

- **Body forces.** These type of forces are caused by a force field, e.g. gravitational force, electromagnetic force, etc. This type of force acts on every mass element contained in the body, thus is proportional to the mass. For \( \Omega \) this body force is given by:

\[ \int_\Omega \rho \phi f \vec{g} dV, \tag{3.16} \]

where \( \vec{g} \) is the gravity vector.

- **Surface forces.** These forces act on the surface of an element, are conveniently expressed per unit area and can deform the body on which they are acting, examples of this type force are the shear stress and longitudinal stress well known in the theory of solid mechanics. To describe this force, we need to use the fluid stress tensor \( \tau \), defined as follows. Let \( \vec{f} \) be the force vector per unit area acting on a surface, then:

\[ f_i = \sum_j \tau_{ij} n_j \tag{3.17} \]

where \( f_i \) is the \( i_{th} \) component of the force vector \( \vec{f} \) and \( n_j \) is the \( j_{th} \) component of the unit outwards normal vector at the surface where the force is acting, Equation (3.17) is the definition for the fluid stress tensor. Recalling that we are working at Darcy scale, we need to introduce a factor that “upscales” the stress tensor, we achieve this by introducing an approximate ratio between fluid
surface area and bulk surface area similar to what is done in Darcy’s law where the relationship between interstitial velocity (\( \vec{v} \)) and Darcy’s velocity (\( \vec{u} \)) is given as \( \vec{u} = \vec{v}\phi \). Taking this into account the sum of all surface force acting on \( \Omega \) is:

\[
\int_{\partial\Omega} \phi \tau \cdot \vec{n} dS.
\]  

(3.18)

The amount of momentum inside \( \Omega \) is \( \int_{\Omega} \phi \rho_f \vec{v} dV \), where \( \vec{v} \) is the fluid interstitial velocity, thus we can reexpress the amount of momentum using Darcy’s velocity as \( \int_{\Omega} \rho \vec{u} dV \). Now Newton’s conservation of linear momentum tells us that the rate of change of momentum inside \( \Omega \) is equal to the sum of all acting forces, which leads to the following integral equation:

\[
\frac{d}{dt} \int_{\Omega} \rho_f \vec{u} dV = \int_{\Omega} \rho_f \phi g dV + \int_{\partial\Omega} \phi \tau \cdot \vec{n} dS.
\]  

(3.19)

It is convenient to recover a differential form of Equation (3.19), this can be achieved by using Leibniz rule to pass the time derivative inside the integral. Is important to recall that \( \Omega \) is moving with the fluid such that the total amount of mass inside \( \Omega \) is always conserved, thus an extra term will appear due to this motion and can be interpreted as a convective mechanism,

\[
\frac{d}{dt} \int_{\Omega} \rho_f \vec{u} dV = \int_{\Omega} \frac{\partial (\rho_f \vec{u})}{\partial t} dV + \int_{\partial\Omega} \rho_f \vec{u} (\vec{v} \cdot \vec{n}) dS.
\]  

(3.20)

Using the divergence theorem on Equation (3.20) and on the surface integral of Equation (3.19) we obtain:

\[
\frac{\partial (\rho_f \vec{u})}{\partial t} + \nabla \cdot (\rho_f \vec{v} \vec{u}) = \rho_f \phi g + \nabla \cdot (\phi \tau).
\]  

(3.21)

Now using the product rule of differentiation:

\[
\vec{v} \left( \frac{\partial (\rho_f \phi)}{\partial t} + \nabla \cdot (\rho_f \vec{u}) \right) + \rho_f \phi \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot (\nabla \vec{v}) \right) = \rho_f \phi g + \nabla \cdot (\phi \tau).
\]  

(3.22)
Using Equation (3.6) we see that the first parenthesis is identically zero. Using the material derivative (Equation (3.7)) the equation of linear momentum is:

$$\rho_f \phi \frac{D\vec{v}}{Dt} = \rho_f \phi \vec{g} + \nabla \cdot (\phi \tau).$$  \hspace{1cm} (3.23)

Taking the inner product of Equation (3.23) with the interstitial velocity and noticing that \( \vec{v} \cdot \frac{D\vec{v}}{Dt} = \frac{D}{Dt} \left( \frac{\vec{v} \cdot \vec{v}}{2} \right) \) we obtain the kinetic energy equation. Let \( e_K = \frac{\vec{v} \cdot \vec{v}}{2} \) be the kinetic energy of the fluid per unit mass, then the new equation is:

$$\rho_f \phi \frac{De_K}{Dt} = \rho_f \phi (\vec{v} \cdot \vec{g}) + \vec{v} \cdot (\nabla \cdot (\phi \tau)).$$  \hspace{1cm} (3.24)

Before proceeding is important to study the term \( \vec{v} \cdot (\nabla \cdot (\phi \tau)) \), we know that \( \nabla \cdot (\phi \tau) \) is the force per unit volume applied to the fluid, thus \( \vec{v} \cdot (\nabla \cdot (\phi \tau)) \) is the mechanical work done to the fluid that increases its kinetic energy. In general we interpret the total work done to the fluid as the sum of the work that contributes to the kinetic energy (mechanical work) and the part that contributes to another type of energy, if we use the product rule of differentiation we can rewrite this term as follows:

$$\vec{v} \cdot (\nabla \cdot (\phi \tau)) = \sum_{i,j} v_i \partial_{x_j} \phi \tau_{ij} = \sum_{i,j} \partial_{x_j} (\phi \tau_{ij} v_i) - \phi \tau_{ij} \partial_j v_i.$$  \hspace{1cm} (3.25)

From this is clear that \( \partial_{x_j} (\phi \tau_{ij} v_i) \) represents the total amount of work done to the fluid and \( \phi \tau_{ij} \partial_j v_i \) the part that does not contributes to the kinetic energy.

### 3.3 Energy conservation

The energy conservation equation, comes from the first law of thermodynamics, in simple terms it states that the rate of change of energy is equal to the sum of all the heat added minus the work done to the system. Here we consider a quasi-static thermodynamic state, such that the work done by the fluid to the system is equal to the work done by the system to the fluid.
3.3.1 Fluid Energy Equation

The total fluid energy contained in the control volume $\Omega$ can be expressed as the sum of the internal energy $e_I$, kinetic energy $e_K$ and potential energy $e_P = -\vec{g} \cdot \vec{r}$, where $\vec{r}$ is the position vector of every particle of fluid inside $\Omega$. The minus sign is added such that the potential energy is always positive and all these quantities are defined per unit mass of fluid, thus the total fluid energy in $\Omega$ is:

$$\int_{\Omega} \phi \rho_f(e_I + e_K + e_P) dV. \quad (3.26)$$

The total amount of heat added to the fluid inside $\Omega$ per unit area per unit time is expressed using the heat flux vector $\vec{q}_f$ as:

$$-\int_{\partial \Omega} \vec{q}_f \cdot \vec{n} dS. \quad (3.27)$$

The minus sign is added because the unit normal is pointing out of the control volume $\Omega$. The total amount of work done by the fluid per unit area was discussed in the previous section and is $-\vec{v} \cdot \tau$, then the total work done by the system to the fluid is:

$$\int_{\partial \Omega} \phi (\vec{v} \cdot \tau) \cdot \vec{n} dS. \quad (3.28)$$

If we combine all these relationships into the energy balance we obtain the first law of thermodynamics in integral form:

$$\frac{d}{dt} \int_{\Omega} \phi \rho_f (e_I + e_K + e_P) dV = -\int_{\partial \Omega} \vec{q}_f \cdot \vec{n} dS + \int_{\partial \Omega} \phi (\vec{v} \cdot \tau) \cdot \vec{n} dS. \quad (3.29)$$

Using the manipulations employed before to move from the integral form to the differential form we obtain:

$$\frac{\partial}{\partial t} (\phi \rho_f (e_I + e_K + e_P)) + \nabla \cdot (\phi \rho (e_I + e_K + e_P) \vec{v}) = -\nabla \cdot \vec{q}_f + \nabla \cdot (\phi \vec{v} \cdot \tau) \quad (3.30)$$
and using the mass conservation equation and the material derivative operator, we can recover the nonconservative form of the energy equation:

\[ \phi \rho_f \frac{D}{Dt}(e_I + e_K + e_P) = -\nabla \cdot \mathbf{q}_f + \nabla \cdot (\phi \mathbf{v} \cdot \tau) \]  \hspace{1cm} (3.31)

noticing that:

\[ \frac{De_p}{Dt} = -\frac{D(\mathbf{g} \cdot \mathbf{r})}{Dt} = -\mathbf{g} \cdot \frac{D\mathbf{r}}{Dt} = -\mathbf{g} \cdot \frac{D\mathbf{r}}{Dt} = -\mathbf{g} \cdot \mathbf{v} \]  \hspace{1cm} (3.32)

where we used the fact that the material derivative of the position vector is the fluid’s velocity and that the gravity vector is a constant. We can combine Equations (3.31), (3.32) and (3.24) to obtain the energy conservation equation in terms of internal energy only:

\[ \phi \rho_f \frac{De_I}{Dt} = -\nabla \cdot \mathbf{q}_f + \nabla \cdot (\phi \mathbf{v} \cdot \tau) - \mathbf{v} \cdot (\nabla \cdot (\phi \tau)). \]  \hspace{1cm} (3.33)

The term that appears in the right hand side is precisely the amount of work that does not contribute to the increase of kinetic energy as discussed previously in Section 3.2. Clearly it contributes to the increase of internal energy and can be seen as a heat source. Recalling the constitutive equation for a Newtonian fluid that relates the fluid stress to the fluid velocity:

\[ \tau_{ij} = -P \delta_{ij} + \mu (\partial_j v_i + \partial_i v_j) \]  \hspace{1cm} \text{incompressible fluid} \hspace{1cm} (3.34)

\[ \tau_{ij} = \left(-P - \frac{2}{3} \nabla \cdot \mathbf{v}\right) \delta_{ij} + \mu (\partial_j v_i + \partial_i v_j) \]  \hspace{1cm} \text{compressible fluid} \hspace{1cm} (3.35)

where \( \delta_{ij} \) is the Kronecker delta tensor and has the special property that:

\[ \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \]  \hspace{1cm} (3.36)

The work done that contributes to the internal energy can be expressed as

\[ \nabla \cdot (\phi \mathbf{v} \cdot \tau) - \mathbf{v} \cdot (\nabla \cdot (\phi \tau)) = -P \nabla \cdot \mathbf{u} + \Phi = -P \nabla \cdot \mathbf{u} - \mathbf{u} \nabla P \]  \hspace{1cm} (3.37)
where the function $\Phi$ is the so-called viscous dissipation function, and a thorough explanation is given in Appendix A. Using this result the equation for the internal energy is:

$$
\phi \rho_f \frac{D e_I}{D t} = -\nabla \cdot \vec{q}_f - P \nabla \cdot \vec{u} - \vec{u} \cdot \nabla P.
$$

(3.38)

Even though this is a very important equation, the internal energy ($e_I$) is not easy to handle. To transform this equation we use the thermodynamic relationship:

$$
d e_I = d h_f - d \left( \frac{P}{\rho_f} \right),
$$

(3.39)

where $h_f$ stands for fluid specific enthalpy. Assuming that the fluid’s compressibility ($c_f$) is constant, introducing Equation (3.39) into Equation (3.31) and using the chain rule:

$$
\phi \rho_f \frac{D h_f}{D t} = \phi \frac{D P}{D t} - \frac{P \phi}{\rho_f} \frac{D \rho_f}{D t} - \nabla \cdot \vec{q}_f - P \nabla \cdot \vec{u} - \vec{u} \cdot \nabla P
$$

(3.40)

expanding the density material derivative, using the fact that the porosity is constant and grouping terms:

$$
\phi \rho_f \frac{D h_f}{D t} = \phi \frac{D P}{D t} - \frac{P \phi}{\rho_f} \left( \frac{\partial (\phi \rho_f)}{\partial t} + \vec{u} \nabla \rho_f + \rho_f \nabla \cdot \vec{u} \right) - \vec{u} \cdot \nabla P - \nabla \cdot \vec{q}_f
$$

(3.41)

recognizing that the term in parenthesis is zero by the mass conservation equation and expanding the pressure material derivative:

$$
\phi \rho_f \frac{D h_f}{D t} = \phi \frac{\partial P}{\partial t} + \vec{u} \cdot \nabla P - \vec{u} \cdot \nabla P - \nabla \cdot \vec{q}_f
$$

(3.42)

canceling the common term and using the definition of the pressure equation (Equation (3.15)) to change the partial derivative of pressure leads to:

$$
\phi \rho_f \frac{D h_f}{D t} = \nabla \cdot \left( \frac{k}{\mu_c} \nabla P \right) - \nabla \cdot \vec{q}_f.
$$

(3.43)

Notice that because in Equation (3.15) we assumed that the partial derivative is equal to the material derivative to linearize the equation, and that the density changes are
only due to pressure changes (isothermal flow) we have the following relationships:

\[ \phi \frac{\partial P}{\partial t} = \phi \frac{DP}{Dt} = \phi \frac{D\rho_f}{\rho_f c_f} \frac{D\rho_f}{Dt} \]  \hspace{1cm} (3.44)

thus we can write the following equation for the change in enthalpy:

\[ \phi \rho_f \frac{Dh_f}{Dt} = \phi \frac{\rho_f c_f D\rho_f}{Dt} - \nabla \cdot \vec{q}_f. \]  \hspace{1cm} (3.45)

Noteworthily, Equations (3.43) and (3.45) tells us that in the absence of heat fluxes the change in enthalpy experienced by a moving particle is equal to the volumetric deformation that the particle is experiencing and inversely proportional to its compressibility. This rather important observation will be used later for the mathematical analysis of temperature transients within reservoirs.

Previous works have introduced a more explicit formulation in terms of pressure and temperature ([7], [16], [15]). Recalling the thermodynamic relationship between enthalpy and pressure and temperature:

\[ dh_f = c_{pf} dT_f - c_{pf} \mu_{JT} dP, \]  \hspace{1cm} (3.46)

where \( c_{pf} \) is the fluid specific heat capacity at constant pressure, \( T_f \) is the fluid temperature and \( \mu_{JT} \) is the Joule-Thomson coefficient defined as:

\[ \mu_{JT} = \left( \frac{\partial T_f}{\partial P} \right)_{h_f} = \frac{(1 - \beta_f T_f)}{\rho_f c_{pf}}. \]  \hspace{1cm} (3.47)

The Joule-Thomson coefficient expresses the change in temperature due to a unit pressure drop when the fluid is under an isenthalpic process (constant enthalpy); a typical isenthalpic process is the one that occurs when fluid flows through a valve in the absence of heat or work added. Notice that if the Joule-Thomson coefficient is positive, something common in highly compressible fluids (gases), it means that a decrease in pressure or a fluid expansion produces a decrease in temperature; such an effect can be observed when using an aerosol spray, when the fluid cools while coming out of the spray. On the other hand, if the Joule-Thomson coefficient is negative
(common in liquids) a fluid expansion warms the fluid, thus we expect that for a production well inside an oil reservoir the Joule-Thomson effect will cause the oil to increase its temperature since the oil Joule-Thomson coefficient is usually negative. Using Equation (3.46) and the chain rule to transform Equation (3.42) and after some simplifications we obtain:

$$\phi \rho_f c_p f \frac{D T_f}{D t} = \phi \beta_f T_f \frac{D P}{D t} - \vec{u} \cdot \nabla P - \nabla \cdot \vec{q}_f$$

(3.48)

or in terms of the Joule-Thomson coefficient and the adiabatic heat expansion coefficient:

$$\frac{D T_f}{D t} = \eta \frac{\partial P}{\partial t} + \mu_{JT} \vec{v} \cdot \nabla P - \frac{1}{\phi \rho_f c_p f} \nabla \cdot \vec{q}_f$$

(3.49)

where the adiabatic heat expansion coefficient is defined as:

$$\eta = \frac{\beta_f T_f}{\rho_f c_p f}$$

(3.50)

**Joule Thomson and Adiabatic Heat Expansion coefficients**

We have mentioned that for liquids the Joule-Thomson coefficient is usually negative, this is due to the fact that the volumetric heat expansion coefficient ($\beta_f$) is very small such that $1 - \beta_f T_f > 0$, thus producing a negative Joule-Thomson coefficient. Instead the adiabatic heat expansion coefficient is always positive since $\beta_f > 0$ that is the fluid expands with a temperature increase.

Now, consider a well producing from an oil formation. The pressure is decreasing throughout the reservoir, thus:

$$\eta \frac{\partial P}{\partial t} < 0 \quad \text{during production}$$

and the role of the adiabatic heat expansion coefficient during **production** is to **decrease** the fluid temperature. Instead if we are **injecting** liquid into the formation
we conclude that:

\[ \eta \frac{\partial P}{\partial t} > 0 \]  

during injection

and the adiabatic heat expansion process **warms** the fluid.

For the Joule-Thomson coefficient we need to study the term \( \vec{v} \cdot \nabla P \), using Darcy’s law (Equation (3.12)) we can conclude that:

\[ \vec{v} \cdot \nabla P = -\mu \phi \vec{v} \cdot (k^{-1} \vec{v}) = -\mu \phi \vec{v}^T k^{-1} \vec{v} < 0 \]

where in the second equality we wrote the equation using linear algebra nomenclature and for the inequality we used the fact that the permeability tensor is symmetric and positive definite. From the latter we conclude that irrespective of the type of well (production or injection) a negative Joule-Thomson coefficient warms the fluid and a positive Joule-Thomson coefficient cools the fluid just as expected from the very definition Equation (3.47). Furthermore, it is common practice to specify the pressure gradient as boundary condition for the wells, if we rewrite \( \vec{v} \cdot \nabla P \) in terms of pressure gradients only:

\[ \vec{v} \cdot \nabla P = -\frac{1}{\mu \phi} (\nabla P)^T k \nabla P < 0 \]

We can see that at the well, the importance of the Joule-Thomson coefficient is proportional to the rock’s permeability and the pressure gradient specified at the well boundary; therefore, in high flow capacity zones (fractures) or in wells with high flow rates (high pressure gradients) the importance of the Joule-Thomson coefficient in temperature transients is critical and should not be neglected. On the other hand, for shut-in periods or buildup tests the pressure gradient at the well is identically zero (after wellbore storage) and the Joule-Thomson coefficient does not produce any temperature change. This unusual characteristic of the Joule-Thomson coefficient that its influence changes abruptly during wellbore storage has been used in well test analysis to identify when the wellbore storage dominated flow period ends (see [3]).
In general the Joule-Thomson coefficient dominates the high flow regions meanwhile the adiabatic heat expansion coefficient dominates the low flow regions, the effect on the fluid temperature depends on both the flow conditions (drawdown or buildup) and the type of fluid (high or low thermal volumetric expansion) under study as discussed previously.

**Heat Flow**

Notice that because in the energy balanced performed to arrive to Equation (3.29) we considered a control volume Ω that is moving with the fluid, the only heat mechanism considered in the Equation (3.27) is the heat flow due to conduction. Furthermore, the energy balanced is performed for the fluid only, thus we should consider not only the heat added by adjacent fluid particles but the heat added due to the contact between solid grains and fluid particles inside the control volume Ω (recall that Ω contains both fluid and solid).

Heat conduction is modeled using Fourier’s law, which relates the heat flux vector to the temperature gradient with a proportionality constant \( \lambda \) that is a second order symmetric tensor and represents the heat conductivity of the medium (is the analogous of the permeability tensor):

\[
\vec{q} = -\lambda \nabla T.
\] (3.51)

At the interface between fluid and solid, from a pore scale point of view, we can model the heat that flows from the solid grains to the fluid particles as a boundary condition. Nevertheless, here we are considering a Darcy scale point of view, thus we need to express the flow of heat from the solid to the fluid per unit of rock volume, this can be done using Newton’s law of cooling:

\[
q_{fs} = \alpha_{fs}(T_s - T_f)
\] (3.52)

where \( q_{fs} \) is the amount of heat that flows from the solid to the fluid per unit rock
volume, $\alpha_{fs}$ is a rock parameter, such that the following equation is valid:

$$\int_{\Omega} \alpha_{fs}(T_s - T_f) dV = -\int_{A_{fs}} \vec{q}_{fs} \cdot \vec{n} dS$$  \hspace{1cm} (3.53)$$

where $A_{fs}$ is the interface between solid and fluid inside $\Omega$, $\vec{q}_{fs}$ is the heat flux vector at the interface, $\vec{n}$ is the unit normal pointing towards the solid and $T_f, T_s$ are the average fluid and solid temperatures respectively.

It is important to mention that Equation (3.52) is similar to the steady-state interporosity flow proposed by Barenblatt and Zheltov [4] to model double porosity reservoirs and is one of many interesting similarities between the heat flow and fluid flow in porous media mathematical theory. Therefore the solid grains act as an additional distributed heat source to the fluid similar to how the matrix behaves as a fluid source to the fracture.

This implies that we can divide the total fluid heat flux vector $\vec{q}_f$ as the sum of the heat flux vector between fluid particles $\vec{q}_{ff}$ that acts over the interface between fluid particles inside and outside $\Omega$ represented by $A_{ff}$ and the heat flux vector between solid and fluid $\vec{q}_{fs}$ acting over the solid-fluid interface inside $\Omega$ represented by $A_{fs}$, thus we can represent the total heat added to the fluid as:

$$-\int_{\partial \Omega} \vec{q}_f \cdot \vec{n} dS = -\int_{A_{ff}} \vec{q}_{ff} \cdot \vec{n} dS - \int_{A_{fs}} \vec{q}_{fs} \cdot \vec{n} dS.$$  \hspace{1cm} (3.54)$$

Using Equation (3.53) we can transform the second surface integral into a volume integral and noticing that the surface integral over $A_{ff}$ can be represented as surface integral over the total boundary $\partial \Omega$ if we introduce the porosity in the integrand:

$$-\int_{\partial \Omega} \vec{q}_f \cdot \vec{n} dS = -\int_{\partial \Omega} \phi \vec{q}_{ff} \cdot \vec{n} dS + \int_{\Omega} \alpha_{fs}(T_s - T_f) dV,$$  \hspace{1cm} (3.55)$$

then using Fourier’s law and the divergence theorem we obtain:

$$-\int_{\Omega} \nabla \cdot \vec{q}_f dV = \int_{\Omega} \nabla \cdot (\phi \lambda_f \nabla T_f + q_{fs}) dV.$$  \hspace{1cm} (3.56)$$
and since this is valid for any control volume we conclude that:

\[-\nabla \cdot \vec{q}_f = \nabla \cdot (\phi \lambda_f \nabla T_f) + \alpha f_s (T_s - T_f)\]  \hspace{1cm} (3.57)

Finally the fluid energy equation in differential form expressed as function of temperature and pressure only is given by:

\[\frac{DT_f}{Dt} = \eta \frac{\partial P}{\partial t} + \mu J_T \vec{v} \cdot \nabla P + \frac{1}{\phi \rho_f c_p} \nabla \cdot (\phi \lambda_f \nabla T_f) + \frac{\alpha f_s}{\phi \rho_f c_p} (T_s - T_f)\]  \hspace{1cm} (3.58)

### 3.3.2 Solid Energy Equation

The solid energy equation can be derived similarly to how the fluid energy equation was derived, but notice that because we are assuming constant porosity, we are implicitly considering that the solid grains are not moving, thus some terms that appear in the fluid energy equation do not appear in the solid equation.

In simple terms we know that the rate of change in internal energy is equal to the heat added to the system plus the work done to the system that does not contribute in the increase of the kinetic energy. In integral form this is:

\[\frac{d}{dt} \int_\Omega \left(1 - \phi\right) \rho_s e_{Is} dV = -\int_{\partial \Omega} \vec{q}_s \cdot \vec{n} dS + \int_\Omega W_s dV\]  \hspace{1cm} (3.59)

where \(W_s\) stands for the work done to the solid grains inside \(\Omega\) per unit rock volume that does not contribute to the kinetic energy and \(e_{Is}\) is the internal energy of the solid per unit mass. We know that \(W_s\) is proportional to the solid displacement, but because we are assuming that the solid grains are not moving we must have that \(W_s = 0\). Besides, \(\Omega\) has always the same amount of solid mass by constraint, and again due to the absence of solid motion \(\Omega\) is not moving, then we can interchange the order of the integral and the derivative to obtain:

\[\int_\Omega \frac{\partial((1 - \phi) \rho_s e_{Is})}{\partial t} dV = -\int_{\partial \Omega} \vec{q}_s \cdot \vec{n} dS.\]  \hspace{1cm} (3.60)
Similarly to what occurs to the fluid, the total heat added to the solid, can be de-
composed as the heat added due to adjacent solid particles outside $\Omega$ and due to the
fluid that is in contact to the solid grains inside $\Omega$. Consequently we can rewrite the
surface integral in Equation (3.59) as follows:

$$- \int_{\partial \Omega} \vec{q}_s \cdot \vec{n} dS = - \int_{A_{ss}} \vec{q}_{ss} \cdot \vec{n} S - \int_{A_{fs}} \vec{q}_{fs} \cdot \vec{n}_s dS$$

(3.61)

where $\vec{q}_{ss}$ is the heat flux between solid particles that flows through the solid part of
$\partial \Omega$ ($A_{ss}$) and $\vec{n}_s$ is the normal at the solid fluid boundary pointing towards the fluid.
Noticing that $\vec{n}_s = - \vec{n}_f$, using Equation (3.53) to transform the surface integral over
the solid-fluid interface into a volume integral and noticing that the surface integral
over $A_{ss}$ can be represented as a surface integral over the total boundary $\partial \Omega$ by
introducing the factor $1 - \phi$ we obtain:

$$- \int_{\partial \Omega} \vec{q}_s \cdot \vec{n} dS = - \int_{\partial \Omega} (1 - \phi) \vec{q}_{ss} \cdot \vec{n} dS + \int_{A_{fs}} \vec{q}_{fs} \cdot \vec{n}_s dS.$$ 

(3.62)

Then using the divergence theorem and Fourier’s law (Equation (3.51)) for a solid
medium we obtain the total heat added to the solid in terms of the solid and fluid
temperature:

$$- \int_{\Omega} \nabla \cdot \vec{q}_s dV = \int_{\Omega} \left( \nabla \cdot ((1 - \phi) \lambda_s \nabla T_s) - \alpha_{fs} (T_s - T_f) \right) dV.$$ 

(3.63)

Finally for an incompressible material, the internal energy satisfy the following rela-
tionship:

$$de_{fs} = c_{ps} dT_s$$

(3.64)

Consequently the energy equation for the solid inside the reservoir is:

$$\frac{\partial ((1 - \phi) \rho_s c_{ps} T_s)}{\partial t} = \nabla \cdot ((1 - \phi) \lambda_s \nabla T_s) - \alpha_{fs} (T_s - T_f).$$

(3.65)

Notice that if there is an absence of thermal equilibrium between the fluid and solid
($T_f \neq T_s$), Equation (3.65) is coupled to Equation (3.58). It has been shown by Bejan
[7] that for low Reynolds numbers, that is for low velocity flow, it is appropriate to assume that solid and fluid temperatures are equal, such that both equations are decoupled.

### 3.3.3 Reservoir Energy Equation

Consider both rock and fluid are in thermal equilibrium and let $T = T_f = T_s$. We can lump the energy equation for the solid and fluid into one energy equation for the reservoir rock (solid and fluid) to obtain:

$$\frac{\partial T}{\partial t} + \varepsilon_f \bar{u} \cdot \nabla T = \bar{\alpha}_T \nabla^2 T + \phi \bar{\eta} \frac{\partial P}{\partial t} + \bar{\mu}_{JT} \bar{u} \cdot \nabla P$$

(3.66)

where:

- $\bar{\rho c}_p = \phi \rho_f c_{pf} + (1 - \phi) \rho_s c_{ps}$
  reservoir thermal storage

- $\varepsilon_f = \frac{\rho_f c_{pf}}{\bar{\rho c}_p}$
  fluid thermal storativity

- $\bar{\lambda} = \phi \lambda_f + (1 - \phi) \lambda_s$
  reservoir thermal conductivity

- $\bar{\alpha}_T = \frac{\bar{\lambda}}{\bar{\rho c}_p}$
  reservoir thermal diffusivity

- $\bar{\eta} = \eta \varepsilon_f$
  reservoir Adiabatic heat expansion

- $\bar{\mu}_{JT} = \mu_{JT} \varepsilon_f$
  reservoir Joule-Thomson coefficient

This is the approach used by Duru [15], [16] and has proven to be a robust model to study the temperature transients inside the reservoir.

It is useful to derive the reservoir energy equation in conservative form, note that using the mass conservation equation in differential form Equation (3.6) we have the following relationship for any scalar fluid property $\psi_f$:

$$\phi \rho_f \left( \frac{\partial \psi_f}{\partial t} + \bar{v} \cdot \nabla \psi_f \right) = \left( \frac{\partial (\rho_f \phi) \psi_f}{\partial t} + \nabla \cdot (\rho_f \phi \bar{v} \psi_f) \right)$$

(3.73)
thus after some manipulations, recalling that the rock density and porosity are constant and that rock and fluid are in thermal equilibrium we can write the reservoir energy equation as:

\[
\frac{\partial (\rho c_p T)}{\partial t} + c_p f \nabla \cdot (\rho_f \bar{u} T) = \nabla \cdot (\mathbf{X} \nabla T) + \phi \frac{\partial P}{\partial t} + c_{pf} \mu_T \left( \frac{\partial (\phi \rho_f) P}{\partial t} + \nabla \cdot (\rho \bar{u} P) \right)
\]

(3.74)

This equation will be useful later to derive the fracture and matrix energy equations.
Chapter 4

Mathematical Analysis

For the mathematical analysis we need to solve the mass and conservation equations for a double porosity reservoir to obtain the temperature and pressure as function of position and time inside the reservoir. The approach described in this chapter is simplifies the equations derived in Chapter 3 to obtain a deterministic model that relates the fluid and rock properties to the pressure and temperature. One of the most important simplifications that has been done in many previous studies is to assume that the pressure equation (Equation (3.15)) is valid, thus enabling us to determine the pressure field first and then use this known pressure field to determine the temperature field.

As shown by Kazemi [20], is possible to replicate the double porosity behavior by assuming a layered reservoir where the fracture layers are placed between matrix layers. This is also the same model assumed by de Swaan [14] for strata matrix and by Streltsova [28] in her gradient flow model. Throughout this work, we based the analysis of double porosity reservoirs on the model proposed by Kazemi. The most important feature of the Kazemi model, which greatly simplifies the mathematical complexity, is the matrix and fracture uniform spacing and dimensions; this regularity produces symmetry lines at the middle of each fracture and matrix layers as shown in Figure 4.1. The importance of the symmetry lines is that because fluid can not flow across them and neither does heat, the symmetry lines isolate the analysis of the total reservoir to a single repetitive element shown in Figure 4.2.
CHAPTER 4. MATHEMATICAL ANALYSIS

Figure 4.1: Double porosity model, with symmetry lines.

Figure 4.2: Repetitive element
4.1 Pressure Equation

For simplicity in the mathematical analysis, we considered the following assumptions:

- All the matrix rock properties are assumed constant within the matrix layer.
- All the fracture rock properties are assumed constant within the fracture layer.
- Matrix and fracture porosities and permeability are allowed to be distinct, meanwhile the thermal rock properties are equal at the fracture and matrix.
- Fluid properties are constant and equal, whether the fluid is at the fracture or the matrix.
- Temperature changes are neglected in the mass conservation equation.
- The fluid is slightly compressible such that the material derivative of pressure can be accurately approximated by the time partial derivative.
- Gravity effects are neglected, horizontal flow only.

All these assumptions allow us to use the pressure equation (Equation (3.15)) to determine the pressure field.

To arrive to the fracture equation, first let us define a coordinate system. We will employ cylindrical coordinates to take advantage of the fact that flow is only radial. The horizontal plane that corresponds to the value of $z = 0$ is placed exactly at the position of the fracture symmetry line shown in Figure 4.2. Now let the fracture pressure $P_F$ be defined as follows:

$$P_F = \frac{1}{h_F} \int_{-h_F/2}^{h_F/2} P \, dz$$

(4.1)

where $h_F$ is the fracture thickness (Figure 4.1). Note that this definition for $P_F$ can be seen as the volumetric average of the pressure at the fracture and will enable us
to reduce the cylindrical problem into a radial problem. To achieve this let us take the volumetric average of the pressure equation over the fracture thickness:

\[
\frac{1}{h_F} \int_{-h_F/2}^{h_F/2} \phi c_f \frac{\partial P}{\partial t} dz = \frac{1}{h_F} \int_{-h_F/2}^{h_F/2} \left( \frac{k_r}{\mu} \nabla_r^2 P + \frac{k_z}{\mu} \nabla_z^2 P \right) dz
\]  \tag{4.2}

where we have divided the divergence term into a radial component and a vertical component. Notice that because the rock and fluid properties are constant, and the partial derivative is with respect to time we have:

\[
\frac{1}{h_F} \int_{-h_F/2}^{h_F/2} \phi c_f \frac{\partial P}{\partial t} dz = \phi c_f \frac{\partial P_F}{\partial t}.
\]  \tag{4.3}

Similarly for the radial term in the right hand side:

\[
\frac{1}{h_F} \int_{-h_F/2}^{h_F/2} k_r \nabla_r^2 P dz = k_r \nabla_r^2 P_F
\]  \tag{4.4}

but for the vertical term in the divergence we obtain:

\[
\frac{1}{h_F} \int_{-h_F/2}^{h_F/2} k_z \nabla_z^2 P dz = \frac{1}{h_F} \int_{-h_F/2}^{h_F/2} \frac{\partial u_z}{\partial z} dz = -\frac{u_z(h_F/2) + u_z(-h_F/2)}{h_f} = -\frac{2u_z(h_F/2)}{h_f}
\]  \tag{4.5}

where we used Darcy’s law and the fact that \(u_z(h_F/2) = -u_z(-h_F/2)\) because both quantities are at the same distance from the symmetry line \(z = 0\). Note that \(u_z(h_F/2)\) is nothing more than the velocity of the fluid at the fracture-matrix interface or the interporosity volumetric flow per unit area. Then the mass equation for the fracture is given by:

\[
\phi_F c_f \frac{\partial P_F}{\partial t} = \frac{k_F}{\mu} \nabla_r^2 P_F - 2 \frac{u_z(h_F/2)}{h_F}.
\]  \tag{4.6}
subject to the boundary and initial conditions:

\[ P_F(r, t = 0) = P_i \]  \hspace{1cm} (4.7)
\[ \frac{\partial P_F}{\partial r}(r = r_w, t) = \frac{q_w B_o \mu}{2 \pi r_w k_F N h_F} \]  \hspace{1cm} (4.8)
\[ P_F(r \to \infty, t) = P_i \]  \hspace{1cm} (4.9)

where \( P_i \) is the initial constant pressure throughout the reservoir, \( q_w \) is the total well flow rate at standard conditions, \( B_o \) is the oil formation volume factor and \( N \) is the number of fracture layers from which the well is producing. If we express the problem as function of the cumulative pressure drop \( \Delta P_F = P_i - P_F \) :

\[ \phi_F c_f \frac{\partial \Delta P_F}{\partial t} = \frac{k_F}{\mu} \nabla^2 \Delta P_F + 2 \frac{u_z (h_F/2)}{h_F} \]  \hspace{1cm} (4.10)
\[ \Delta P_F(r, t = 0) = 0 \]  \hspace{1cm} (4.11)
\[ \frac{\partial \Delta P_F}{\partial r}(r = r_w, t) = -\frac{q_w B_o \mu}{2 \pi r_w k_F N h_F} \]  \hspace{1cm} (4.12)
\[ \Delta P_F(r \to \infty, t) = 0. \]  \hspace{1cm} (4.13)

This is the same problem as the one proposed by de Swaan [14] to model transient interporosity flow. Notice that if \( u_z = \vec{u} \cdot \hat{z} < 0 \) where \( \hat{z} \) is the unit vector pointing in the vertical direction then the fluid is flowing from the matrix to the fracture thus decreasing the fracture cumulative pressure drop \( \Delta P_F \) or increasing the fracture pressure \( P_F \) just as expected.

To complete the problem we need an expression for \( u_z \), for this we will consider
the following problem for the matrix medium:

\[ \phi_M c_f \frac{\partial \Delta P_M}{\partial t} = \frac{k_M}{\mu} \frac{\partial^2 \Delta P_M}{\partial z^2} \quad \text{for} \quad \frac{h_F}{2} < z < \frac{h_F + h_M}{2}, \quad t > 0 \]  
\[ \frac{\partial \Delta P_M}{\partial z} = 0 \quad \text{at} \quad z = \frac{h_F + h_M}{2}, \quad t > 0 \]  
\[ \Delta P_M = \Delta P_F \quad \text{at} \quad z = \frac{h_F}{2}, \quad t > 0 \]  
\[ \Delta P_M = 0 \quad \text{for} \quad \frac{h_F}{2} < z < \frac{h_F + h_M}{2}, \quad t = 0 \]  

(4.14)  
(4.15)  
(4.16)  
(4.17)

Thus, for the matrix we are assuming that the flow is only in the vertical direction (towards the fracture) and neglecting flow in the horizontal direction, this assumption was shown to be valid for reservoir where the matrix permeability \( k_M \) is very low by Kazemi [20]. The matrix problem can be solved by considering the following auxiliary problem:

\[ \phi_M c_f \frac{\partial \psi}{\partial t} = \frac{k_M}{\mu} \frac{\partial^2 \psi}{\partial z^2} \quad \text{for} \quad \frac{h_F}{2} < z < \frac{h_F + h_M}{2}, \quad t > 0 \]  
\[ \frac{\partial \psi}{\partial z} = 0 \quad \text{at} \quad z = \frac{h_F + h_M}{2}, \quad t > 0 \]  
\[ \psi = 1 \quad \text{at} \quad z = \frac{h_F}{2}, \quad t > 0 \]  
\[ \psi = 0 \quad \text{for} \quad \frac{h_F}{2} < z < \frac{h_F + h_M}{2}, \quad t = 0 \]  

(4.18)  
(4.19)  
(4.20)  
(4.21)

The function \( \psi(z,t) \) can be seen as the function that represents the effect of creating a unit change disturbance at the interface between the matrix and the fracture and is referred as the transfer function. The solution to the auxiliary problem has been studied in the theory of heat flow and the solution can be found by means of separation
of variables or the Laplace transform technique (see [12], [28]) and is given by:

$$\psi(z*, t) = 1 - \frac{4}{h_M} \sum_{j=0}^{\infty} \frac{1}{\lambda_j} \exp \left(-\frac{\lambda_j^2 t}{\alpha_{HM}}\right) \sin (\lambda_j z*)$$

(4.22)

$$\tilde{\psi}(z*, s) = \frac{\cosh \left(\sqrt{\frac{s}{\alpha_{HM}}} \left(\frac{h_M}{2} - z*\right)\right)}{s \cosh \left(\sqrt{\frac{s}{\alpha_{HM}}} \frac{h_M}{2}\right)}$$

(4.23)

where $\lambda_j = (2j + 1)\pi/h_M$, $\alpha_{HM} = k_M/(\mu c_f \phi_M)$ is the matrix hydraulic diffusivity, $z* = z - h_F/2$, $s$ is the Laplace parameter and $\tilde{\psi}$ is the solution in Laplace domain.

Once the auxiliary problem is solved, Duhammel's principle can be used to give the solution to the problem for the matrix as:

$$\Delta P_M(r, z*, t) = \int_0^t \frac{\partial \Delta P_F(r, \tau)}{\partial \tau} \psi(z*, t - \tau) d\tau$$

(4.24)

$$\Delta \tilde{P}_M(r, z*, s) = \Delta \tilde{P}_F(r, s) \tilde{\psi}(z*, s)$$

(4.25)

Now the fluid velocity inside the matrix is obtained applying Darcy’s law to Equation (4.24):

$$u_z(z, t) = -\frac{k_M}{\mu} \frac{\partial P_M}{\partial z} = \frac{k_M}{\mu} \frac{\partial \Delta P_M}{\partial z} = \frac{k_M}{\mu} \int_0^t \frac{\partial \Delta P_F(\tau)}{\partial \tau} \frac{\partial \psi(z, t - \tau)}{\partial z} d\tau$$

(4.26)

Introducing this definition of the matrix velocity into Equation (4.10) we obtain the fracture equation expressed only as function of the fracture pressure:

$$\phi_{Fc} \frac{\partial \Delta P_F}{\partial t} = \frac{k_F}{\mu} \nabla_r^2 (\Delta P_F) + 2 \frac{k_M}{\mu h_F} \int_0^t \frac{\partial \Delta P_F(\tau)}{\partial \tau} \frac{\partial \psi(z* = 0, t - \tau)}{\partial z} d\tau.$$  

(4.27)

Applying the Laplace transform to the previous equation we obtain:

$$s \Delta \tilde{P}_F \left(\frac{1}{\alpha_{HF}} - 2 \frac{k_M}{k_F h_F} \frac{\partial \tilde{\psi}(z* = 0, s)}{\partial z}\right) = \frac{1}{r} \frac{d}{dr} \left( r \frac{d \Delta \tilde{P}_F}{dr}\right)$$

(4.28)
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The solution of this ordinary differential equation subject to the transformed boundary conditions is given in terms of the modified Bessel functions of the second kind $K_\nu$ as:

$$\Delta \tilde{P}_F(r, s) = \frac{c_2}{s\sqrt{c_1(s)s}} \frac{K_0(r\sqrt{c_1(s)s})}{K_1(r_w\sqrt{c_1(s)s})}$$

$$c_1(s) = \frac{1}{\alpha_{HF}} - 2 \frac{k_M}{k_F h_F} \frac{\partial \tilde{\psi}(z^* = 0, s)}{\partial z}$$

$$c_2 = \frac{q_w B_o \mu}{2\pi r_w k_F N h_F}.$$  

Unfortunately there is no analytical inversion of this expression, even so several authors have derived approximations of the Laplace space solution and inverted the approximations analytically (see [14], [21]) while others used a numerical inversion algorithm to invert the exact solution from Laplace space (see [28]).

**Early-time behavior**

At early time the matrix influence has not been felt, and the flow is dominated by the fracture properties, thus we have $c_1(s) = 1/\alpha_{HF}$ and assuming a line source well such that $r_w \to 0$ Van Everdingen and Hurst [30] gave the solution in time domain as:

$$\Delta P_F(r, t) = -\frac{q_w B_o \mu}{4\pi k_F N h_F} E_i \left( -\frac{r^2}{4\alpha_{HF} t} \right)$$

where $E_i$ is the exponential integral function.

**Late-time behavior**

For late time, we will take the average of the matrix pressure equation over the matrix half thickness, and using the fact that $\partial \Delta P_M / \partial z = 0$ at $(h_M + h_F)/2$ we obtain

$$\phi_M c_f \frac{\partial \Delta P_M}{\partial t} = -\frac{2}{h_M} u_z(h_F/2)$$
combining this equation to Equation (4.10) to cancel \( u_z \), and assuming that \( \Delta P_F = \Delta P_M \) we obtain the equivalent reservoir pressure equation:

\[
c_f \varphi_R \frac{\partial \Delta P_F}{\partial t} = \frac{k_{Fb}}{\mu} \nabla^2 \Delta P_F
\]  

(4.34)

where \( \chi_F = h_F/(h_F + h_M) \) can be seen as the ratio of fracture volume per unit volume of reservoir rock, \( \phi_R = \phi_M \chi_M + \phi_F \chi_F \) is the average reservoir porosity and \( k_{Fb} = k_F \chi_F \) is the effective reservoir permeability. This equation has the same form as the pressure equation for an homogeneous medium with a hydraulic conductivity \( \alpha_{HR} \) given by:

\[
\alpha_{HR} = \frac{k_{Fb}}{\mu \phi_R c_f}
\]  

(4.35)

and the solution is:

\[
\Delta P_F(r, t) = -\frac{q_w B_o \mu}{4\pi k_F N h_F} E_i \left( -\frac{r^2}{4\alpha_{HR} t} \right).
\]  

(4.36)

These are the same solutions derived by de Swaan [14] for early and late time behavior.

**Intermediate-time behavior**

For intermediate time, the matrix pressure transient have not felt the matrix boundary at \( z = (h_M + h_F)/2 \) and the solution to the auxiliary problem can be obtained assuming that the matrix has infinite thickness. Then under these assumptions the transfer function \( \psi \) becomes:

\[
\psi(z^*, t) = 1 - \text{erf} \left( \frac{z^*}{\sqrt{4\alpha_{HM} t}} \right)
\]  

(4.37)

\[
\frac{\partial \psi(z^* = 0, t)}{\partial z} = -\frac{1}{\sqrt{\pi \alpha_{HM} t}}
\]  

(4.38)
where erf is the error function. Using this consideration we have:

$$c_1(s) = \frac{1}{\alpha_{HF}} + 2 \frac{k_M}{k_F h_F \sqrt{\alpha_{HM} s}}.$$ (4.39)

If we approximate for late-time or \( s \to 0 \) we obtain that \( c_1(s) \approx \frac{2k_M}{k_F h_F \sqrt{\alpha_{HM} s}} \)
and the solution for the cumulative fracture pressure drop for a line source well in Laplace domain is:

$$\Delta \tilde{P}_F(r, s) = \frac{q_w B_o \mu}{2\pi k_F N h_F s} K_0 \left( r s^{1/4} \sqrt{\frac{2k_M}{k_F h_F \sqrt{\alpha_{HM}}}} \right)$$ (4.40)

using the property that for small arguments \( K_0(x) \approx -\gamma - \ln(x/2) \):

$$\Delta \tilde{P}_F(r, t) = -\frac{q_w B_o \mu}{2\pi k_F N h_F s} \left( \frac{1}{4} \ln s + \frac{1}{2} \ln \left( \frac{r^2 k_M}{2k_F h_F \sqrt{\alpha_{HM}}} \right) + \gamma \right)$$ (4.41)

and in time domain the solution is:

$$\Delta P_F(r, t) = \frac{q_w B_o \mu}{4\pi k_F N h_F} \left( \frac{1}{2}(\gamma + \ln t) - \ln \left( \frac{r^2 k_M}{2k_F h_F \sqrt{\alpha_{HM}}} \right) - \gamma \right)$$ (4.42)

where \( \gamma = 0.57722\ldots \) is Euler’s constant.

For the general case, Najurieta [21] derived an approximate solution for all practical positions and time given by:

$$\Delta P_F(r, t) = -\frac{q_w B_o \mu}{4\pi k_F N h_F} E_i \left( -\frac{r^2}{4\alpha_{HRo} t} \right)$$ (4.43)

$$\alpha_{HRo} = \frac{k_{fb}}{\mu e_f \phi_F \chi_F + \phi_M \chi_M \sqrt{\frac{1}{\tau} \tanh \sqrt{\frac{\tau}{t}}}}$$ (4.44)

where \( \tau = \frac{h_M^2}{4\gamma \alpha_{HM}} \). Notice that this equation is in agreement with the equations derived for early-time and late-time behavior.
4.2 Energy Equation

Similar to what was done to obtain the fracture pressure equation, the energy equation for the fracture medium is obtained by averaging the energy equation across the fracture thickness.

Fracture Energy Equation

To derive the fracture energy equation we used the same methodology performed to derive the fracture pressure equation. Let us define the fracture temperature as:

$$T_F = \frac{1}{h_F} \int_{-h_F/2}^{h_F/2} T \, dz$$  \hspace{1cm} (4.45)

Then, taking the average over the fracture thickness of the reservoir energy equation in conservative form (Equation (3.74)) we obtain the following individual relationships:

$$\frac{1}{h_F} \int_{-h_F/2}^{h_F/2} \frac{\partial(p_c p T)}{\partial t} \, dz = \frac{\partial(p_c p F T_F)}{\partial t}$$  \hspace{1cm} (4.46)

$$\frac{1}{h_F} \int_{-h_F/2}^{h_F/2} c_{pf} \nabla \cdot (\rho_f \vec{u} T) \, dz = c_{pf} \nabla \cdot (\rho_f \vec{u}_r T_F) + \frac{2}{h_F} c_{pf} \rho_f u_z (h_F/2) T_F$$  \hspace{1cm} (4.47)

$$\frac{1}{h_F} \int_{-h_F/2}^{h_F/2} \nabla \cdot (\lambda \nabla T) \, dz = \nabla \cdot (\lambda_F \nabla T_F) - \frac{2}{h_F} q_z$$  \hspace{1cm} (4.48)

$$\frac{1}{h_F} \int_{-h_F/2}^{h_F/2} \phi \frac{\partial P}{\partial t} \, dz = \phi_F \frac{\partial P_F}{\partial t}$$  \hspace{1cm} (4.49)

$$\frac{1}{h_F} \int_{-h_F/2}^{h_F/2} \nabla \cdot (\rho \vec{u} P) \, dz = \nabla \cdot (\rho \vec{u}_r P_F) + \frac{2}{h_F} \rho_f u_z (h_F/2) P_F$$  \hspace{1cm} (4.50)
where \( q_z \) is the interporosity heat flow per unit area due to conduction. Combining this relationships we obtain the fracture energy equation:

\[
\frac{\partial (\rho_c T_F)}{\partial t} + c_p F \rho_f \bar{u}_r \cdot \nabla_r T_F - \frac{2}{h_F} c_p_f u_z (h_F/2) T_F = \nabla_r \cdot (k_F \nabla_r T_F)
\]

\[- \frac{2}{h_F} q_z + \phi_F \frac{\partial P_F}{\partial t} + c_{p f} \mu_JT \left( \frac{\partial (\phi_F \rho_f P_F)}{\partial t} + \nabla_r \cdot (\rho \bar{u}_r P_F) \right) + \frac{2}{h_F} \rho_f u_z (h_F/2) P_F.\]  

(4.51)

As we can see, there are several extra terms appearing in the fracture equation due to the interaction between the fracture and the matrix. The term \( c_{p f} \nabla_r \cdot (\rho_f \bar{u}_r T_F) \) represents the transport of heat along the fracture by convection and \( \nabla_r \cdot (k_F \nabla_r T_F) \) represents the fracture of heat along the fracture by conduction. Similarly \( \frac{2}{h_F} c_{p f} \rho_f u_z (h_F/2) T_F \) is the transport of heat from the matrix to the fracture due to convection and \( \frac{2}{h_F} q_z \) the transport of heat from the matrix to the fracture due to conduction.

If we also average the mass conservation equation in terms of density (Equation (3.6)) we arrive to the following equation:

\[
\frac{\partial (\phi_F \rho_f)}{\partial t} + \nabla_r \cdot (\rho_f \bar{u}_r) + \frac{2}{h_F} \frac{\partial}{\partial t} u_z (h_F/2) = 0.
\]  

(4.52)

Using this relationship the following equation holds for any property \( \psi_F \):

\[
\frac{\partial (\phi_F \rho_f \psi_F)}{\partial t} + \nabla_r \cdot (\rho_f \bar{u}_r \psi_F) + \frac{2}{h_F} \frac{\partial}{\partial t} u_z (h_F/2) \psi_F = \phi_F \rho_f \frac{\partial \psi_F}{\partial t} + \rho_f \bar{u}_r \cdot \nabla_r \psi_F.
\]  

(4.53)

Therefore, the fracture energy equation in nonconservative form is given by:

\[
\frac{\partial T_F}{\partial t} + c_{p f} \rho_f \bar{u}_r \cdot \nabla_r T_F = \nabla_r \cdot (k_F \nabla_r T_F) - \frac{2}{h_F} q_z + \phi_F \frac{\partial P_F}{\partial t}
\]

\[+ \phi_F \rho_f c_{p f} \mu_JT \left( \frac{\partial P_F}{\partial t} + \bar{v}_r \cdot \nabla_r P_F \right).\]  

(4.54)

Noteworthy, the only extra term that appears in the nonconservative form is the heat flow due to conduction between the matrix and the fracture, the other terms are
taken into account in the solution of the fracture pressure due to the assumption of slightly compressible fluid in the mass conservation equation.

**Equivalent reservoir formulation**

For late time behavior, we consider that matrix and fracture are in thermal equilibrium. This can occur “rapidly” if the matrix blocks are sufficiently small or if the matrix thermal diffusivity is very high. To obtain the average matrix equation, we consider the following problem for the matrix medium:

\[
\frac{\partial (\rho c_p T)}{\partial t} + c_f \frac{\partial}{\partial z} (u_z \rho_f T) = -\frac{\partial q_z}{\partial z} + \phi \frac{\partial P}{\partial t} + c_f \muJT \left( \frac{\partial (\phi \rho_f P)}{\partial t} + \frac{\partial}{\partial z} (u_z \rho_f P) \right)
\]  
(4.55)

\[q_z = 0 = u_z \text{ at } z = \frac{h_F + h_M}{2}, \quad t > 0\]  
(4.56)

Integrating the equation over the half matrix thickness we get:

\[
\frac{\partial (\rho c_p M T_M)}{\partial t} - c_f u_z \rho_f T_M = \frac{2}{h_M} q_z + \phi M \frac{\partial P_M}{\partial t} + c_f \mu J_T \left( \frac{\partial (\phi M \rho_f P_M)}{\partial t} - u_z \rho_f P_M \right)
\]  
(4.57)

and using the same procedure as before the nonconservative form is given by:

\[
\frac{\partial T_M}{\partial t} = \frac{2}{\rho c_p M} \frac{\partial q_z}{\partial t} + \phi M \frac{\partial P_M}{\partial t} + \phi_M \rho_f c_p \mu J_T \frac{\partial P_M}{\partial t}.
\]  
(4.58)

Assuming thermal equilibrium between matrix and fracture and combining Equations (4.54) and (4.58) to cancel the interporosity heat flux \(q_z\) we obtain the energy equation for an equivalent homogeneous reservoir:

\[
\frac{\partial T_F}{\partial t} + \chi_F c_p \rho_f \vec{u}_r \cdot \nabla r T_F = \nabla r \cdot (\chi_F \lambda T_F \nabla r T_F) + \phi R \frac{\partial P_F}{\partial t} + \rho_f \chi_F \mu J_T \left( \frac{\partial P_F}{\partial t} + \chi_F \vec{u}_r \cdot \nabla r P_F \right)
\]  
(4.59)
where

\[
\bar{\rho}c_{pR} = \chi_F \bar{\rho}c_{pF} + (1 - \chi_F)\bar{\rho}c_{pM}
\]

\[
= \rho_f c_p f(\phi_f \chi_f + \phi_M \chi_M) + \rho_s c_p s(1 - \phi_f \chi_f - \phi_M \chi_M)
\]

(4.60)

For the analysis is useful to introduce the following dimensionless variables

\[
\omega_H = \frac{\phi_2}{\phi_1 + \phi_2} = \frac{\phi_f \chi_f}{\phi_F \chi_F + \phi_M \chi_M}
\]

(4.62)

\[
\varepsilon_T = \frac{(\phi_1 + \phi_2)\rho_f c_p f + \rho_s c_p s(1 - \phi_1 - \phi_2)}{\phi_2 \rho_f c_p f}
\]

(4.63)

\[
\alpha_H = \omega_H \frac{k_{Fb}}{\mu c_f \phi_2}
\]

(4.64)

\[
\alpha_T = \frac{\lambda_f \phi_2 + \lambda_s (\chi_F - \phi_2)}{(\phi_1 + \phi_2)\rho_f c_p f + \rho_s c_p s(1 - \phi_1 - \phi_2)}
\]

(4.65)

\[
P_D = \frac{2\pi k_{Fb} H_w}{q_w B_o \mu}(P_i - P_F(r,t))
\]

(4.66)

\[
T_D = (T_i - T_F(r,t)) \frac{c_f}{\varepsilon_T \alpha_H} \frac{\varepsilon_T \alpha_H}{\mu JT} \omega_H \alpha_T
\]

(4.67)

\[
P_e = \frac{q_w B_o}{2\pi H_w \phi_2 \alpha_T} \frac{\varepsilon_T}{\varepsilon_T \alpha_T}
\]

(4.68)

\[
r_D = \frac{r}{r_w}
\]

(4.69)

\[
t_D = \frac{t}{T_w^2}
\]

(4.70)

\[
\zeta = \frac{\varepsilon_T}{\omega_H \mu JT}
\]

(4.71)

Notice that \(k_{Fb} = k_F \chi_F\) is the effective reservoir permeability and \(H_w = (h_F + h_M)N\) where \(N\) is the total number of horizontal fractures from which the well is producing, \(\phi_1, \phi_2\) are the porosities defined by Warren and Root[32], \(\omega_H\) is the hydraulic storativity ratio also defined by Warren and Root[32], \(P_e\) is the dimensionless Peclet number, \(\varepsilon_T\) is the ratio of the thermal storage of the fluid inside the fracture to the reservoir thermal storage, \(\alpha_H\) is the reservoir hydraulic diffusivity, \(\alpha_T\) is the reservoir thermal diffusivity and \(\zeta\) is the parameter that takes into account the ratio of influence between the Joule-Thomson coefficient and the adiabatic heat expansion coefficient in
the reservoir.

Using these dimensionless variables the energy equation for an equivalent homogeneous reservoir is given by:

\[
\frac{\partial T_D}{\partial t_D} + \frac{\partial P_D}{\partial r_D} \frac{\partial T_D}{\partial r_D} = \frac{1}{P_e} \nabla^2_{r_D} T_D - P_e \left( \zeta \frac{\partial P_D}{\partial t_D} + \left( \frac{\partial P_D}{\partial r_D} \right)^2 \right). \tag{4.72}
\]

Notice that if \( \omega = 1 \) this equation reduces to an early-time approximation where we neglect the heat flow due to conduction between the matrix and the fracture medium; therefore, we will consider only this formulation to do the mathematical analysis.

**Analytical Solution**

A typical value for the dimensionless Peclet number \( P_e \) defined in Equation (4.68) is usually very large for high flow rates. Consequently the diffusion term is very small compared to all the others and can be neglected. Now the resulting equation has the general form:

\[
\frac{\partial T_D}{\partial t_D} + w(r_D, t_D) \frac{\partial T_D}{\partial r_D} = f(r_D, t_D) \tag{4.73}
\]

where

\[
w(r_D, t_D) = \frac{\partial P_D}{\partial r_D} \tag{4.74}
\]

\[
f(r_D, t_D) = -P_e \left( \zeta \frac{\partial P_D}{\partial t_D} + \left( \frac{\partial P_D}{\partial r_D} \right)^2 \right). \tag{4.75}
\]

This generic equation is called the convection equation and can be solved using the method of characteristics. The idea behind the method of characteristics comes from the fact that if we consider the vector field in \( \mathbb{R}^2 \):

\[
V(r_D, t_D) = (1, w(r_D, t_D)), \tag{4.76}
\]
then $\partial T_D/\partial t_D + w(r_D, t_D)\partial T_D/\partial r_D$ is the directional derivative of $T_D$ along $V$. Thus we can parameterize all the integral curves ($\Gamma(\xi, s)$) of $V$ and solve the equation along each integral curve. Consequently the method of characteristics enable us to reduce a partial differential equation into several ordinary differential equations. The integral curves of $V$ are $\Gamma(\xi, s) = (t_D(\xi, s), r_D(\xi, s))$, where $\xi$ is a parameter, distinct for each integral curve and $s$ is the variable that controls the position along the curve. For fixed $\xi$, each integral curve must satisfy the following property:

$$\frac{d}{ds}\Gamma(\xi, s) = \left(\frac{dt_D}{ds}, \frac{dr_D}{ds}\right) = (1, w), \quad (4.77)$$

then the problem along each integral curve is formulated as follows:

$$\frac{dT_D}{ds}(\xi, s) = f(r_D(\xi, s), t_D(\xi, s)) \quad T_D(\xi, 0) = 0 \quad (4.78)$$

$$\frac{dr_D}{ds}(\xi, s) = w(r_D(\xi, s), t_D(\xi, s)), \quad r_D(\xi, s = 0) = \xi \quad (4.79)$$

$$\frac{dt_D}{ds}(\xi, s) = 1, \quad t_D(\xi, 0) = 0 \quad (4.80)$$

Once the integral curves are computed, which can be done using a numerical method if necessary, the ordinary differential equation for $T_D$ can be solved easily and the solution is given by:

$$T_D(\xi, s) = \int_0^s f(\xi, s^*)ds^*. \quad (4.81)$$

To obtain a more explicit representation note that we can rewrite:

$$f(r_D, t_D) = -P_e \left( \zeta \left( \frac{\partial P_D}{\partial t_D} + w \frac{\partial P_D}{\partial r_D} \right) + \left( \frac{\partial P_D}{\partial r_D} \right)^2 (1 - \zeta) \right) \quad (4.82)$$

$$f(r_D, t_D) = -P_e \left( \frac{dP_D}{ds} + \left( \frac{\partial P_D}{\partial r_D} \right)^2 (1 - \zeta) \right) \quad (4.83)$$
Therefore the dimensionless temperature drop is given by:

\[ T_D(\xi, s) = -P_e \zeta P_D(\xi, s) - P_e(1 - \zeta) \int_0^s \left( \frac{\partial P_D}{\partial r_D} \right)^2 (r_D(\xi, s^*), s^*) ds^*. \] (4.84)

To compute the temperature at the wellbore \((r_D = 1)\) we introduce the variable \(\xi_w(t_D)\) that tells us what was the initial position \(\xi\) of the particle that at time \(t_D\) is arriving at the well. Using this new variable and realizing that \(t_D = s\), the temperature at the well is given by:

\[ T_{wD}(t_D) = T_D(\xi_w(t_D), t_D) = -P_e \zeta P_{wD}(t_D) - P_e(1 - \zeta) \int_0^{t_D} \left( \frac{\partial P_D}{\partial r_D} \right)^2 (r_D(\xi_w(t_D), s^*), s^*) ds^*. \] (4.85)

To compute the dimensionless temperature derivative let us denote \(\sigma(t_D) = \xi_w'(t_D)\), then using the chain rule, the derivative of dimensionless temperature with respect to time is:

\[ \frac{dT_{wD}}{dt_D} = -P_e \zeta \frac{dP_{wD}}{dt_D} - P_e(1 - \zeta) \left( \frac{\partial P_D}{\partial r_D} \right)^2 (r_D = 1, t_D) - 2P_e(1 - \zeta) \int_0^{t_D} \frac{\partial P_D}{\partial r_D} \frac{\partial^2 P_D}{\partial r_D^2} \frac{\partial r_D}{\partial \xi} \sigma ds^*. \] (4.86)

For an equivalent homogeneous reservoir the dimensionless pressure drop caused by a line source well is given by:

\[ P_D = -\frac{1}{2} E_i \left( -\theta P_e \frac{r_D^2}{4t_D} \right) \] (4.87)

where \(\theta = \alpha_T/\alpha_H\) is the ratio of thermal and hydraulic diffusivities. The derivatives
of the dimensionless pressure drop are:

\[
\frac{\partial P_D}{\partial t_D} = \frac{1}{2} \frac{e^{-\theta P_e \frac{r_D^2}{4t_D}}}{t_D} \tag{4.88}
\]

\[
\frac{\partial P_D}{\partial r_D} = -\frac{e^{-\theta P_e \frac{r_D^2}{4t_D}}}{r_D} \tag{4.89}
\]

Recalling that \( \frac{\partial P_D}{\partial r_D} = \frac{dr_D}{ds} \) and because \( 0 < e^{-\theta P_e \frac{r_D^2}{4t_D}} \leq 1 \) for \( t_D > 0 \) we have the following estimates:

\[
\int_0^{t_D} \left( \frac{\partial P_D}{\partial r_D} \right)^2 (\xi_w(t_D), s^*) ds^* = \int_{\xi_w(t_D)}^{1} \frac{\partial P_D}{\partial r_D} dr_D \tag{4.90}
\]

\[
= \int_{1}^{\xi_w(t_D)} e^{-\theta P_e \frac{r_D^2}{4t_D}} \frac{1}{r_D} dr_D \tag{4.91}
\]

\[
\leq \int_{1}^{\xi_w(t_D)} d \ln(r_D) = \ln(\xi_w(t_D)) \tag{4.92}
\]

Therefore we have the following approximations for the dimensionless temperature drop at the wellbore for \( t_D \to \infty \):

\[
T_{wD}(t_D) = -P_e \xi P_{wD}(t_D) - P_e (1 - \zeta) \ln(\xi_w(t_D)) \tag{4.93}
\]

\[
t_D \frac{dT_{wD}}{dt_D} = -\frac{P_e \xi}{2} - P_e (1 - \zeta) \frac{\sigma(t_D)}{\xi_w(t_D)} \tag{4.94}
\]

Because we are already using the long time approximation, then the integral curves \( \Gamma(\xi, t_D) = (t_D, r_D(\xi, t_D)) \) must satisfy the following equation for long times:

\[
\frac{dr_D}{dt_D} = \frac{\partial P_D}{\partial r_D} \approx \frac{1}{r_D} \tag{4.95}
\]

therefore:

\[
r_D(\xi, t_D) = \sqrt{\xi^2 - 2t_D}. \tag{4.96}
\]
Solving for $\xi$ and evaluating at $r_D = 1$:

$$\xi_w = \sqrt{1 + 2t_D}. \quad (4.97)$$

Therefore $\sigma(t_D)$ becomes:

$$\sigma(t_D) = \frac{d\xi_w(t_D)}{dt_D} = \frac{1}{\xi_w(t_D)} \quad (4.98)$$

and the derivative of the dimensionless temperature drop is given by:

$$t_D \frac{dT_{wD}}{dt_D} = -P_e \left( \frac{\xi}{2} + t_D \frac{1 - \xi}{1 + 2t_D} \right). \quad (4.99)$$

Taking the limit as $t_D \to \infty$:

$$t_D \frac{dT_{wD}}{dt_D} = -\frac{P_e}{2} \quad (4.100)$$

or in terms of real variables:

$$t \frac{dT_w}{dt} = -\mu_{JT} \frac{q_w B_o \mu}{4\pi k F_b H_w}. \quad (4.101)$$

Recalling that for long times the derivative of the dimensionless pressure drop is:

$$t_D \frac{dP_{wD}}{dt_D} = \frac{1}{2} \quad (4.102)$$

or in terms of real variables:

$$t \frac{dP_w}{dt} = -\frac{q_w B_o \mu}{4\pi k F_b H_w} \quad (4.103)$$

This implies that the ratio of the temperature logarithmic derivative to the pressure logarithmic derivative is:

$$t \frac{dT_w}{dt} \left( t \frac{dP_w}{dt} \right)^{-1} = \mu_{JT}. \quad (4.104)$$
Therefore if the Joule-Thomson coefficient is constant throughout the analysis and the heat conduction influence is negligible, the value of the apparent Joule-Thomson coefficient of the fluid inside the reservoir can be easily obtained from Equation (4.104).

In general for late times and under convection dominated heat flow, the dimensionless temperature drop at any position is given by the following equation:

\[
T_D(r_D, t_D) = -P_e \zeta P_D(r_D, t_D) + \frac{P_e (1 - \zeta)}{2} \ln \left( \frac{r_D^2}{r_D^2 + 2t_D} \right)
\]  

(4.105)

4.3 Chapter Summary

The mathematical analysis of the mass and energy equations was performed for a double porosity reservoir. We assumed that the mass conservation equation can be accurately approximated by the pressure equation for a slightly compressible fluid in isothermal flow. We derived the pressure and energy equations for the fracture medium by taking averages of the equations over the fracture thickness. This gave us the same formulation as the one introduced by de Swaan [14] for the transient interporosity flow model. Notably the interporosity heat flow due to convection does not appear explicitly due to the mass conservation equation.

We gave the exact solution for the fracture pressure equation in Laplace domain and derived approximate solutions for early-time, intermediate-time and late-time behavior by invoking physical reasoning to decouple the matrix and fracture equations or to simplify the transfer function.

To analyze the energy equation, we appealed to physical reasoning to obtain an equivalent reservoir equation where the matrix and fracture media behave as one equivalent medium. We introduced dimensionless variables that fully described the behavior of the equivalent reservoir and are equally applicable for a single porosity reservoir with the appropriate modifications. The most important parameters that govern the energy equation are the dimensionless Peclet number \( P_e \) and the parameter \( \zeta \). To solve the energy equation, we considered the scenario were the Peclet number is sufficiently large such that convection is the dominant heat transport mechanism and obtained the solution of this approximate problem. The most important characteristic
is that for late-time behavior the ratio of the temperature derivative to the pressure derivative is equal to the Joule-Thomson coefficient; therefore the temperature by itself gives sufficient information to determine at least the same parameters as the pressure and obviously complement the pressure analysis.
Chapter 5

Numerical Model

While the approximate solution given in Chapter 4 is important and explains the temperature behavior for late times and high fluid velocities, we need to understand the range of applicability and accuracy of the approximate solution. To achieve this, we developed a numerical model that simulates the temperature behavior considering both convection and conduction transport mechanisms, as well as the interporosity heat flux. The numerical simulator of the double porosity model is also based on the reservoir geometry proposed by Kazemi [20] for a double porosity reservoir.

5.1 Dimensionless Equations

To compare the results between the numerical solution and the analytical approximation, we considered the partial differential equations in dimensionless form. To achieve this, we introduced the new dimensionless variables:
where \( \omega_T \) is the ratio of fracture thermal storage to the reservoir thermal storage and is analogous to the storativity ratio \( \omega_H \) defined by Warren and Root, \( \bar{k}_M \) is the bulk permeability ratio and \( \gamma_M \) is the thermal conductivity ratio. Note that using the new variable \( x_D \) greatly simplifies the radial component of the laplacian. Using the chain rule the radial component of the laplacian becomes:

\[
\nabla^2_{r_D} = \frac{1}{r_D} \frac{\partial}{\partial r_D} \left( r_D \frac{\partial}{\partial r_D} \right) \tag{5.6}
\]

\[
= e^{-2x_D} \frac{\partial^2}{\partial x_D^2} \tag{5.7}
\]

This coordinate transformation allowed us to use all the theory developed in the numerical analysis of partial differential equations for Cartesian coordinate systems. In fact the reservoir domain \( \Omega \) in \((x_D, z_D)\) coordinates is a rectangle with dimensions

\[
\Omega = \{(x_D, z_D)|0 \leq x_D \leq x_{De}, 0 \leq z_D \leq z_{De}/2\}. \tag{5.8}
\]

where \( x_{De} \) is the outer boundary of the reservoir and \( z_{De} = (h_M + h_F)/h_F \) is the dimensionless repetitive element thickness.
5.1.1 Pressure Equation

Using these new variables the dimensionless pressure equation for the matrix medium \((1/2 \leq z_D \leq z_{De}/2)\) is:

\[
(1 - \omega_H) \frac{\partial P_D}{\partial t_D} = \frac{\overline{k}_M}{P_c \theta} \left( e^{-2x_D} \frac{\partial^2 P_D}{\partial x_D^2} + \frac{r_w^2}{h_F^2} \frac{\partial^2 P_D}{\partial z_D^2} \right)
\]  
(5.9)

subject to the boundary conditions:

\[
\frac{\partial P_D}{\partial x_D}(x_D = 0, z_D, t_D) = 0
\]  
(5.10)

\[
\frac{\partial P_D}{\partial x_D}(x_{De} = 0, z_D, t_D) = 0
\]  
(5.11)

\[
\frac{\partial P_D}{\partial z_D}(x_D, z_D = z_{De}/2, t_D) = 0
\]  
(5.12)

Similarly, the dimensionless pressure for the fracture equation \((z_D = 0)\) is given by:

\[
\omega_H \frac{\partial P_D}{\partial t_D} = \frac{1}{P_c \theta} \left( e^{-2x_D} \frac{\partial^2 P_D}{\partial x_D^2} + 2\overline{k}_M \frac{r_w^2}{h_F^2} \chi_M \frac{\partial P_D}{\partial z_D} \right)_{z_D=1/2}
\]  
(5.13)

subject to the boundary conditions:

\[
\frac{\partial P_D}{\partial x_D}(x_D = 0, z_D = 0, t_D) = -1
\]  
(5.14)

\[
\frac{\partial P_D}{\partial x_D}(x_{De} = 0, z_D = 0, t_D) = 0
\]  
(5.15)

\[
\frac{\partial P_D}{\partial z_D}(x_D, z_D = 0, t_D) = 0
\]  
(5.16)

To complete the pressure problem the initial condition for the pressure at any point inside the reservoir is \(P_D(x_D, z_D, t_D = 0) = 0\).
5.1.2 Energy Equation

The dimensionless energy equation for the matrix ($1/2 \leq z_D \leq z_{De}/2$) is given by:

\[
(1 - \omega_T) \frac{\partial T_D}{\partial t_D} + \bar{K}_M \left( e^{-2x_D} \frac{\partial P_D}{\partial x_D} \frac{\partial T_D}{\partial x_D} + \frac{r_w^2}{h_F^2} \frac{\partial P_D}{\partial z_D} \frac{\partial T_D}{\partial z_D} \right) = \\
+ \frac{\gamma_M}{P_e} \left( e^{-2x_D} \frac{\partial^2 T_D}{\partial x_D^2} + \frac{r_w^2}{h_F^2} \frac{\partial^2 T_D}{\partial z_D^2} \right) - (1 - \omega_H) P_e \zeta \frac{\partial P_D}{\partial t_D} \\
- P_e \bar{K}_M \left( e^{-2x_D} \left( \frac{\partial P_D}{\partial x_D} \right)^2 + \frac{r_w^2}{h_F^2} \left( \frac{\partial P_D}{\partial z_D} \right)^2 \right) 
\]

subject to the boundary conditions:

\[
\frac{\partial T_D}{\partial x_D}(x_D = 0, z_D, t_D) = 0 \quad (5.18) \\
\frac{\partial T_D}{\partial x_D}(x_{De}, z_D, t_D) = 0 \quad (5.19) \\
\frac{\partial T_D}{\partial z_D}(x_D, z = z_{De}/2, t_D) = 0 \quad (5.20)
\]

Similarly the dimensionless energy equation for the fracture ($z_D = 0$) is:

\[
\omega_T \frac{\partial T_D}{\partial t_D} + e^{-2x_D} \frac{\partial P_D}{\partial x_D} \frac{\partial T_D}{\partial x_D} = \frac{1}{P_e} \left( e^{-2x_D} \frac{\partial^2 T_D}{\partial x_D^2} + 2 \gamma_M \frac{r_w^2}{h_F^2} \frac{\chi_F}{\chi_M} \left( \frac{\partial T_D}{\partial z_D} \right)_{z_D=1/2} \right) \\
- P_e \left( \omega_H \zeta \frac{\partial P_D}{\partial t_D} + e^{-2x_D} \left( \frac{\partial P_D}{\partial x_D} \right)^2 \right) 
\]

subject to the boundary conditions:

\[
\frac{\partial T_D}{\partial x_D}(x_D = 0, z_D = 0, t_D) = P_e \quad (5.22) \\
\frac{\partial T_D}{\partial x_D}(x_{De}, z_D = 0, t_D) = 0 \quad (5.23)
\]
To understand why the temperature spatial derivative at the well \((x_D = 0)\) is equal to the Peclet number, recall the enthalpy constitutive equation:

\[
dh = c_p dT - c_p \mu J_T dP
\]  

(5.24)

The fluid flow from the reservoir to the wellbore can be seen as fluid moving through a porous plug. Because there is no heat added to the reservoir from the wellbore, it follows that the fluid enthalpy at the reservoir must be equal to the enthalpy inside the wellbore, thus:

\[
\left( \frac{\partial h}{\partial r} \right)_{r=r_w} = 0
\]  

(5.25)

which implies that at \(r = r_w\):

\[
\frac{\partial T}{\partial r} = \mu J_T \frac{\partial P}{\partial r}
\]  

(5.26)

or in dimensionless form:

\[
\frac{\partial T_D}{\partial r_D} = -P_e \frac{\partial P_D}{\partial r_D}
\]  

(5.27)

and after substituting the pressure boundary condition at the well \((\partial P_D/\partial r_D = -1)\) we obtain Equation (5.22).

### 5.2 Discretization

#### 5.2.1 Domain

To discretize the reservoir domain \(\Omega\) we introduced \(M + 1\) equally spaced grid points in the \(x_D\) direction:

\[
0 = x_0 < x_1 < \cdots < x_M = x_{De}
\]  

(5.28)
and \( N + 1 \) equally spaced grid points in the \( z_D \) direction:

\[
0 = z_0 < z_1 < \cdots < z_N = z_{De}/2
\]

(5.29)

The nomenclature used here is the one proposed by Larsson and Thomée [22]. Let \( j_1 \) and \( j_2 \) be two variables that take the values \( j_1 = 0, \ldots, M \), \( j_2 = 0, \ldots, N \), then we can express \( x_{j_1} = j_1 h_x \) and \( z_{j_2} = j_2 h_z \) where \( h_x \) and \( h_z \) are the uniform spacing between two adjacent grid points in the \( x_D \) and \( z_D \) directions respectively. Time is discretized in a similar way as \( t_k = k h_t \) for \( k \geq 0 \) an integer. Now let \( j = (j_1, j_2) \) be a multi-index such that:

\[
T_D(x_{j_1}, z_{j_2}, t_k) = w_k^j
\]

(5.30)

\[
P_D(x_{j_1}, z_{j_2}, t_k) = u_k^j
\]

(5.31)

where \( w_k^j \) and \( u_k^j \) are the numerical solutions of dimensionless temperature and dimensionless pressure respectively.

### 5.2.2 Pressure Equation

For the discretization of the pressure equation we used a second-order central finite difference scheme, and an implicit first-order scheme for the time derivatives. Denoting \( e_x = (1, 0) \) and \( e_z = (0, 1) \) the discretize pressure equation for the fracture \( j_2 = 0 \) is:

\[
\omega H \frac{u_j^{k+1} - u_j^k}{h_t} = \frac{1}{P_e \theta} \left( e^{-2i h_x} \frac{u_{j+e_x}^{k+1} - 2u_j^{k+1} + u_{j-e_x}^{k+1}}{h_x^2} + 2k_M \frac{r_w x_F u_{j+e_z}^{k+1} - u_j^{k+1}}{h_F M h_z} \right)
\]

(5.32)

and for the matrix \( j_2 > 0 \):

\[
(1 - \omega H) \frac{u_j^{k+1} - u_j^k}{h_t} = k_M \frac{e^{-2i h_x} \frac{u_{j+e_x}^{k+1} - 2u_j^{k+1} + u_{j-e_x}^{k+1}}{h_x^2}}{P_e \theta} + \frac{r_w x_F u_{j+e_z}^{k+1} - 2u_j^{k+1} + u_{j-e_z}^{k+1}}{h_F^2 M h_z^2}
\]

(5.33)
To discretize the boundary and initial conditions we introduced ghost cells outside Ω to obtain the following relations:

\[ u_j^0 = 0 \]  
\[ u_{j=-1,j_2>0}^k = u_{j=1,j_2>0}^k \]  
\[ u_{j=-1,j_2=0}^k = u_{j=1,j_2=0}^k + 2h_x \]  
\[ u_{j=M-1,j_2>0}^k = u_{j=M+1,j_2>0}^k \]  
\[ u_{j\geq0,j_2=N+1}^k = u_{j\geq0,j_2=N-1}^k \]  

For simplicity we define the following variables:

\[ D_{Fx} = \frac{h_x e^{-2j_1 h_x}}{P_e \theta h_x^2} \]  
\[ D_{Fz} = \frac{h_x k M r_w^2 X_F}{P_e \theta h_x^2 X_M h_z} \]  
\[ D_{Mx} = \frac{h_x k M e^{-2j_1 h_x}}{P_e \theta h_x^2} \]  
\[ D_{Mz} = \frac{h_x k M r_w^2}{P_e \theta h_x^2 h_z^2} \]  

and now the fracture equation \((0 \leq j_1 \leq M, j_2 = 0)\) can be expressed as:

\[ (-D_{Fx})u_{j_2 \epsilon_{x}}^{k+1} + (\omega_H + 2D_{Fx} + 2D_{Fz})u_j^{k+1} + (-D_{Fx})u_{j_2 + \epsilon_{x}}^{k+1} + (-2D_{Fz})u_{j_2 + \epsilon_{x}}^{k+1} = \omega_H u_j^k \]  

Similarly the matrix equation \((0 \leq j_1 \leq M, 0 \leq j_2 \leq N)\) is:

\[ -D_{Mz}u_{j_2 \epsilon_{x}}^{k+1} - D_{Mx}u_{j_2 + \epsilon_{x}}^{k+1} + (1 - \omega_H + 2D_{Mx} + 2D_{Mz})u_j^{k+1} - D_{Mx}u_{j_2 + \epsilon_{x}}^{k+1} - D_{Mz}u_{j_2 + \epsilon_{x}}^{k+1} = (1 - \omega_H)u_j^k, \]  

If we order the unknowns in a vector, cycling first through \(j_1\) with fixed \(j_2\) this system of equations leads to a block tridiagonal matrix, where the diagonal blocks are tridiagonal and the off-diagonal blocks are diagonal. This is a very common result
for the numerical solution of second-order partial differential equations in Cartesian grids.

5.2.3 Energy Equation

Unlike the pressure diffusion equation, the discretization of the energy equation is slightly more delicate due to its dual nature involving convective and diffusion terms. In general diffusion smooths the solution (adding regularity) meanwhile first-order derivatives may cause discontinuities. Therefore for a numerical scheme such as the Finite Difference method, that relies in a smooth solution for convergence the treatment of the convective terms must be done with cautious. It has been shown that to avoid spurious solutions the numerical procedure should mimic the analytical solution at least for simple cases. Because we know that for pure convection the solution is given along the characteristics or the integral curves of the velocity field, several methods have been proposed in the literature that replicate this behavior (for a discussion on this topic see [23]). One of the simplest methods is the so-called upwinding scheme for first order derivatives. Instead of using the central difference approximation, the upwinding scheme uses one-sided approximations depending on the velocity direction. Because we know that the “velocity” that transports the temperature signal is always from the matrix medium to the fracture medium and the from the fracture medium to the well for production, the upwinding discretization of first-order spatial derivatives is:

\[
\frac{\partial T_D}{\partial l_D} = \frac{w_{j+e_l}^k - w_j^k}{h_l}, \quad l = x, z \quad (5.45)
\]

Denoting as:

\[
\partial_l u_j^k = \frac{w_{j+e_l}^k - w_j^k}{h_l}, \quad l = x, z \quad (5.46)
\]

\[
\partial_t u_j^k = \frac{u_j^{k+1} - u_j^k}{h_t} \quad (5.47)
\]
the discretized energy equation for the fracture $j_2 = 0$ is:

$$\frac{w_{j}^{k+1} - w_{j}^{k}}{\omega} + e^{-2ih_{x}} \mu_{x}u_{j}^{k+1} \frac{w_{j+e_{x}}^{k+1} - w_{j}^{k+1}}{h_{x}} = e^{-2ih_{x}} \frac{w_{j+e_{x}}^{k+1} - 2w_{j}^{k+1} + w_{j-e_{x}}^{k+1}}{P_{e}}$$

$$+ \frac{2\gamma_{M}r_{w}^{2}}{P_{e}h_{F}^{2}X_{M}} \frac{w_{j+e_{x}}^{k} - w_{j}^{k}}{h_{z}} - P_{e} \left( \omega_{H}\zeta\partial_{t}u_{j}^{k+1} + e^{-2ih_{x}}(\partial_{x}u_{j}^{k+1})^{2} \right)$$

(5.48)

and for the matrix $j_2 > 0$:

$$\left(1 - \omega_{T}\right) \frac{w_{j}^{k+1} - w_{j}^{k}}{\omega} + \bar{k}_{M} \left( e^{-2ih_{x}} \mu_{x}u_{j}^{k+1} \frac{w_{j+e_{x}}^{k+1} - w_{j}^{k+1}}{h_{x}} + \frac{r_{w}^{2}}{h_{F}^{2}} \mu_{x}u_{j}^{k+1} \frac{w_{j+e_{x}}^{k+1} - w_{j}^{k+1}}{h_{z}} \right) =$$

$$+ \frac{\gamma_{M}}{P_{e}} \left( e^{-2ih_{x}} \frac{w_{j+e_{x}}^{k} - 2w_{j}^{k+1} + w_{j-e_{x}}^{k+1}}{h_{x}^{2}} + \frac{r_{w}^{2}}{h_{F}^{2}} \frac{w_{j+e_{x}}^{k} - 2w_{j}^{k+1} + w_{j-e_{x}}^{k+1}}{h_{z}^{2}} \right)$$

$$- P_{e} \left( 1 - \omega_{H}\zeta\partial_{t}u_{j}^{k+1} + \bar{k}_{M}e^{-2ih_{x}}(\partial_{x}u_{j}^{k+1})^{2} + \bar{k}_{M}\frac{r_{w}^{2}}{h_{F}^{2}}(\partial_{x}u_{j}^{k})^{2} \right)$$

(5.49)

The boundary conditions are:

$$w_{j}^{0} = 0$$

$$w_{j}^{k} \left( j_{1} = -1, j_{2} > 0 \right) = w_{j}^{k} \left( j_{1} = 1, j_{2} > 0 \right)$$

$$w_{j}^{k} \left( j_{1} = -1, j_{2} = 0 \right) = w_{j}^{k} \left( j_{1} = 1, j_{2} = 0 \right) - 2P_{e}h_{x}$$

$$w_{j}^{k} \left( j_{1} = M - 1, j_{2} \geq 0 \right) = w_{j}^{k} \left( j_{1} = M + 1, j_{2} \geq 0 \right)$$

$$w_{j}^{k} \left( j_{1} \geq 0, j_{2} = N - 1 \right) = w_{j}^{k} \left( j_{1} \geq 0, j_{2} = N + 1 \right)$$

(5.50)

(5.51)

(5.52)

(5.53)

(5.54)
Defining the following variables:

\[
C_{Fx} = \frac{h_t e^{-2 j_1 h_z}}{h_x} \partial_x u^{k+1}_j \quad (5.55)
\]

\[
D_{TFx} = \frac{h_t e^{-2 j_1 h_z}}{P_e h_z^2} \quad (5.56)
\]

\[
D_{TFz} = \frac{h_t r_w^2 \gamma_M X_F}{P_e h_z^2 X_M h_z} \quad (5.57)
\]

\[
f_{F_j}^{k+1} = h_t P_e \left( \omega_H \partial_t u^{k+1}_j + e^{-2 i h_z} (\partial_x u^{k+1}_j)^2 \right) \quad (5.58)
\]

\[
C_{Mx} = \frac{h_t e^{-2 j_1 h_z}}{h_x} \tilde{k}_M \partial_x u^{k+1}_j \quad (5.59)
\]

\[
C_{Mz} = \frac{h_t r_w^2 \tilde{k}_M}{h_z^2} \partial_z u^{k+1}_j \quad (5.60)
\]

\[
D_{TMx} = \frac{h_t r_w^2 \gamma_M}{P_e h_x^2} \quad (5.61)
\]

\[
D_{TMz} = \frac{h_t r_w^2 \gamma_M}{P_e h_x^2} \quad (5.62)
\]

\[
f_{M_j}^{k+1} = h_t P_e \left( (1 - \omega_H) \tilde{k}_M \partial_t u^{k+1}_j + \tilde{k}_M e^{-2 i h_z} (\partial_x u^{k+1}_j)^2 + \tilde{k}_M \frac{r_w^2}{h_F^2} (\partial_z u^{k+1}_j)^2 \right) \quad (5.63)
\]

The fracture equation \(0 \leq j_1 \leq M, j_2 = 0\) is given by:

\[
-D_{TFx} w^{k+1}_{j-\epsilon_x} + (\omega_T - C_{Fx} + 2 D_{TFx} + 2 D_{TFz}) w^{k+1}_j + (C_{Fx} - D_{TFx}) w^{k+1}_{j+\epsilon_x} - 2 D_{TFz} w^{k+1}_{j+\epsilon_x} = \omega_T w^{k}_j - f_{M_j}^{k+1} \quad (5.64)
\]

and the matrix equation \(0 \leq j_1 \leq M, 0 < j_2 \leq N\) is:

\[
-D_{TMz} w^{k+1}_{j-\epsilon_z} - D_{TMx} w^{k+1}_{j-\epsilon_x} + \left( 1 - \omega_T - C_{Mx} + 2 D_{TMx} - C_{Mz} + 2 D_{TMz} \right) w^{k+1}_j + (C_{Mx} - D_{TMx}) w^{k+1}_{j+\epsilon_x} + (C_{Mz} - D_{TMz}) w^{k+1}_{j+\epsilon_x} = (1 - \omega_T) w^{k}_j - f_{M_j}^{k+1} \quad (5.65)
\]

Again, if we organize the unknown variables in a vector cycling first through the \(j_1\) index we end up we a block tridiagonal matrix.
Chapter 6

Results and Discussion

In this Chapter we used the numerical simulator developed in Chapter 5 to compare and determine the accuracy of the approximate solution developed in Chapter 4:

\[ T_D(r_D = 1, t_D) = -P_e \zeta P_D(r_D = 1, t_D) - P_e \frac{1 - \zeta}{2} \ln(1 + 2t_D) \] (6.1)

As mentioned in Chapter 4, for long times the logarithmic derivative of dimensionless temperature approaches \(-P_e/2\); therefore, for the analysis we referred to the normalized dimensionless temperature:

\[ -\frac{T_D}{P_e}(r_D = 1, t_D) \] (6.2)

because the logarithmic derivative of the normalized dimensionless temperature behaves similarly to the logarithmic derivative of the dimensionless pressure.
6.1 Base case

Let us consider a base case scenario with the properties shown in Tables 6.4 through 6.3. For this case we have that:

\[ P_e = 980.49 \] (6.3)
\[ \zeta = -7.7 \times 10^{-3} \] (6.4)
\[ \omega_H = 2.43 \times 10^{-2} \] (6.5)
\[ \omega_T = 3 \times 10^{-3} \] (6.6)
\[ \bar{k}_M = 4.989 \times 10^{-4} \] (6.7)
\[ \gamma_M = 596.24 \] (6.8)
\[ \theta = 2.82 \times 10^{-9} \] (6.9)

Notice that because the well produces from a very thin fracture layer the Peclet number is very large even for a modest production flow rate (115 STBPD). Furthermore, the permeability ratio \( \bar{k}_M \) is very small (common characteristic of all double porosity reservoirs) but the thermal conductivity ratio \( \gamma_M \) is quite large. This occurs because one of the main differences between the pressure and the energy equation is that in the energy equation both solid and fluid contribute to the heat conduction and the thermal conductivity of the fluid and the solid is very similar.

Figures 6.1 to 6.4 show the numerical results of the base case scenario. As can be seen from Figure 6.3 the early-time and late-time approximate solutions developed in Chapter 4 describe the expected behavior very well. Furthermore Figure 6.4 shows that for late times the normalized dimensionless temperature derivative \(-t_D T_D'/P_e\) stabilizes at the value of 1/2 just as predicted from the late-time approximate solution. A comparison between the pressure and the temperature diagnostic plots Figures 6.2 and 6.4 respectively, shows that the transition period is more important and last longer in the temperature transient just as expected from the large thermal storativity ratio \( \omega_T \) and even approximates the value of 1/4 that is expected from a transition period showing a linear flow regime.
### Table 6.1: Solid properties.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density, $\rho_s$</td>
<td>1400</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>Thermal conductivity, $\lambda_s$</td>
<td>5</td>
<td>W/(°Km)</td>
</tr>
<tr>
<td>Heat capacity, $c_{p_s}$</td>
<td>750</td>
<td>J/(°Kkg)</td>
</tr>
</tbody>
</table>

### Table 6.2: Reservoir properties.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fracture porosity, $\phi_F$</td>
<td>0.45</td>
<td></td>
</tr>
<tr>
<td>Matrix porosity, $\phi_M$</td>
<td>0.05</td>
<td></td>
</tr>
<tr>
<td>Fracture thickness, $h_F$</td>
<td>0.05</td>
<td>ft</td>
</tr>
<tr>
<td>Matrix thickness, $h_M$</td>
<td>18.05</td>
<td>ft</td>
</tr>
<tr>
<td>Fracture permeability, $k_F$</td>
<td>7236</td>
<td>mD</td>
</tr>
<tr>
<td>Matrix permeability, $k_M$</td>
<td>0.01</td>
<td>mD</td>
</tr>
</tbody>
</table>

### Table 6.3: Well properties.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flow rate, $q_w$</td>
<td>115</td>
<td>STBD</td>
</tr>
<tr>
<td>Formation volume factor, $B_o$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Well radius, $r_w$</td>
<td>0.375</td>
<td>ft</td>
</tr>
<tr>
<td>Total thickness, $H_w$</td>
<td>18.1</td>
<td>ft</td>
</tr>
</tbody>
</table>

### Table 6.4: Fluid properties.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Viscosity, $\mu$</td>
<td>1.0</td>
<td>cp</td>
</tr>
<tr>
<td>Density, $\rho_f$</td>
<td>800</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>Compressibility, $c_f$</td>
<td>$1 \times 10^{-6}$</td>
<td>psi$^{-1}$</td>
</tr>
<tr>
<td>Thermal conductivity, $\lambda_f$</td>
<td>0.3</td>
<td>W/(°Km)</td>
</tr>
<tr>
<td>Heat capacity, $c_{p_f}$</td>
<td>1600</td>
<td>J/(°Kkg)</td>
</tr>
<tr>
<td>Joule-Thomson coefficient, $\mu_{JT}$</td>
<td>$-4.5 \times 10^{-7}$</td>
<td>°K/Pa</td>
</tr>
<tr>
<td>Adiabatic heat expansion coefficient, $\eta$</td>
<td>$5.625 \times 10^{-8}$</td>
<td>°K/Pa</td>
</tr>
<tr>
<td>Formation volume factor, $B_o$</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
Figure 6.1: Semilog analysis of $P_D$ for the base case scenario.

Figure 6.2: Diagnostic plot of $P_D$ for the base case scenario.
CHAPTER 6. RESULTS AND DISCUSSION

Figure 6.3: Semilog analysis of $-T_D$ for the base case scenario.

Figure 6.4: Diagnostic plot of $T_D$ for the base case scenario.
6.2 Single Porosity Analysis

It is also important to study how the double porosity behavior differs from the temperature transient of a single porosity reservoir. For this case we considered a single porosity reservoir with average permeability $k = 10mD$, porosity $\phi = 0.2$ and all other properties equal to the base case scenario. This single porosity reservoir is characterized by the following dimensionless variables:

$$P_e = 1.74$$  \hspace{1cm} (6.10)

$$\zeta = -2.9 \times 10^{-2}$$  \hspace{1cm} (6.11)

$$\theta = 1.088 \times 10^{-4}$$  \hspace{1cm} (6.12)

Then we performed a sensitivity analysis of the Peclet number, by changing the well flow rate and keeping all other quantities constant. Figure 6.5 shows that for low Peclet numbers the approximate analytical solution underestimates the normalized dimensionless temperature. It is important to notice that for a negative Joule-Thomson coefficient the fluid is increasing its temperature as the Peclet number increases but the normalized temperature change decreases as the Peclet number increases.

Noteworthy for all values of Peclet numbers the Figure 6.5 shows that all curves are parallel. This observation is confirmed in Figure 6.6 with all the curves converging to a value of $t_D T'_D / P_e = 1/2$. Therefore we can conclude that the temperature behavior for a wide range of Peclet numbers has a $-(P_e/2) \ln(t_D)$ behavior. Nevertheless because the dimensionless time is also a function of the Peclet number as shown in Equation (4.70), the time that is needed to reach this behavior is inversely proportional to the Peclet number; therefore, for low Peclet numbers the well test must be impractically long to observe this behavior. Moreover because the temperature change is proportional to the Peclet number, small Peclet numbers produce a temperature change difficult to measure which might lead to erroneous temperature transient analysis.
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Figure 6.5: Sensitivity of $-T_D/P_e$ to Peclet number for a single porosity reservoir.

Figure 6.6: Sensitivity of $-t_D T_D'/P_e$ to Peclet number for single porosity reservoir.
6.3 Applicability of Approximate Solution

To determine the range of applicability of the approximate solution obtained in Chapter 4 we performed a sensitivity analysis by varying the Peclet number $P_e$ and the thermal conductivity ratio $\gamma_M$.

6.3.1 Peclet number

The Peclet number sensitivity analysis was conducted by changing the well flow rate while keeping all other quantities constant and equal to the base case scenario. Figure 6.7 shows the semilog analysis of the normalized dimensionless temperature for different values of Peclet numbers. As can be seen for $P_e = 9.8 \times 10^3$ (which corresponds to a well flow rate of $q_w = 1150 \text{STBD}$) and larger values of Peclet numbers the early-time and late-time analytical solutions are in perfect agreement to the results obtained from the numerical simulation.

For low Peclet numbers the temperature transient enters the transition period before actually developing the early-time behavior predicted by the approximate analytical solution. For late times, the late-time behavior solution overestimates the temperature change, thus the effect of conduction cools the fluid temperature. The difference between the late time behavior predicted from the approximate analytical solution and the numerical solution increases as the Peclet number decreases (poor accuracy for low Peclet numbers).

In contrast to what was observed in the single porosity reservoir, for sufficiently low Peclet numbers ($P_e = 98.05$), the temperature transient does not show a characteristic $\ln(t_D)$ behavior. Nevertheless Figure 6.8 shows that for low Peclet numbers, the logarithmic derivative of the normalized dimensionless temperature presents a minimum similar to the one shown in the diagnostic plot for pressure. Moreover Figure 6.8 shows that for high Peclet number the transition period approximates a value of the dimensionless normalized logarithmic derivative of:

$$\frac{t_D \partial T_D}{P_e \partial t_D} (r_D = 1, t_D) \approx \frac{1}{4}$$

(6.13)
and it last at that value for approximate three log cycles. Also as predicted by the approximate solution, all curves with $P_e \geq 980$ reach the value of:

$$\left.\frac{t_D}{P_e} \frac{\partial T_D}{\partial t_D} \right|_{(r_D = 1, t_D)} \approx \frac{1}{2}$$

Converting the value of dimensionless time to real time, we realized that the time needed to reach this late time convective dominated behavior is inversely proportional to the Peclet number. Thus the higher the Peclet number, the sooner the temperature transient reaches the late-time behavior.

### 6.3.2 Thermal Conductivity ratio

The sensitivity analysis of the thermal conductivity ratio was done simply by taking the base case scenario values, and changing the thermal conductivity ratio while keeping all other variables constant.

Figures 6.9 and 6.10 show the results of changing the thermal conductivity ratio.
It seems that reducing the thermal conductivity ratio by some amount is nearly equivalent as increasing the Peclet number by the same amount in terms of the normalized dimensionless temperature. From this observation we conclude that the convective dominated analytical solution for long times developed in Chapter 4 is valid for:

$$\frac{P_e}{\gamma_M} \geq 2.$$  \hspace{1cm} (6.15)

Nevertheless observing the form of the dimensionless energy equation for the fracture medium Equation (5.21) it is clear that the overall parameter that governs the influence of the interporosity heat flux is:

$$\nu = 2\gamma_M \frac{\chi F w^2}{\chi M h_F^2}.$$  \hspace{1cm} (6.16)

this parameter plays the same role as the interporosity coefficient $\lambda$ introduced by
Warren and Root [32] and will be referred to as the thermal interporosity coefficient. Thus, it is more meaningful to give a range of values considering the thermal interporosity coefficient $\nu$ and Peclet number $P_e$. From the values considered in the analysis we can conclude that the analytical solution is valid for:

$$\frac{P_e}{\nu} \geq 50.$$  

(6.17)

For example, for the base case scenario we have that $\nu = 18.5$ and $P_e/\nu = 53$. The validity of the analytical solutions can be confirmed form Figures 6.3 and 6.4 that show the normalized dimensionless temperature as function of time.
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6.4 Thermal and Hydraulic Storativity ratios

The hydraulic and thermal storativity ratios play a very important role in the theory of double porosity reservoirs. Essentially both storativity ratios relate a fracture volumetric property (hydraulic storage $\phi c_f$ or thermal storage $\rho c_p$) to a reservoir property, thus the storativity ratios are related to each other. Figure 6.11 shows the relationship between the storativity ratios for different values of matrix and fracture intrinsic porosities using the solid and fluid properties considered in the base case scenario. For small values of $\omega_H$ we have the following relationship:

$$\ln(\omega_T) = \ln(\omega_H) + \ln \left( \frac{\phi_M \rho_f c_{pf} \phi_F + \rho_s c_{ps} (1 - \phi_F)}{\phi_F \rho_f c_{pf} \phi_M + \rho_s c_{ps} (1 - \phi_M)} \right)$$

(6.18)

if $\rho_f c_{pf} \approx \rho_s c_{ps}$ the relationship simplifies to:

$$\ln(\omega_T) = \ln(\omega_H) + \ln \left( \frac{\phi_M}{\phi_F} \right)$$

(6.19)
Is common that the fracture intrinsic porosity $\phi_F$ is quite large due to vugs and caverns and typical values considered in the literature are $\phi_F \geq 0.5$. Meanwhile the theoretical maximum intrinsic porosity achievable by the matrix (without vugs or caverns) is obtained from a cubic grain packing with $\phi_M = 0.476$. Therefore for typical values of intrinsic porosities the hydraulic storativity ratio is smaller than the thermal storativity ratio.

To honor the fact that the storativity ratios are related, we performed a sensitivity analysis by only changing the matrix layer thickness $h_M$ and keeping all other variables constant and equal to the base case scenario. To model the case where convection dominates we chose a value of $P_e = 9805$ or $q_w = 1150 STBPD$.

Figures 6.12 and 6.13 show the numerical results of the dimensionless pressure and its logarithmic derivative as function of $t_D/(P_e \theta)$. Recall that $t_D/(P_e \theta)$ is the usual dimensionless time used in the theory of pressure transient analysis. These figures show the typical behavior already studied in the literature of a double porosity reservoir for different values of the hydraulic storativity ratio.

Figures 6.14 and 6.15 show the behavior of normalized dimensionless temperature as function of dimensionless time. We can see that for late times all curves converge to the late-time analytical solution, and the relationship between $\omega_T$ and $T_D$ is the same as the relationship between $\omega_H$ and $P_D$. In fact from Chapter 4 we know that for early-time behavior the dimensionless pressure and dimensionless temperature is given by:

$$P_{D_{\text{early}}}(r_D = 1, t_D) = \frac{1}{2} \ln \left( \frac{1.781}{P_e \theta \omega_H} \frac{4 t_D}{P_e \theta \omega_H} \right)$$

$$T_{D_{\text{early}}}(r_D = 1, t_D) = -\frac{P_e}{2} \left( \frac{\omega_H}{\omega_T} \left[ \ln \left( \frac{1.781}{P_e \theta \omega_H} \frac{4 t_D}{P_e \theta \omega_H} \right) \right. \right. \left. \left. - \ln \left( 1 + 2 \frac{t_D}{\omega_T} \right) \right] \right) + \ln \left( 1 + 2 \frac{t_D}{\omega_T} \right)$$

where we have used the property that for small arguments $Ei(-x) \approx \ln(1.781x)$.
Figure 6.11: Relationship between $\omega_H$ and $\omega_T$ for different values of matrix and fracture intrinsic porosities.
Similarly for late-time behavior we have that:

\[ P_{D_{\text{late}}}(r_D = 1, t_D) = \frac{1}{2} \ln \left( 1.781 \frac{4t_D}{Pe \theta} \right) \]  
(6.22)

\[ T_{D_{\text{late}}}(r_D = 1, t_D) = -\frac{P_e}{2} \left( \zeta \left[ \ln \left( 1.781 \frac{4t_D}{Pe \theta} \right) - \ln (1 + 2t_D) \right] + \ln (1 + 2t_D) \right) \]  
(6.23)

Therefore if we equate the early-time and late-time relationships such that:

\[ P_{D_{\text{early}}}(r_D = 1, t_{D1p}) = P_{D_{\text{late}}}(r_D = 1, t_{D2p}) \]  
(6.24)

\[ T_{D_{\text{early}}}(r_D = 1, t_{D1t}) = T_{D_{\text{late}}}(r_D = 1, t_{D2t}) \]  
(6.25)

then we can conclude that:

\[ \ln(t_{D2p}) - \ln(t_{D1p}) = -\ln(\omega_H) \]  
(6.26)

\[ \ln(t_{D2t}) - \ln(t_{D1t}) = -\ln(\omega_T) + \zeta \ln \left( \left[ \frac{2}{1.781Pe \theta} \right] \frac{\omega_H}{\omega_T}^{-1} \left( \frac{\omega_T}{\omega_H} \right)^{\frac{\omega_H}{\omega_T}} \right) \]  
(6.27)

For the base case scenario the second term of Equation (6.27) is of order \(O(10^{-1})\) meanwhile the first term is \(O(1)\). This explains why in Figure 6.14 it seems that the relationship between \(\omega_T\) and \(-T_D/P_e\) is the same as the relationship between \(\omega_H\) and \(P_D\).

Noteworthy Figure 6.15 shows that at the transition period the derivative approximate the value of \(-t_D P_D'/P_e \approx 1/4\) for all different values of \(\omega_T\).
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Figure 6.12: Sensitivity of $P_D$ to $\omega_H$.

Figure 6.13: Sensitivity of $t_D P_D'$ to $\omega_H$. 
CHAPTER 6. RESULTS AND DISCUSSION

Figure 6.14: Sensitivity of $-T_D/P_e$ to $\omega_T$.

Figure 6.15: Sensitivity of $-t_D T_D'/P_e$ to $\omega_T$. 
6.5 Permeability ratio

The influence of the permeability ratio $k_M / k_F$ was studied by changing the matrix permeability $k_M$ while keeping all other parameters constant. All the other parameters are equal to the base case scenario but the well flow rate was changed to $q_w = 1150 STBD$ to model the convective dominated scenario. As can be seen from Figures 6.16 and 6.17 the influence of the permeability ratio in the dimensionless pressure is to change the time at which the transition period starts, this result has been thoroughly studied in the literature (see [13],[19]). Nevertheless Figure 6.17 shows that there is no apparent influence of the permeability ratio on the normalized dimensionless temperature.
CHAPTER 6. RESULTS AND DISCUSSION

Figure 6.16: Sensitivity of $P_D$ to the permeability ratio $k_M/k_F$.

Figure 6.17: Sensitivity of $-T_D/P_e$ to the permeability ratio $k_M/k_F$. 
6.6 Chapter Summary

In this chapter, we studied the influence of the Peclet number, thermal storativity ratio, permeability ratio and thermal conductivity ratio on the dimensionless temperature.

For reference we investigated the behavior of a single porosity reservoir and found that even for low Peclet numbers the logarithmic derivative of the normalized dimensionless temperature $-t_D T'_D/P_e$ is equal to $-t_D T'_D/P_e = 1/2$ for sufficiently long times. Nevertheless only for $P_e \geq 2$ the analytical solution derived in Chapter 4 is a good approximation to the real temperature behavior.

Then we determined the range of applicability of the analytical solution for convective dominate flow. We introduced a new dimensionless variable $\nu$ that governs the interporosity heat flow and concluded that the analytical solution derived in Chapter 4 for convective dominated flow is only valid for values of $P_e/\nu \geq 50$. Within this range of applicability we discovered that the influence of the thermal storativity ratio on the dimensionless temperature is nearly the same as the influence of the hydraulic storativity ratio on the dimensionless pressure, and derived relationships that relate the separation of the early and late time approximations with the rock and fluid properties.

Furthermore we observed that during the transition period the logarithmic time derivative of the normalized dimensionless temperature $-t_D T'_D/P_e$ approaches the value of $-t_D T'_D/P_e = 1/4$ and for late times is equal to $-t_D T'_D/P_e = 1/2$. This is remarkably similar to the behavior shown by the logarithmic derivative of the dimensionless pressure and proves that the temperature is a promising and useful source of reservoir information.

Finally we found that the permeability ratio has no apparent influence on the dimensionless temperature.
Chapter 7

Concluding Remarks

7.1 Conclusions

The objective of this work was to study and identify the relevant characteristics of the temperature transient that occurs due to a well producing from a naturally fractured reservoir.

To study this problem, we considered two approaches:

- Analytical approach. In Chapter 4 we derived and gave a thorough explanation of the pressure behavior of a double porosity reservoir and an approximate equation to predict the temperature behavior when convection is the governing heat transport mechanism.

- Numerical. In Chapter 5 we showed the matrix and fracture equations in dimensionless form, to develop a numerical model that considers all heat transport mechanism (convection and diffusion) as well as the heat flow due to conduction between the matrix and the fracture.

As a result of these two approaches, we have the following principal results:

- The temperature behavior of a double porosity reservoir can be described fully with the following dimensionless parameters:
  1. Peclet number $P_e$
2. Thermal storativity ratio $\omega_T$

3. Interporosity thermal coefficient $\nu$

4. Dimensionless parameter $\zeta$

• The normalized dimensionless temperature $-T_D/P_e$ behaves similarly to the dimensionless pressure, in particular it shows a $\ln(t_D)$ behavior.

• The normalized dimensionless temperature when plotted against dimensionless time in logarithmic scale, shows two parallel lines for the early and late-time behavior with slope equal to $1/2$. This behavior is in agreement with the analytical solutions derived for these periods.

• Even though there is no mathematical proof, for convective-dominated heat flow the transition period in a plot of normalized dimensionless temperature against dimensionless time in logarithmic scale, shows a straight line with slope approximately equal to $1/4$.

• The analytical solutions derived for convective-dominated heat flow are valid for values of $P_e/\nu \geq 50$.

• The separation between the parallel lines that appear in the semilog analysis are related to the thermal storativity ratio in the sense that for $\omega_T = 0.1$ the lines are approximately separated by one log cycle.

• The ratio of the derivative of temperature to the derivative of pressure for convective-dominated heat flow and long times is equal to the Joule-Thomson coefficient.

• Because the temperature behavior responds to the behavior of the solid and the fluid and not only the fluid, the temperature is an excellent source of information of fracture and matrix volumes as can be seen from the fact that all the thermal properties are volumetric averages of the fluid and solid properties.

Based on these results we can conclude that the temperature behavior is an excellent source of information to characterize a double porosity reservoir. Also, in combination
with pressure information storage properties such as fracture volumes can be obtained by combining the results obtained from the analysis of the temperature transient.

7.2 Future work

The work found some important relationships between the temperature behavior and the rock properties. Nevertheless this work only gave mathematical equations for early and late-time behavior but not for the transient period. Moreover there seems to be a similar behavior in the derivative plot between the dimensionless temperature and pressure that should be investigated and can be source of important information for reservoir characterization specifically for the rock porosity in the matrix and the fracture media.

In addition this work considered several assumptions that may not be always valid in the application of this theory. The main assumptions that should be tested are the considerations that the pressure is independent of the temperature and that the thermal properties such as the Joule-Thomson coefficient and the adiabatic heat expansion coefficient can be assumed constant. From an analytical point of view, perhaps a good approach is to consider an integral transform similar to the one used to linearize the pressure equation for a highly compressible fluid.

One topic that was not discussed is the presence of boundaries, all the simulations were specifically considered to replicate the behavior of an infinite-acting reservoir. Because the temperature boundary conditions can be very different from the pressure boundary conditions, the difference between the temperature and pressure response to a boundary could contain valuable information of the reservoir boundaries.

Finally we gave a relationship to compute the in-situ Joule-Thomson coefficient from the ratio between the pressure and temperature derivatives. In general for a multiphase flow the Joule-Thomson coefficient will be varying according to the saturations of reservoir fluids; therefore, if a similar procedure is derived for multiphase flow the combined use of pressure and temperature can lead to important information of phase fractions in the vicinity of the wellbore.
Appendix A

Viscous Dissipation

The viscous dissipation function arises when considering the amount of work done that does not transforms into mechanical energy, thus this work only contributes to a change in the internal energy.

The total amount of work done by a fluid from a pore scale point of view is given by:

$$\nabla \cdot (\bar{v} \cdot \tau)$$ (A.1)

and the amount of work that contributes to the mechanical energy is given by:

$$\bar{v} \cdot (\nabla \cdot \tau)$$ (A.2)

where $\bar{v}$ is the fluid velocity and $\tau$ is the symmetric second-order stress tensor that is related to the fluid velocity of a Newtonian fluid by the constitutive relationship:

$$\tau_{ij} = -\left(P + \frac{2}{3}\mu \nabla \cdot \bar{v}\right) \delta_{ij} + \mu(\partial_{x_i} v_j + \partial_{x_j} v_i)$$ (A.3)

where $\delta_{ij}$ is the Kronecker delta and $\mu$ is the fluid viscosity. Then, using this relationship the amount of work that only contributes to the change in internal energy is
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given by:

\[ \nabla \cdot (\vec{v} \cdot \tau) - \vec{v} \cdot (\nabla \cdot \tau) = \sum_j \partial_{x_j} \left( \sum_i v_i \tau_{ij} \right) - \sum_i v_i \left( \sum_j \partial_{x_j} \tau_{ji} \right) = \sum_{ij} (\partial_{x_j} (v_i \tau_{ij}) - \tau_{ij} \partial_{x_j} v_i) = \sum_{ij} \tau_{ij} \partial_{x_j} v_i \quad (A.4) \]

substituting the definition for the stress tensor:

\[ \nabla \cdot (\vec{v} \cdot \tau) - \vec{v} \cdot (\nabla \cdot \tau) = - \sum_{ij} \left( P + \frac{2}{3} \mu \nabla \cdot \vec{v} \right) \delta_{ij} \partial_{x_j} v_i + \mu \sum_{ij} (\partial_{x_j} v_i + \partial_{x_i} v_j) \partial_{x_j} v_i \]

\[ = -P \nabla \cdot \vec{v} - \frac{2}{3} \mu (\nabla \cdot \vec{v})^2 + \mu \sum_{ij} (\partial_{x_j} v_i + \partial_{x_i} v_j) \partial_{x_j} v_i \]

\[ = -P \nabla \cdot \vec{v} + \tilde{\Phi} \quad (A.5) \]

where \( \tilde{\Phi} \) is the so-called fluid viscous dissipation from a pore scale point of view. For an incompressible fluid the viscous dissipation in a Cartesian coordinate system is given by:

\[ \tilde{\Phi} = 2 \mu \left[ (\partial_x v_x)^2 + (\partial_y v_y)^2 \right] + \mu (\partial_x v_y + \partial_y v_x)^2 \quad (A.6) \]

As can be seen, the viscous dissipation function is a strictly positive quantity; therefore, it increases the internal energy of the fluid irrespective of the flow regime and the rate of heat generation is proportional to the fluid viscosity. Because of this, the pore scale viscous dissipation can be interpreted as the heat generation due to the friction between the fluid particles.

The viscous dissipation for a Darcy scale point of view \( \Phi \), can be obtained by upscaling the pore scale viscous dissipation over a rock volume \( \Omega \) containing a pore volume \( \Omega_p \):

\[ \Phi = \frac{\int_{\Omega_p} \tilde{\Phi} dV}{\int_{\Omega} dV} \quad (A.7) \]
Figure A.1: Idealized pore configuration.

To obtain a more useful representation let us consider the scenario where the reservoir pore topology can be represented as a collection of horizontal tubes as shown in Figure A.1. Then the rock volume is given by $V = lwh$ and considering $N$ horizontal tubes the pore volume is $V_p = N\pi R^2 l$. Furthermore for this scenario we can assume that the flow is the same for all tubes and the velocity vector within each tube is given by:

$$\vec{v} = (0, v_y(r), 0)$$  \hspace{1cm} (A.8)

where $r$ represents the radial distance from the center of each tube in a local coordinate system. For this simplified scenario the pore scale viscous dissipation function becomes

$$\tilde{\Phi} = \mu \left( \frac{dv_y}{dr} \right)^2$$  \hspace{1cm} (A.9)
Then the Darcy scale viscous dissipation is given by:

\[
\Phi = \frac{N2\pi l \mu}{lwh} \int_0^R \left( \frac{dv_y}{dr} \right)^2 r dr \tag{A.10}
\]

considering Poiseuille flow, we know that the velocity is given by:

\[
v_y(r) = -\frac{1}{4\mu} \left( \frac{\partial P}{\partial y} \right) (R^2 - r^2) \tag{A.11}
\]

and assuming a constant pressure gradient we can conclude that:

\[
\Phi = \frac{N2\pi \mu}{wh} \left( \frac{1}{2\mu} \frac{\partial P}{\partial y} \right)^2 R^4 \tag{A.12}
\]

noticing that the rock porosity is given by:

\[
\phi = \frac{N\pi R^2}{wh} \tag{A.13}
\]

we can rewrite:

\[
\Phi = \phi \frac{R^2}{8\mu} \left( \frac{\partial P}{\partial y} \right)^2 \tag{A.14}
\]

Now recalling that the average velocity across a tube \( \tilde{v}_y \) in the \( y \) direction is:

\[
\tilde{v}_y = \frac{2}{R^2} \int_0^R v_y(r) r dr = -\frac{R^2}{8\mu} \left( \frac{\partial P}{\partial y} \right) \tag{A.15}
\]

and that the Darcy velocity \( u_y \) is related to the average velocity in the tube by \( u_y = \tilde{v}_y \phi \) we finally obtain:

\[
\Phi = -u_y \left( \frac{\partial P}{\partial y} \right) \tag{A.16}
\]

or for an arbitrary coordinate system:

\[
\Phi = -\vec{u} \cdot \nabla P = \mu(\vec{u} \cdot k^{-1} \vec{u}) \tag{A.17}
\]
Thus, the viscous dissipation for Darcy scale can be interpreted as the upscaled heat generated due to friction between the fluid and the rock pore structure which is captured by the rock second-order symmetric permeability tensor $k$. 
Bibliography


