DYNAMIC DATA INTEGRATION FOR TRANSPORT
USING STATISTICAL MOMENT EQUATIONS

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Errata

Chapter 2

In Section 2.2, the velocity for streamline tracking has to be interstitial velocity. Therefore, Eq. 2.9 has to be

\[ v_{x_i}(\vec{x}) = -\frac{e^{Y(\vec{x})}}{\phi} \frac{\partial P(\vec{x})}{\partial x_i}. \]

As a result, the moment equations (Eqs. 2.10-2.11) become

\[ \langle v_{x_i}(\vec{x}) \rangle = -\frac{e^{Y(\vec{x})}}{\phi} \frac{\partial P(\vec{x})}{\partial x_i} + \left \langle Y'(\vec{x}) \frac{\partial P'(\vec{x})}{\partial x_i} \right \rangle + O(\sigma_Y^3), \]

\[ C_{vv} = \langle v'(\vec{x})v'(\vec{\chi}) \rangle \\
= \frac{KK}{\phi \phi}(\vec{x},\vec{\chi}) \left[ P_x(\vec{x})P_x(\vec{\chi})C_{YY}(\vec{x},\vec{\chi}) + \frac{\partial^2 C_{PP}(\vec{x},\vec{\chi})}{\partial x_i^2} \right] + \frac{KK}{\phi \phi}(\vec{x},\vec{\chi}) \left[ P_x(\vec{x}) \frac{\partial C_{YP}(\vec{x},\vec{\chi})}{\partial x_i} + P_x(\vec{\chi}) \frac{\partial C_{YP}(\vec{\chi},\vec{x})}{\partial x_i} \right] + O(\sigma_Y^3). \]

Chapter 3

There are a few equations in Section 3.2 have incorrect signs. The following are the corrected version.

Travel time fluctuation (Eq. 3.6):

\[ \tau'(\xi) \approx -\int_{\xi_0}^{\xi} \frac{1}{\langle v_\xi(\xi,\eta)\rangle^2} \left[ v_\xi'(\xi,\langle \eta(\xi) \rangle) + \eta'(\xi) \frac{d\langle v_\xi(\xi,\eta)\rangle}{d\langle \eta(\xi) \rangle} \right] d\xi. \]
Cross-covariance between travel time and log-permeability (Eq. 3.7):

\[ C_{\tau Y}(\xi, \bar{x}) = \langle Y'(\bar{x})\tau'(\xi) \rangle \approx -\int_{\xi_0}^{\xi} \frac{1}{\langle v_\xi(\xi, \eta(\xi)) \rangle^2} \left[ \langle Y'(\bar{x})v'_\xi(\xi, \eta(\xi)) \rangle + \langle Y'(\bar{x})\eta'(\xi) \rangle \frac{d\langle v_\xi(\xi, \eta(\xi)) \rangle}{d\langle \eta(\xi) \rangle} \right] d\xi. \]

Fluctuation of velocity in \( \xi \) direction (Eq. 3.8):

\[ v'_\xi(\xi, \eta) = v'_{\xi_1}(\bar{x}) \cos \theta + v'_{\xi_2}(\bar{x}) \sin \theta = -e^{(y_{\xi_1}(\bar{x}))} \left[ \frac{\partial (P(\bar{x}))}{\partial x_1} Y'_{\xi_1}(\bar{x}) + \frac{\partial P'(\bar{x})}{\partial x_1} \right] \cos \theta \]
\[ -e^{(y_{\xi_2}(\bar{x}))} \left[ \frac{\partial (P(\bar{x}))}{\partial x_2} Y'_{\xi_2}(\bar{x}) + \frac{\partial P'(\bar{x})}{\partial x_2} \right] \sin \theta. \]

Cross-covariance between velocity in \( \xi \) direction and log-permeability (Eq. 3.9):

\[ C_{v\xi Y}(\xi, \eta; \bar{x}) = \langle Y'(\bar{\chi})v'_\xi(\xi, \eta) \rangle \]
\[ = -e^{(y_{\xi_1}(\bar{x}))} \left[ \frac{\partial (P(\bar{x}))}{\partial x_1} C_{Y \xi_1}(\bar{\chi}, \bar{x}) + \frac{\partial C_{Y P}(\bar{\chi}, \bar{x})}{\partial x_1} \right] \cos \theta \]
\[ -e^{(y_{\xi_2}(\bar{x}))} \left[ \frac{\partial (P(\bar{x}))}{\partial x_2} C_{Y \xi_2}(\bar{\chi}, \bar{x}) + \frac{\partial C_{Y P}(\bar{\chi}, \bar{x})}{\partial x_2} \right] \sin \theta. \]

Fluctuation of velocity in \( \eta \) direction (Eq. 3.12):

\[ v'_\eta(\xi, \eta) = -v'_{\eta_1}(\bar{x}) \sin \theta + v'_{\eta_2}(\bar{x}) \cos \theta = +e^{(y_{\eta_1}(\bar{x}))} \left[ \frac{\partial (P(\bar{x}))}{\partial x_1} Y'_{\eta_1}(\bar{x}) + \frac{\partial P'(\bar{x})}{\partial x_1} \right] \sin \theta \]
\[ -e^{(y_{\eta_2}(\bar{x}))} \left[ \frac{\partial (P(\bar{x}))}{\partial x_2} Y'_{\eta_2}(\bar{x}) + \frac{\partial P'(\bar{x})}{\partial x_2} \right] \cos \theta. \]

Cross-covariance between velocity in \( \eta \) direction and log-permeability (Eq. 3.13):

\[ C_{v\eta Y}(\xi, \eta; \bar{x}) = \langle Y'(\bar{\chi})v'_\eta(\xi, \eta) \rangle \]
\[ = +e^{(y_{\eta_1}(\bar{x}))} \left[ \frac{\partial (P(\bar{x}))}{\partial x_1} C_{Y \eta_1}(\bar{\chi}, \bar{x}) + \frac{\partial C_{Y P}(\bar{\chi}, \bar{x})}{\partial x_1} \right] \sin \theta \]
\[ -e^{(y_{\eta_2}(\bar{x}))} \left[ \frac{\partial (P(\bar{x}))}{\partial x_2} C_{Y \eta_2}(\bar{\chi}, \bar{x}) + \frac{\partial C_{Y P}(\bar{\chi}, \bar{x})}{\partial x_2} \right] \cos \theta. \]

Note that the errors in signs of these equations do not have any consequences in the inversion results since they cancel out one another.
I certify that I have read this report and that in my opinion it is fully adequate, in scope and in quality, as partial fulfillment of the degree of Master of Science in Petroleum Engineering.

Prof. Hamdi Tchelepi  
(Principal advisor)
Abstract

Only limited information is usually available about the properties (e.g., permeability) of oil reservoirs, which are usually heterogeneous with high variability levels and complex spatial correlation structures. Incomplete knowledge about these heterogeneous natural systems leads to uncertainty in the reservoir description model (i.e., permeability distribution). The uncertainty in the reservoir description (input), leads to uncertainty in predictions of flow performance (output). Available information (e.g., measurements of permeability, pressure, saturation, and production rate) can be used to reduce the level of uncertainty in both the input and output parameters. We describe an inversion algorithm for integrating saturation measurements directly into the Statistical Moments Equations (SME) of immiscible two-phase flow in heterogeneous porous media. The approach employs a geostatistical (Kriging) scheme and makes use of an existing SME simulator of the forward problem for two-phase flow and transport. We demonstrate the iterative sequential algorithm using simple examples for the quarter of a five-spot geometry. The behavior of the first two conditional moments of log-permeability and predicted saturation due to the presence of saturation measurement is analyzed.
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Chapter 1

Introduction

Oil reserves are found in subsurface geologic formations, which are often thousands of feet deep. The architecture and stratigraphy of these natural systems can be quite complex. Moreover, formation properties, such as porosity and permeability, are usually heterogeneous with high variability levels and complex spatial correlations that span multiple scales. In order to manage oil field operations, accurate predictions of the flow behavior in the subsurface are required. Numerical reservoir simulation serves this purpose. The predictive reliability of the numerical simulations is a strong function of the quality and quantity of the input information. Here, we focus on the reservoir description information (i.e., the permeability distribution) of the particular reservoir under study.

Unfortunately, only limited data is usually available for the construction of the reservoir description model for the purpose of making flow performance predictions. In the field, most of the available information comes from wells. Examples include core data, electronic logging, pressure and rate measurements. More recently, use of permanent down-hole pressure and temperature sensors has increased. Nevertheless, well locations provide a very sparse sampling of the heterogeneous subsurface oil-bearing geologic formation. Transient well testing and seismic surveys provide additional bulk, or coarse-scale, information of varying quality and resolution. This is a common theme in reservoir management: we know that the reservoir is heterogeneous with complex spatial correlation structures, but we only have limited data that comes from different sources and is of varying quality and resolution. This incomplete knowledge (uncertainty) about the property distribution of the porous formation means that significant uncertainty accompanies predictions of flow performance.
A reservoir description model is used by a numerical simulator to predict the flow response, which is subsequently used in an economic model. The results are used to support operational and strategic business decisions. Since there is uncertainty associated with the (input) reservoir description, a single scenario of the most likely case should not be used as a basis for making decisions. To improve the quality of the decisions, we cannot rely on the most likely case or a specific realization, for that matter. Instead, we must quantify the uncertainty associated with the predictions, subject, of course, to the assumptions of the probabilistic model itself.

The traditional, and widely used, method to quantify the uncertainty associated with performance predictions is Monte Carlo Simulation (MCS). The idea is to generate equiprobable realizations of the reservoir description based on all available information (e.g., core measurements, mean, variance, spatial correlation structure). For the recovery process under study, predictions of the dependent variables (e.g., pressure and saturation) are obtained, for one realization at a time, using a ‘deterministic’ numerical flow simulator. Afterwards, statistical moments of the quantities of interest are computed by post processing the simulated ensemble of realizations. MCS is a flexible and robust approach, since if a large enough ensemble is used, the results provide the full description of any dependent variable (e.g., probability density function of ultimate oil recovery). MCS is only limited by the capabilities of the conventional numerical simulator available. In spite of its wide applicability, MCS computations can be very expensive since a large ensemble of highly detailed realizations of the reservoir description may be needed to generate accurate statistical information about the dependent variables (e.g., pressure, saturation, production rate, etc.). In most cases, the full description of the probability density function of a dependent variable is not necessary; instead, only the first few statistical moments, usually the first and second, are of interest.

An alternative approach to quantify prediction uncertainty is by using Statistical Moment Equations (SME). This approach was developed by subsurface hydrologists, and it has been applied to reservoir engineering problems recently. Zhang and Winter [1] developed statistical moment equations for single-phase, steady-state flow in bounded, heterogeneous reservoirs, using low-order perturbation expansions. Zhang and Tchelepi [2] extended the SME approach to nonlinear immiscible two-phase flow in heterogeneous domains. Their approach decouples the flow and transport problems. The flow part is solved in Eudorian space using finite difference as in reference [1]. While the transport part is solved in
Lagrangian space along streamlines. The moment equations were developed for one dimensional (1D) and two-dimensional (2D) problems with uniform mean flow, using expansions for the mean-removed equations. Later, Zhang, Li, and Tchelepi[3] formulated the transport moment equations for more general 2D problems. By applying perturbation expansions to the flow and transport problems, the exact moment equations can be derived. However, the resulting equations cannot be solved due to the presence of unavailable higher order terms. To have a closed set of equations, low order approximations are usually employed. The effect of dropping the high order terms was investigated by Li et al. [4].

Thus far, all the SME related developments are for the so-called forward problem. The forward problem refers to predicting the dependent variables (e.g., pressure, velocity, travel time, saturation) by solving the governing equations, which are constructed using information about the independent variables (e.g., porosity and permeability). On the other hand, a process that uses measurements of the dependent variables to determine (improve the estimate) of the independent variables, is referred to as an inverse problem. Several studies on the inverse flow problem in heterogeneous porous media have been conducted. One important example is the ensemble Kalman filter method [16] for the integration of dynamic data, which fits nicely within an MCS framework. Our focus here, however, is on developing an SME-based method for the inverse problem in the presence of dynamic (saturation) measurement.

Several investigations related to the inverse problem in groundwater systems have been reported [5, 6, 7, 8, 9]. Li and Tchelepi [10] studied the behavior of bounded domains in the presence of wells (i.e., non-stationary, non-uniform mean flow) problems, which are of strong interest in reservoir engineering. They presented results of the forward problem for the first two conditional statistical moments of pressure and velocity in the presence of permeability measurements. Then, they extended the conditioning process to the inverse flow problem in the presence of pressure measurements [11]. Quite recently, they formulated the equations to integrate dynamic measurements (i.e., saturation) into the transport problem [12].

In this report, we describe the inversion procedure for the integration of saturation measurements directly into the SME equations of two-phase immiscible flow in heterogeneous formations. The implementation of the iterative algorithm into the SME simulator is described. Using simple examples, the conditional first two statistical moments of both saturation and permeability are presented and analyzed.
Chapter 2

Forward Problem: Governing Equations

We are interested in the two-phase problem. Specifically, we assume incompressible two-phase flow. The flow equation can be written as

\[-\nabla \cdot \vec{v} = \nabla \cdot (\mathbb{K} \cdot \nabla P) = \tilde{q},\] (2.1)

where \(v\) is total velocity, \(K\) is the effective permeability tensor, \(P\) is pressure, and \(\tilde{q}\) is flow rate of sink/source per unit volume. The transport problem can be written as

\[\frac{\partial S_w}{\partial t} + \nabla \cdot (\vec{v} \times f_w) = \tilde{q}_w,\] (2.2)

where \(S_w\) is water saturation, \(t\) is time, \(f_w\) is fractional flow of water, and \(\tilde{q}_w\) is water flow rate of sink/source per unit volume.

We assume that the only source of uncertainty is incomplete knowledge of the permeability field, \(K\). That is, the available information about \(K\) is in the form of a covariance function and perhaps a few measurements. In this context, we take \(K\) as a random field. As a result, Eqs. 2.1 and 2.2 are stochastic differential equations. In Appendix A, a short description about random functions is provided.

The solution strategy is to decouple the problem in two parts, flow and transport. The flow part is solved by using finite differences on Cartesian grid, while the transport problem is solved along streamlines. The computation is in sequential order: pressure, velocity, travel-time, and saturation.
2.1 Pressure

As Li et al. [4] demonstrated, the results of the statistical moment equations are more accurate if the equations are derived using log-permeability \( Y = \ln K \), instead of permeability \( K \). Therefore, the formulation here is based on log-permeability.

The partial differential equation of pressure can be written as
\[
\frac{\partial^2 P(x)}{\partial x_i^2} + \frac{\partial Y(x) \partial P(x)}{\partial x_i} = 0, \tag{2.3}
\]
where \( P \) is pressure, and \( Y \) is log-permeability. Both \( Y \) and \( P \) are random variables. By decomposing the random variables into a mean component and a zero-mean fluctuation, and taking expectation, the moment equation of pressure is obtained as
\[
\frac{\partial^2 \langle P(x) \rangle}{\partial x_i^2} + \frac{\partial \langle Y(x) \rangle \partial \langle P(x) \rangle}{\partial x_i} + \left\langle \frac{\partial Y'(x) \partial P'(x)}{\partial x_i} \right\rangle = 0, \tag{2.4}
\]
where \( \langle P(x) \rangle \) is pressure mean, \( P'(x) \) is pressure zero-mean fluctuation, \( \langle Y(x) \rangle \) is log-permeability mean, and \( Y'(x) \) is log-permeability zero-mean fluctuation. The equation cannot be solved for \( \langle P(x) \rangle \) due to the presence of the high order term, \( \left\langle \frac{\partial Y'(x) \partial P'(x)}{\partial x_i} \right\rangle \). By dropping this term, the equation can be solved; and the result is a first-order equation for the mean pressure.

Fig. 2.1 demonstrates the problem set up, which is a quarter five-spot pattern with a unit-square domain. The log-permeability mean, \( \langle \ln K \rangle \), is 0 and the variance, \( \sigma^2_{\ln K} \), is 1. The dimensionless correlation length of the log-permeability is 0.4. The injection well is put on pressure control, while the producer well is on rate control. Fig. 2.1 shows a smooth surface representing the mean pressure field. This result is equivalent to averaging the results obtained from multiple realizations, which is referred as the Monte Carlo Simulation (MCS) approach.

To derive the second moment equations, Eq. 2.3 is subtracted from Eq. 2.4, and the resulting equation is multiplied by fluctuation of log-permeability at another point. Taking expectation of this mean-removed pressure fluctuation equation leads to
\[
\frac{\partial^2 C_{YP}(\bar{x}, \bar{\chi})}{\partial \chi_i^2} + \frac{\partial C_{YP}(\bar{x}, \bar{\chi}) \partial \langle Y(\bar{\chi}) \rangle}{\partial \chi_i} + \frac{\partial C_{YY}(\bar{x}, \bar{\chi}) \partial \langle P(\bar{\chi}) \rangle}{\partial \chi_i} + \left\langle Y'(\bar{x}) \frac{\partial Y'(\bar{\chi}) \partial P'(\bar{\chi})}{\partial \chi_i} \right\rangle = 0, \tag{2.5}
\]
where \( C_{YP} \) is cross-covariance between log-permeability and pressure, and \( C_{YY} \) denotes covariance of log-permeability.
Similarly, Eq. 2.3 is subtracted from Eq. 2.4, multiplied by fluctuation of pressure at another point, and after taking expectation of this mean-removed pressure fluctuation equation, we get

\[
\frac{\partial^2 C_{PP}(\tilde{x}, \tilde{\chi})}{\partial x_i^2} + \frac{\partial C_{PP}(\tilde{x}, \tilde{\chi})}{\partial x_i} \frac{\partial Y(\tilde{x})}{\partial x_i} + \frac{\partial C_{PY}(\tilde{\chi}, \tilde{x})}{\partial x_i} \frac{\partial P(\tilde{x})}{\partial x_i} + \left( P'(\tilde{\chi}) \frac{\partial Y'(\tilde{x})}{\partial x_i} \frac{\partial P'(\tilde{x})}{\partial x_i} \right) = 0, \tag{2.6}
\]

where \( C_{PP} \) is covariance of pressure, and \( C_{PY} \) is cross-covariance between pressure and log-permeability; it is equivalent to \( C_{YP} \).

Again, high order terms appear in Eq. 2.5 and Eq. 2.6. We drop the high order terms to obtain low-order approximation of the second moments. Consequently, Eq. 2.5 and Eq. 2.6 can be rewritten as

\[
\frac{\partial^2 C_{YP}(\tilde{x}, \tilde{\chi})}{\partial \chi_i^2} + \frac{\partial C_{YP}(\tilde{x}, \tilde{\chi})}{\partial \chi_i} \frac{\partial Y(\tilde{\chi})}{\partial \chi_i} + \frac{\partial C_{YY}(\tilde{\chi}, \tilde{x})}{\partial \chi_i} \frac{\partial P(\tilde{x})}{\partial \chi_i} = 0, \tag{2.7}
\]

and

\[
\frac{\partial^2 C_{PP}(\tilde{x}, \tilde{\chi})}{\partial x_i^2} + \frac{\partial C_{PP}(\tilde{x}, \tilde{\chi})}{\partial x_i} \frac{\partial Y(\tilde{x})}{\partial x_i} + \frac{\partial C_{PY}(\tilde{\chi}, \tilde{x})}{\partial x_i} \frac{\partial P(\tilde{x})}{\partial x_i} = 0, \tag{2.8}
\]

respectively. Eq. 2.7 and Eq. 2.8 are solved sequentially to obtain \( C_{YP} \) and \( C_{PP} \).
Fig. 2.2 shows the pressure variance in the quarter five-spot problem described earlier (i.e., \( \langle \ln K \rangle = 0, \sigma_{\ln K}^2 = 1 \)). As mentioned, the injection well has pressure control, which explains the zero variance at the well. On the other hand, the production well with rate control is the location of the highest pressure variance.

### 2.2 Velocity

The pressure field obtained on the Eulerian grid is used to compute the velocity field by Darcy’s law

\[
v_{x_i}(\vec{x}) = -e^{Y(\vec{x})} \frac{\partial P(\vec{x})}{\partial x_i}.
\]  

(2.9)

Streamlines, which are used for solving the transport problem, are then generated by launching particles in the velocity field. In Eq. 2.9, the Darcy velocity \( v \), \( Y \), and \( P \) are random variables. Here again, the random variables are decomposed into mean and zero-mean fluctuation components. After taking expectation, the equation for the first-moment of velocity is obtained as

\[
\langle v_{x_i}(\vec{x}) \rangle = -e^{Y(\vec{x})} \left[ e^{-\frac{\sigma_Y^2}{2}} \frac{\partial P(\vec{x})}{\partial x_i} + \langle Y'(\vec{x}) \frac{\partial P'(\vec{x})}{\partial x_i} \rangle + O(\sigma_Y^3) \right].
\]  

(2.10)
The high order term, $O(\sigma_Y^3)$, is then dropped, and the low-order equation is solved for the mean velocity field. With the mean velocity field, the second moment of velocity can be derived as

$$C_{vv} = \langle v'(\vec{x})v'(\vec{\chi}) \rangle$$

$$= \mathbb{K}\mathbb{K}(\vec{x}, \vec{\chi}) \left[ \mathcal{P}_{x_i}(\vec{x}) \mathcal{P}_{x_i}(\vec{\chi}) C_{YY}(\vec{x}, \vec{\chi}) + \frac{\partial^2 C_{PP}(\vec{x}, \vec{\chi})}{\partial x_i^2} \right] + \mathcal{K}(\vec{x}, \vec{\chi}) \left[ \mathcal{P}_{x_i}(\vec{x}) \frac{\partial C_{YP}(\vec{x}, \vec{\chi})}{\partial x_i} + \mathcal{P}_{x_i}(\vec{\chi}) \frac{\partial C_{YP}(\vec{\chi}, \vec{x})}{\partial x_i} \right] + O(\sigma_Y^3), \tag{2.11}$$

where $\mathbb{K}(\vec{x}, \vec{\chi}) = e^{(Y(\vec{x}))} e^{(Y(\vec{\chi}))}$, and $\mathcal{P}_{x_i}(\vec{x}) = \frac{\partial P(\vec{x})}{\partial x_i}$. Again, only terms up to second order are shown. The higher order terms (e.g., $O(\sigma_Y^3)$ and higher) are dropped when applying the low-order approximation.

### 2.3 Travel-Time

The computations of the flow part are performed in Eulerian space, but the transport part is solved in Lagrangian space. This is done by launching particles to trace path lines. Then, the transport equations will be solved along the streamlines as a 1-D problem. The travel-time is a coordinate along the streamlines in Lagrangian space, and it is defined as

$$\tau(\vec{x}; \vec{x}^0) = \int_{x_1^0}^{x_1} \frac{dx_1}{v_{x_1}[x_1, x_2 = \eta(x_1; x_1^0)]}, \tag{2.12}$$

where $\tau$ is travel-time, and $\eta$ is transverse displacement. The travel-time and the transverse displacement are also random variables. The travel-time is the time required for a particle to move from $\vec{x}^0$ to $\vec{x}$ in a given velocity field. The transverse displacement is the other coordinate in Lagrangian space; more details can be found in [3].

By decomposing the variables in Eq. 2.12 and dropping high order terms, the mean travel-time can be expressed as

$$\langle \tau(\vec{x}; \vec{x}^0) \rangle = \int_{x_1^0}^{x_1} \frac{dx_1}{v_{x_1}[x_1, x_2 = \langle \eta \rangle]} \right), \tag{2.13}$$
2.4. SATURATION

and the second moment of travel-time can be expressed as

$$C_{rr}(\vec{x}; \vec{x}^0, \vec{\chi}; \chi^0) = \langle r'(\vec{x}; \vec{x}^0) r'(\vec{\chi}; \chi^0) \rangle$$

$$= \int_{x_1}^{x_1} \int_{x_1}^{x_1} \frac{dx_1 d\chi_1}{\langle v_{x_1}[x_1, x_2 = \langle \eta_1 \rangle] \rangle^2 \langle v_{x_1}[\chi_1, \chi_2 = \langle \eta_2 \rangle] \rangle^2} \times \left[ \langle v'_{x_1}[x_1, x_2 = \langle \eta_1 \rangle] \rangle \langle v'_{\chi_1}[\chi_1, \chi_2 = \langle \eta_2 \rangle] \rangle \right]$$

$$+ b(x_1, x_2 = \langle \eta_1 \rangle) \langle v'_{x_1}[\chi_1, \chi_2 = \langle \eta_2 \rangle] \rangle \eta_1'$$

$$+ b(\chi_1, \chi_2 = \langle \eta_2 \rangle) \langle v'_{\chi_1}[x_1, x_2 = \langle \eta_1 \rangle] \rangle \eta_2'$$

$$+ b(x_1, x_2 = \langle \eta_1 \rangle) b(\chi_1, \chi_2 = \langle \eta_2 \rangle) \langle \eta_1 \eta_2 \rangle,$$  \hspace{1cm} (2.14)

where $\eta_1 = \eta(x_1; x_0^0), \eta_2 = \eta(\chi_1; \chi_0^0)$, and $b(x_1, x_2 = \langle \eta \rangle) = \frac{\partial (v_{x_1}[x_1, x_2 = \eta])}{\partial \eta}$ at $\eta = \langle \eta \rangle$.

2.4 Saturation

In Lagrangian space, the transport part becomes a 1-D problem along a streamline. The problem is described by the following partial differential equation.

$$\frac{\partial S}{\partial t} + f_w'(S) \frac{\partial S}{\partial \tau} = 0,$$  \hspace{1cm} (2.15)

where $S$ is saturation. The solution of the Buckley-Leverett problem, Eq. 2.15, is

$$S(\tau, t) = s(\tau/t) \text{H}(f_w'(S) - \tau/t),$$  \hspace{1cm} (2.16)

where $s(\tau/t) = f_w^{-1}(\tau/t)$, $\text{H}[\ ]$ is Heaviside function, and $f_w'(S) = [f_w(S) - f_w(S_{wr})]/(S_s - S_{wr})$. The solution shows that saturation is a nonlinear function of time and travel-time. The first and the second moments of saturation can be obtained using the statistical definitions. The mean saturation at a particular time can be obtained by integrating the saturation field, as a sole function of travel-time, weighted by the probability density function of travel-time for the entire range. That is,

$$\langle S(\vec{x}, t) \rangle = \int_{-\infty}^{\infty} S(\tau(x_1; x_1^0), t) \ f(\tau) \ d\tau,$$  \hspace{1cm} (2.17)

where $f(\tau)$ is the probability density function of travel-time. At this point, a certain form of the travel-time probability density function is needed; and it is assumed to be a log-normal distribution, that is

$$f(\tau) = \text{Log-N}(\langle \ln \tau \rangle, \sigma_{\ln \tau}^2).$$  \hspace{1cm} (2.18)
While the results of the travel-time moment equations are \( \langle \tau \rangle \) and \( \sigma^2_\tau \), the distribution requires parameters of \( \langle \ln \tau \rangle \) and \( \sigma^2_{\ln \tau} \). These parameters can be determined from the transformations shown in Eqs. B.3 and B.4.

\[
\begin{align*}
C_{SS}(\tilde{x}, t_1; \tilde{x}, t_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S'(\tau_1(x_1; x_1^0), t_1) \ S'(\tau_2(\chi_1; \chi_1^0), t_2) \ f(\tau_1, \tau_2) \ d\tau_1 \ d\tau_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S'(\tau_1(x_1; x_1^0), t_1) \ S'(\tau_2(\chi_1; \chi_1^0), t_2) \ f(\tau_1, \tau_2) \ d\tau_1 \ d\tau_2 \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S'(\tau_1(x_1; x_1^0), t_1) \ S'(\tau_2(\chi_1; \chi_1^0), t_2) \ f(\tau_1, \tau_2) \ d\tau_1 \ d\tau_2 \quad (2.19)
\end{align*}
\]

In addition to the assumption that the travel-time probability density function of each particle is a log-normal distribution, the joint distribution of the two-particle travel-times is assumed to be a bivariate log-normal distribution, namely,

\[
f(\tau_1, \tau_2) = \text{Log-N}(\langle \ln \tau_1 \rangle, \sigma^2_{\ln \tau_1}; \langle \ln \tau_2 \rangle, \sigma^2_{\ln \tau_2}; \rho_{\ln \tau_1 \ln \tau_2}). \quad (2.20)
\]
The term, $\rho_{\ln \tau_1 \ln \tau_2}$, in Eq. 2.20 is the correlation coefficient between $\ln \tau_1$ and $\ln \tau_2$. This parameter can be determined by transforming $\rho_{\tau_1 \tau_2}$ using Eq. B.8.

Figure 2.4: Saturation Variance in a Quarter Five-Spot System

The result of the saturation variance, Fig. 2.4, indicates that the highest variance is around the mean shock front. This can be explained by analogy to MCS. With multiple realizations, most of the realizations are likely to have the shock front in the neighborhood of the mean shock front location; only a few realizations would have fronts that are far from that mean shock front location. This fact leads to the highest variance of the ensemble in the mean shock front area.

The first and the second moment equations of all the dependent variables, including $P$, $v$, $\tau$, and $S$, are developed. As a result, we can compute predictions of saturation. The predictions, which depend on the spatial distribution of the log-permeability field, are consequently compared to available measurements. Therefore, the differences between the predictions and the measurements can be used to condition the log-permeability field by an inverse process.
Chapter 3

Inverse Problem: Governing Equation

At this point, assume that the predictions have been made and compared to available measurements. The predictions (pressure, velocity, travel-time, and saturation) are based on information related to the independent variable, $Y$. Consequently, the deviation of the predictions from the measurements can be used to condition the independent variables. A method that is widely used in geostatistics to utilize measurement data to condition independent variables is Kriging.

3.1 Kriging

The method relies on the assumption that a conditioned independent variable is a linear combination of measurement data. In this study, the measurement data is saturation, $S$, and the independent variable is log-permeability, $Y$. The relation can be written as

$$h_Y(x) = h_Y(x)_{UC} + \lambda^T(x) [S_m(x_m) - \bar{S}(x_m)],$$

(3.1)

where subscript $C =$ conditioned, subscript $UC =$ unconditioned, $\lambda =$ weight vector, superscript $T =$ transpose, and subscript $m =$ measurement.

Along with the linear combination assumption, the conditioned independent variable has to be unbiased, and by minimizing the mean square error, the weight vector ($\lambda$) can be determined from the relation

$$C_{SS}(x_m, x_m) \lambda = C_{SY}(x_m, x),$$

(3.2)
where $C_{SS} = \text{covariance matrix among the measurement data}$, $\tilde{C}_{SY} = \text{cross-covariance vector between the measurement data and an independent variable}$.

With the same weight vector, the second moment of the independent variable can be conditioned as follows

$$C_{YYC}(\tilde{x}, \bar{x}) = C_{YY_{IC}}(\tilde{x}, \bar{x}) - \tilde{x}^T(\tilde{x}) \tilde{C}_{SY}(\tilde{x}_m, \bar{x}), \quad (3.3)$$

where $C_{YY}$ is a covariance of log-permeability, and $\tilde{C}_{SY}$ is the cross-covariance vector between the saturation measurements and a log-permeability.

Although the development of the conditioning process assumes that the independent variable is log-permeability and the dependent variable is saturation, there is no limit to this set up. The independent and the dependent variables could be between $Y$ and $P$, $Y$ and $v$, $P$ and $v$, etc.

### 3.2 Additional Terms

Most of the moments are already available from the forward problem. From Eqs. (3.1), (3.2), and (3.3), the only unavailable moment at this point is $C_{SY}$.

#### 3.2.1 Cross-covariance of saturation and log-permeability, $C_{SY}$

Again, the second moment can be determined by using the definition of cross-covariance

$$C_{SY}(\tilde{x}, t; \bar{x}) = \int_{-\infty}^{\infty} \int_0^{\infty} S'(\tau(x_1; x_0), t) Y'(\bar{x}) f(\tau, Y) \, d\tau \, dY, \quad (3.4)$$

where $f(\tau, Y)$ is the joint probability density function of a travel-time and log-permeability. Since travel-time has a log-normal distribution form and log-permeability has a normal distribution form, the joint distribution is assumed to be of mixed type: log-normal and normal, so we write

$$f(\tau, Y) = \text{Log-N-N}(\ln \tau, \sigma_{\ln \tau}^2; \langle Y \rangle, \sigma_Y^2; \rho_{\ln \tau Y}). \quad (3.5)$$

This distribution requires an expression for $\rho_{\ln \tau Y}$, which is obtained using Eq. B.10. To do so, the cross-covariance between travel-time and log-permeability, $C_{\tau Y}$, needs to be formulated.
3.2.2 Cross-covariance of travel-time and log-permeability, $C_{\tau Y}$

With Eq. 2.12, and 2.13, the zero-mean fluctuation of travel-time can be expressed as

$$\tau'(\xi) \approx \int_{\xi_0}^{\xi} \frac{1}{\langle v_\xi(\xi, \eta(\xi)) \rangle^2} \left[ v_\xi'(\xi, \eta(\xi)) + \eta'(\xi) \frac{d}{d\eta(\xi)} \langle v_\xi(\xi, \eta(\xi)) \rangle \right] d\xi,$$  \hspace{1cm} (3.6)

where $\xi$ = spatial coordinate along a mean streamline, $v_\xi$ = velocity along a mean streamline, and $v_\eta$ = velocity perpendicular to a streamline. By multiplying a zero-mean fluctuation of log-permeability and applying the expectation operator, the cross-covariance between travel-time and log-permeability is obtained as

$$C_{\tau Y}(\xi, \bar{x}) = \langle Y'(\bar{x})\tau'(\xi) \rangle \approx \int_{\xi_0}^{\xi} \frac{1}{\langle v_\xi(\xi, \eta(\xi)) \rangle^2} \left[ \langle Y'(\bar{x})v_\xi'(\xi, \eta(\xi)) \rangle + \langle Y'(\bar{x})\eta'(\xi) \rangle \frac{d}{d\eta(\xi)} \langle v_\xi(\xi, \eta(\xi)) \rangle \right] d\xi.$$  \hspace{1cm} (3.7)

To solve Eq. 3.7, the moments ($C_{v_\xi Y}$ and $C_{\eta Y}$) and the velocity gradient ($\frac{d}{d\eta(\xi)} \langle v_\xi(\xi, \eta(\xi)) \rangle$) need to be determined.

3.2.3 Cross-covariance of velocity and log-permeability, $C_{v_\xi Y}$

The random velocity components in the streamline transformed coordinates can be obtained from the random velocity components in Cartesian coordinates as follows

$$v_\xi'(\xi, \eta) = v_{x_1}'(\bar{x}) \cos \theta + v_{x_2}'(\bar{x}) \sin \theta$$

$$= e^{(Y_{x_1}(\bar{x}))} \left[ \frac{\partial (P(\bar{x}))}{\partial x_1} C_{YY_{x_1}}(\bar{x}) + \frac{\partial P(\bar{x})}{\partial x_1} \right] \cos \theta +$$

$$e^{(Y_{x_2}(\bar{x}))} \left[ \frac{\partial (P(\bar{x}))}{\partial x_2} C_{YY_{x_2}}(\bar{x}) + \frac{\partial P(\bar{x})}{\partial x_2} \right] \sin \theta,$$  \hspace{1cm} (3.8)

where $\theta = \tan^{-1} \frac{v_{x_2}(\bar{x})}{v_{x_1}(\bar{x})}$. Then, the cross-covariance can be formulated by multiplying Eq. 3.8 with a zero-mean fluctuation of log-permeability and taking expectation, which leads to

$$C_{v_\xi Y}(\xi, \eta; \bar{x}) = \langle Y'(\bar{x})v_\xi'(\xi, \eta) \rangle$$

$$= e^{(Y_{x_1}(\bar{x}))} \left[ \frac{\partial (P(\bar{x}))}{\partial x_1} C_{YY_{x_1}}(\bar{x}) + \frac{\partial C_{YP}(\bar{x}, \bar{x})}{\partial x_1} \right] \cos \theta +$$

$$e^{(Y_{x_2}(\bar{x}))} \left[ \frac{\partial (P(\bar{x}))}{\partial x_2} C_{YY_{x_2}}(\bar{x}) + \frac{\partial C_{YP}(\bar{x}, \bar{x})}{\partial x_2} \right] \sin \theta.$$  \hspace{1cm} (3.9)

This moment is required for solving Eq. 3.7.
3.2.4 Cross-covariance of transverse displacement and log-permeability, $C_{\eta Y}$

We start with a zero-mean fluctuation of transverse displacement; the approximation of the fluctuation can be written as

$$\eta'(\xi) \approx \int_0^\xi \frac{v'_n(\xi, \langle \eta \rangle)}{v(\xi, \langle \eta \rangle)} \, d\xi.$$  \hspace{1cm} (3.10)

By multiplying this expression with a zero-mean fluctuation of log-permeability and taking expectation, the cross-covariance between transverse displacement and log-permeability, we get

$$C_{\eta Y}(\xi; \chi) = \langle Y'(\chi)\eta'(\xi) \rangle \approx \int_0^\xi \frac{\langle Y'(\chi) v'_n(\xi, \langle \eta \rangle) \rangle}{\langle v(\xi, \langle \eta \rangle) \rangle} \, d\xi.$$  \hspace{1cm} (3.11)

Eq. 3.11 requires additional information, which is the cross-covariance between the zero-mean fluctuation of velocity perpendicular to the mean streamline and the log-permeability.

3.2.5 Cross-covariance of velocity and log-permeability, $C_{v\eta Y}$

The zero-mean fluctuation of velocity perpendicular to a mean streamline can be written as

$$v'_n(\xi, \eta) = -v'_{x_1}(\bar{x}) \sin \theta + v'_{x_2}(\bar{x}) \cos \theta$$

$$= -e^{\langle Y_{x_1}(\bar{x}) \rangle} \left[ \frac{\partial (P(\bar{x}))}{\partial x_1} Y'_{x_1}(\bar{x}) + \frac{\partial P'(\bar{x})}{\partial x_1} \right] \sin \theta$$

$$+ e^{\langle Y_{x_2}(\bar{x}) \rangle} \left[ \frac{\partial (P(\bar{x}))}{\partial x_2} Y'_{x_2}(\bar{x}) + \frac{\partial P'(\bar{x})}{\partial x_2} \right] \cos \theta.$$  \hspace{1cm} (3.12)

The next step is to multiply Eq. 3.12 by a the zero-mean fluctuation of log-permeability and then apply the expectation operator, which takes the following form (after dropping the high order terms):

$$C_{v\eta Y}(\xi, \eta; \bar{\chi}) = \langle Y'(\bar{\chi}) v'_n(\xi, \eta) \rangle$$

$$= -e^{\langle Y_{x_1}(\bar{x}) \rangle} \left[ \frac{\partial (P(\bar{x}))}{\partial x_1} C_{YY_{x_1}}(\bar{\chi}, \bar{x}) + \frac{\partial C_{Y P}(\bar{\chi}, \bar{x})}{\partial x_1} \right] \sin \theta$$

$$+ e^{\langle Y_{x_2}(\bar{x}) \rangle} \left[ \frac{\partial (P(\bar{x}))}{\partial x_2} C_{YY_{x_2}}(\bar{\chi}, \bar{x}) + \frac{\partial C_{Y P}(\bar{\chi}, \bar{x})}{\partial x_2} \right] \cos \theta.$$  \hspace{1cm} (3.13)

With this expression, Eq. 3.11 can be computed.
3.2.6 Velocity gradient, $\frac{d\langle v_\xi(\eta(\xi)) \rangle}{d\langle \eta(\xi) \rangle}$

This velocity gradient is the last element required to solve Eq. 3.7. Then, the cross-covariance between saturation and log-permeability can be determined, and consequently the inverse problem can be processed. This term can be expressed as

$$\frac{d\langle v_\xi(\eta(\xi)) \rangle}{d\langle \eta(\xi) \rangle} = -\frac{\partial(v_{x_1}(\vec{x}))}{\partial x_1} \cos \theta \sin \theta - \frac{\partial(v_{x_2}(\vec{x}))}{\partial x_1} \sin^2 \theta + \frac{\partial(v_{x_1}(\vec{x}))}{\partial x_2} \cos^2 \theta + \frac{\partial(v_{x_2}(\vec{x}))}{\partial x_2} \sin \theta \cos \theta.$$  \hspace{1cm} (3.14)

At this point, all the terms and the equations have been developed. The details of the implementation are given in the next chapter.
Chapter 4

Inverse Problem: Implementation

The inversion procedure is implemented as an additional module into an existing SME (Statistical Moment Equations) numerical simulator, which was developed by Li in C++ language. This implementation takes full advantage of all the available functionality of this simulator. Therefore, only the additional functions required for accommodating the presence of saturation data to condition the statistical moments of log-permeability were added.

In the conditioning process, measurement data are not used directly; instead, the difference between the measured values and the predicted ones are used to condition the input parameter. The conditioning process can be divided into two parts (Fig. 4.1); one is the forward problem to compute the predictions, the other is the inverse problem conditioning the input parameter by using the difference between the predictions and the measurements.

In this study, the measurement data is limited to saturation; therefore, the forward problem includes flow (pressure and velocity) and transport computations (i.e., saturation predictions), then the inverse problem is performed to condition the statistical moments of the log-permeability field based on the available saturation measurements.

4.1 Algorithm

As shown in Eq. 2.16, saturation is a spatial and temporal function. A measurement of saturation can be at any point in the domain and at any point in time. There are two alternatives to solve the problem. The first alternative is a global scheme, where we treat
CHAPTER 4. INVERSE PROBLEM: IMPLEMENTATION

Figure 4.1: Conditioning Process

the measurement data at all locations and at all times simultaneously. The other alternative, which is pursued in this study, is a sequential scheme. The approach treats measurement data throughout the domain at a particular time. Although applying the sequential scheme does simplify the implementation, there is no guarantee that the measurement data at time \( t_1 \) to time \( t_{n-1} \) will be honored during the conditioning process at time \( t_n \).

In addition to the time loop in the sequential scheme, there is another algorithmic loop. As described in Section 3.1, Kriging has the major assumption of a linear relation, which
is not quite in agreement with the relation in the actual forward problem. However, the result can be improved by applying the conditioning process, Fig. 4.1, iteratively.

As shown in Fig. 4.2, the time is set to zero in each iteration loop so that the log-permeability field is consistent in time. This will not increase the computational expense, since the hyperbolic transport problem along a streamline has an analytical solution which can be determined at any time.

**4.2 Mapping**

In general, a saturation measurement is assigned to be a cell-centered value on the Cartesian grid. Since a prediction of saturation is on a streamline in Lagrangian space, the prediction value needs to be mapped into Cartesian grid, so that it can be compared to the measurement data.

In the inverse problem, the streamlines need not be distributed over the entire domain, because only saturation values at the measurement locations are needed. The better method
is to have a few streamlines around each measurement location for accurate mapping. To avoid complications due to mapping and simplify the problem, we use a single streamline that passes through the center of the grid block to represent the measurement.

Consequently, the ability to select a streamline passing through a certain location in the domain becomes crucial. In the forward problem, the streamlines are constructed by launching particles from an injection well, and forward tracking to a production well. To construct a streamline passing through a certain location, we launch a particle from the location and perform backward tracking to the source.

### 4.3 Discretization

In both Chapters 2 and 3, the governing equations are formulated in partial differential equation form. But, they are too complex to be solved analytically. In Eulerian space, the problem is solved numerically by the finite difference method. In Lagrangian space, although the problem has an analytical solution, in practice the streamlines need to be discretized especially for computation of a joint moment between variables in both Eulerian and Lagrangian space (e.g., $C_{SY}$).

Fig. 4.5 shows the discretization in Eulerian and Lagrangian spaces. The underlying
grids are control volumes in Eulerian space. Pressure in Eulerian space is computed as a cell-center value. While, log-permeability and velocity are variables at the interfaces. The line $[A, B, C]$ represents a section of a mean streamline, noting that the line is approximately illustrated as a straight line; an actual mean streamline is more curvilinear. Travel-time can be determined at any point on a streamline by assuming a linear relation for velocity inside a grid block. Then, saturation can be determined at any point along a streamline as well. Although the solution can be determined analytically at any location, in practice it is discretized into sections. In this implementation, a streamline is discretized with nodes at the intersections of the mean streamline and interfaces of the underlying grid.

![Discretization in Eulerian and Lagrangian Spaces](image)

In computing $C_{SY}$, variables in Lagrangian space, including travel-time and saturation, have a value at point $B$ (Fig. 4.5) representing both $b_1$ and $b_2$ sections. To represent these quantities in Eulerian space, they must be interpolated to point $B$; the interpolated value of grid $(i-1, j)$ is for section $b_1$, and the value of grid $(i, j)$ is for section $b_2$. Thus, a cell-centered value represents the sections of the streamlines within the grid block.

### 4.4 Input File and Data Structure

Most of the main input file remains the same; there is only one flag (Fig. 4.6) added to turn saturation conditioning on. The measurement information is input as a separate file, $\text{trans\_condi.in}$ (Fig. 4.7). The input of $\text{Total points in time}$ is followed by information
about each conditioning step, including the time, number of the measurements, locations
in Eulerian space, and values of the measurements. The last line in the input files contains
the number of maximum iterations, and the acceptable tolerance of the difference between
the conditioned predictions and the measurements.

Fig. 4.8 shows the data structure storing the input information which is \textit{trans\_condi.in}.
The data structure is selected to be nested type of linked list, dynamically allocated. Each
members in the linked list contains the time of the conditioning step, the number of mea-
measurements, and a pointer to a linked list of the measurement locations and values.
Figure 4.8: Saturation Measurement Data Structure
The variables in Fig. 4.8 are

\[\begin{align*}
\text{condiTimePtr} & \quad = \quad \text{pointer of the nested link list} \\
\text{totalCondiTime} & \quad = \quad \text{number of total points in time} \\
\text{ctime} & \quad = \quad \text{measurement time} \\
\text{numSw} & \quad = \quad \text{number of saturation measurement in the measurement time} \\
\text{swPtr} & \quad = \quad \text{pointer of the saturation measurement link list} \\
\text{sw} & \quad = \quad \text{saturation measurement value} \\
\text{i_} & \quad = \quad \text{grid index in } X\text{-direction} \\
\text{j_} & \quad = \quad \text{grid index in } Y\text{-direction} \\
\text{k_} & \quad = \quad \text{grid index in } Z\text{-direction} \\
\text{maxIter} & \quad = \quad \text{maximum iteration} \\
\text{etol} & \quad = \quad \text{acceptable error tolerance}
\end{align*}\]

The data structure was designed such that it can be expanded to accommodate future development of other dynamic measurements.
Chapter 5

Results

In the conditioning process, there are several interesting aspects. One of those is the cross-
covariance structure between saturation and log-permeability. The cross-covariance is the
quantity used in the Kriging system to quantify the relationship between the dependent
variable and the independent variable. Another aspect is the conditioned moments of log-
permeability. It is interesting to observe how the moments respond to the measurement
locations and values. In addition to the conditioned independent variable, the conditioned
predictions of saturation are also observed. Finally, conditioning with multiple points at
the same time, and the sequential scheme behavior are investigated.

Since the computation is quite intensive, we can afford a relatively fine domain for those
aspects where we do not iterate. For the other aspects, a coarse discretization is used
instead.

5.1 Cross-Covariance of Saturation and Log-Permeability

The investigation is focused on the cross-covariance of saturation and log-permeability after
one iteration. Consequently, the discretization can be relatively fine. The set up is a quarter
five-spot system with a unit-square domain. It is discretized into 21x21 grid blocks. The
injection well has pressure control, and the production well has rate control. As a result
of having the rate control at the production well with the incompressible flow assumption,
the overall flux in the system is constant.

The first example, shown in Fig. 5.1, has a saturation measurement at the middle of
the domain. The map reveals that the cross-covariance, as an indicator of the relationship
between saturation and log-permeability, is strongest in the vicinity of the measurement location. Moreover, the cross-covariance shows an orientation that is aligned with the flow direction. Another interesting point is that parts of the domain have negative value of the cross-covariance. This is because the overall flux in the system is constant; changes in the flux density in a certain area cause changes in the opposite direction in other areas.

Figure 5.1: Cross-covariance between log-permeability everywhere and the predicted saturation, obtained from the forward problem, at location (11, 11) where a measurement is available.

The other example has a saturation measurement located at the top-left area of the domain, shown in Fig. 5.2. The result is in agreement with the previous example; the map has the strongest covariance in the vicinity of the measurement location, and the structure is aligned with the flow path. An additional interesting point is that the negative cross-covariance in this example is stronger in magnitude than the first example, Fig. 5.1. Since the unconditional (original) flux density in the neighborhood of the measurement is low, conditioning leads to a positive $C_{SY}$ in the immediate vicinity of the measurement location. However, in order to obtain significant benefit for conditioning at that location, the cross-covariance with respect to locations that are far from the measurement can be significantly
Figure 5.2: Cross-covariance between log-permeability everywhere and the predicted saturation, obtained from the forward problem, at location (4,18) where a measurement is available.
more negative.

5.2 Conditioned Log-Permeability Field

In the last section, the cross-covariances of saturation and log-permeability were demonstrated for two measurements. The Kriging system (Eq. 3.2) can then be solved for the weighting vector. Consequently, log-permeability mean and variance are conditioned by Eq. 3.1 and Eq. 3.3, respectively.

We take the value of the measurement at the middle of the domain to be less than the predicted one and the value of the measurement at the top-left area to be greater than the predicted value. Fig. 5.3 illustrates the conditioned log-permeability mean field after the first iteration. The overestimation of the prediction in the middle of the domain leads to a reduction in the mean log-permeability field in the area; the original (unconditional) log-permeability mean was 0 for the entire domain. Simultaneously, underestimation of the prediction at the top-left location leads to higher value for the conditional mean log-permeability in that region.

![Figure 5.3: Conditioned Log-Permeability Mean with Two $S_w$ Measurements](image)
The other part of the inversion process is the conditioning of the second moments (variance and covariance) of the log-permeability field. Both overestimation and underestimation are expected to reduce the associated uncertainty; in other words, the variance of the log-permeability field is expected to be reduced with measurement data, independent of the measurement values.

Fig. 5.4 shows the conditioned log-permeability variance. The values either remained the same or decreased. The reduction is concentrated around measurement locations; however, it is very modest. As a matter of fact, it is not so surprising since the relation between log-permeability and saturation is expected to be rather weak.

5.3 Convergence of Conditioned Saturation Mean

The purpose of conditioning is to improve the accuracy of the prediction results. In this study, it is anticipated that the predicted saturation mean approaches the measurement value with iterations of the conditioning process.

To investigate the behavior of the conditioned saturation mean, the number of iteration
is set to a high number; consequently, the system has to be relatively coarse. The system is a quarter five-spot in a unit-square domain, discretized into 5x5 grids. The injection well has pressure control, while the production well has rate control. The measurement location is at (4, 3), shown in Fig. 5.5. The measurement is assumed to be less than the unconditioned prediction by 10%.

The conditioning process was conducted for 100 iterations. The evolution of the predicted saturation mean is depicted in Fig. 5.6. The result indicates the algorithm is working; the predicted value asymptotically approaches the measurement value. However, the number of iteration is very high. The high number of iterations is probably due to the weak, yet nonlinear, relation between saturation and log-permeability, as well as the assumption of Gaussianity.

5.4 Behavior of Conditioned Saturation Variance

The purpose of the conditioning process is to reduce the associated uncertainty in the reservoir description as well as in the prediction of the dependent variable. In Section 5.2,
the log-permeability variance is reduced slightly with measurement data. Hence, we expect the conditioned saturation variance to decrease with iterations of the conditioning process.

At first sight of Fig. 5.7, it is very surprising to see that the variance has increased. The curve increases rapidly at first and then the slope decreases; the turning point is around the 40th to the 50th iteration. Although the variance decreases thereafter, it has a very small slope. The variance after 100 iterations is still much higher than the original one. What could be the explanation of this behavior? Could it be because of the fact that the measurement location is fixed in Eulerian space, while the saturation is calculated in Lagrangian space?

Fig. 5.8 is an exaggerated illustration of an actual saturation variance profile along a streamline. In the figure, the saturation variance along a streamline is shown for two distinct iteration steps. The investigation was conducted in Lagrangian space, in which the predicted saturations are computed. The two bell curves represent the saturation variance profile along a streamline in different iteration steps, n and n + 1. The highest saturation variance is around the mean shock front location. The later iteration step has less uncertainty (smaller profile). However, this uncertainty reduction may not be observable in Eulerian
CHAPTER 5. RESULTS

Figure 5.7: Conditioned Saturation Variance

Figure 5.8: Saturation Variance along a Streamline
5.5. MULTIPLE-POINT CONDITIONING

There is no doubt that if the movement in Lagrangian space of the measurement location in Eulerian space is in the direction from $\tau_n$ to $\tau_{n+1}^1$, the observation in Eulerian space will always be a reduction in saturation variance. On the other hand, if the direction of the movement is from $\tau_n$ to either $\tau_{n+1}^2$ or $\tau_{n+1}^3$, the observation could lead to an increase or decrease in the saturation variance; this depends on whether the movement is from $\tau_n$ to $\tau_{n+1}^2$, or to $\tau_{n+1}^3$.

Significant change in the mean travel-time at a measurement location is necessary in one iteration step to move from $\tau_n$ to $\tau_{n+1}^3$. This situation corresponds to the early iteration steps in Fig. 5.6. The saturation mean changes rapidly as a result of significant changes in mean travel-time. The consequence is an apparent increase of saturation variance during the same period, which is shown in Fig. 5.7. Conversely, to move the mean travel-time from $\tau_n$ to $\tau_{n+1}^2$, the change has to be relatively small. This takes place during the high iteration counts where the mean saturation changes very slightly. The movement from $\tau_n$ to $\tau_{n+1}^2$ is reflected in a reduction of the saturation variance during the late iterations. In addition, the turning point in Fig. 5.7 corresponds to the point in Fig. 5.6 where the change in the saturation mean is quite small, around the 40th to the 50th iteration.

5.5 Multiple-Point Conditioning

Given a saturation measurement, we need to compute the cross-covariance between the prediction at the measurement location and the log-permeability at every location (node) in the computational domain. This fact indicates how expensive conditioning with multiple measurements could be. The investigation is conducted on a 5x5 unit-square domain of a quarter five-spot system. The injection well again has pressure control, while the production well has rate control. With these controls, the system has constant overall flux.

The measurements are at four different locations at the same point in time. They are located along the diagonal as shown in Fig. 5.9. The investigation is focused on the convergence of the conditioned mean saturation field. Saturation measurements 1 and 3 are assumed to be less than the prediction, indicating overestimation of the prediction. Saturation measurements 2 and 4 are assumed to be greater than the prediction. This setup is quite challenging due to the different directions of conditioning at locations that are close to each other.
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Figure 5.9: Quarter Five-Spot System with Four Measurements

Figure 5.10: Conditioned $S_w$ Mean at Measurement Location 1
Fig. 5.10 depicts the conditioned saturation mean at measurement location 1. The saturation mean increases during the early iterations, instead of converging to the measurement value. Then, the prediction moves toward the measurement value as the iterations increase. The opposite behavior in the early iteration steps could be influenced by the measurement at location 2. Consider both Fig. 5.10 and Fig. 5.11, the measurement at location 1 tries to reduce the flux in the upper left area; at the same time, measurement 2 tries to increase the flux in the same area. This indicates that the influence of measurement 2 dominates the conditioning between these two measurement. However, the influence of measurement 2 decays with iteration. The influence on the prediction at location 1 becomes insignificant when the prediction at location 2 approaches the measurement value.

In Fig. 5.12, the prediction decreases for the early iterations until the prediction falls below the measurement value. With the conditioning of the measurement 3 alone, the prediction would not have become less than the measurement. This could be another effect from measurement 2 since the flux is reallocated to the upper left part of the domain according to the value of measurement 2. Therefore, both measurements 2 and 3 dictate the prediction at location 3. The conditioned result is improved at later iteration steps, when the effect of measurement 2 becomes small.

Fig. 5.13 indicates that the behavior in the vicinity of measurement 4 could be influenced by measurement 2. The prediction moves in the opposite direction to the measurement.
Figure 5.12: Conditioned $S_{w}$ Mean at Measurement Location 3

Figure 5.13: Conditioned $S_{w}$ Mean at Measurement Location 4
value in the early iteration steps. When the influence of measurement 2 become minimal in the latter iteration steps, the prediction turns back to the direction approaching the measurement value.

In conclusion, measurements 2 and 3 try to increase the flux density in the upper left area. While, measurements 1 and 4 try to reduce the flux density in the area. This is based on the constraint of constant overall flux. The analysis of Figs. 5.10, 5.11, 5.12, and 5.13 suggests that measurement 2 dominates the conditioning over the other measurements at first; as a result, the predictions at the other locations move away from the measurement values. Once the difference between the prediction and the measurement at location 2 is diminished, the predictions at location 1, 3, and 4 eventually approach the measurements.

5.6 Sequential Scheme Behavior

This exercise is to investigate the conditioning process with measurements at different points in time. The system is again a quarter five-spot with two saturation measurements at two distinct locations and two different times, shown in Fig. 5.14.

The conditioning process starts with measurement 1 at time $t_1$ until the prediction converges to the measurement value within an acceptable tolerance. Then, measurement 2 at time $t_2$ is used in the conditioning process. The predictions at both locations are assumed to be below the measurement values. With the log-permeability field evolving in the process of conditioning, the prediction of saturation mean at measurement location 1 is determined at time $t_1$; while the prediction of the saturation mean at measurement location 2 is determined at time $t_2$. The results are shown in Fig. 5.15 and Fig. 5.16, respectively.

At early time, the domain is conditioned by measurement 1. Being an underestimate of the prediction at location 1, the flux in the domain is reallocated such that the flux density around measurement location 1 is increased. As a result, the flux density elsewhere is decreased to preserve a constant overall flux. Fig. 5.15 shows that the prediction at location 1 approaches the measurement value nicely. On the other hand, the prediction at location 2 diverges from the measurement due to the reallocation of the flux density, shown in Fig. 5.16.

Later in time, the behavior is opposite. Measurement 2 is used in the conditioning process. Without knowledge of measurement 1, the conditioning process reallocates flux to location 2 from elsewhere, including location 1. In Fig. 5.15, it can be seen that the
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Figure 5.14: Quarter Five-Spot System with Two Measurements at Different Times

Figure 5.15: Conditioned $S_w$ Mean at Measurement Location 1, Time $t_1$
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Figure 5.16: Conditioned $S_w$ Mean at Measurement Location 2, Time $t_2$

prediction now diverges from the measurement. While, the prediction at location 2 now converges to the measurement value.

As mentioned earlier, the sequential scheme does not guarantee whether the prediction at time $t_1$ to $t_{n-1}$ will be honored when the conditioning process is at time $t_n$. 
Chapter 6

Conclusions

The available information about a subsurface geologic formation is usually limited; as a result, a reservoir description is inevitably incomplete (i.e., uncertain). Uncertainty in the reservoir description leads to uncertainty in the computed predictions of flow performance. This prediction uncertainty can be reduced by utilizing all available information, which can be from various sources (e.g., seismic, logging, well tests, production data). We describe a new inversion procedure to directly condition the first two statistical moments of log-permeability based on saturation measurements. The method employs a geostatistical (Kriging) approach, in which the estimates are represented as linear combinations of the measurements. The formulation make use of the moments of both the independent and dependent variables. The inversion algorithm makes use of an existing Eulerian-Lagrangian SME simulator of the forward problem.

The target was the integration of saturation measurements to condition (update) the first two statistical moments of log-permeability. The Kriging system offers the capability to condition the moments using multiple measurements in both space and time. Moreover, the approach is not limited to conditioning log-permeability based on saturation measurements. To apply the linear approximation (Kriging) for this nonlinear relation, the conditioning process was implemented in an iterative procedure.

The results suggest that in immiscible two-phase flow, the relation between saturation and log-permeability is rather weak. Although the conditional predictions approach the measurement value, a large number of iterations is necessary. Moreover, the variances of log-permeability and the predicted saturation were reduced only slightly. The results of the conditioning with multiple measurements show that a measurement may dominate the
conditioning process for a certain period, and that interference from other measurements can affect the convergence behavior.

Since saturation is a function of space and time, the inverse problem can become prohibitively large, if the number of measurements in both space and time is large. To reduce the size of the Kriging system and simplify the implementation, we devised a sequential scheme for the time dimension. That is the conditioning process is performed for a particular time. The algorithm, however, does not guarantee that the measurements at previous times, $t_1$ to $t_{n-1}$, will be honored during the conditioning process at time $t_n$. This inconsistency can be removed by a global, yet more expensive, scheme.

Many other interesting aspects remain unexplored. These include detailed analysis of the variance of both log-permeability and saturation, conditioning with multiple points at different times, implementation of a global scheme, and extending the method to integrate dynamic production data (i.e., production and injection rates, water cut).
Nomenclature

\[ \langle \rangle = \text{Expectation operator} \]
\[ C = \text{Matrix of a covariance} \]
\[ C_{V_1 V_2} = \text{Covariance (or Cross-covariance) of random variables, } V_1 \text{ and } V_2 \]
\[ f = \text{Probability density function} \]
\[ f_w = \text{Fractional flow curve of water} \]
\[ H(\ ) = \text{Heaviside function} \]
\[ K = \text{Permeability} \]
\[ K = \text{Permeability tensor} \]
\[ \text{Log-N}(\ ) = \text{Probability density function of log-normal distribution} \]
\[ \text{Log-N–N}(\ ) = \text{Probability density function of mixed normal and log-normal distribution} \]
\[ N(\ ) = \text{Probability density function of normal distribution} \]
\[ P = \text{Pressure} \]
\[ \bar{q} = \text{Flow rate per unit volume} \]
\[ S = \text{Saturation} \]
\[ t = \text{Time} \]
\[ v = \text{Velocity} \]
\[ x = \text{Spatial coordinate in Eulerian space} \]
\[ Y = \text{Log-permeability} \]
Greek Letters

η = Spatial coordinate perpendicular to a streamline in Lagrangian space
η = Transverse displacement
θ = Angle of the velocity with respect to the horizontal at a point on the streamline
λ = Kriging weight
ξ = Spatial coordinate parallel to a streamline in Lagrangian space
ρ = Correlation coefficient
σ² = Variance
τ = Travel-time
χ = Spatial coordinate in Eulerian space of another point

Superscripts

' = Indicator of zero-mean fluctuation
\sim = Indicator of vector
0 = Indicator of initial state
T = Transpose

Subscripts

C = Conditioned
i = Integer indices
m = Measurement
r = Irreducible
UC = Unconditioned
w = Water
Bibliography


5


Appendix A

Random Variables

Permeability is considered to be a random variable of space. Since the permeability is an input variable to flow simulation, all the dependent variables (i.e., pressure, total-velocity, travel-time, saturation) are also random variables. A random variable can be decomposed into a mean and a fluctuation around the mean, namely,

\[ V = \langle V \rangle + V', \quad (A.1) \]

where

\[ V \quad \text{is a random variable} \]
\[ \langle V \rangle \quad \text{is the expected value of the random variable, } V \]
\[ V' \quad \text{is the zero-mean fluctuation of the random variable, } V \]

In a probabilistic framework, \( \langle V \rangle \) is the expected value (first moment) of the random variable, \( V \). The second statistical moment is given by

\[ \sigma^2_V = \langle (V - \langle V \rangle)^2 \rangle = \langle (V')^2 \rangle = \int_{-\infty}^{\infty} (V')^2 f(V) \, dV \quad (A.2) \]

where \( \sigma^2_V \) is the variance, or expected square error, of the random variable, \( V \). And the weight, \( f(V) \), is the probability density function (pdf) of the random variable. The other second moment is that between two different random variables. It can be expressed as

\[ C_{V_1V_2} = \langle (V_1 - \langle V_1 \rangle)(V_2 - \langle V_2 \rangle) \rangle = \langle V'_1V'_2 \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (V'_1V'_2) f(V_1, V_2) \, dV_1 \, dV_2 \quad (A.3) \]

The \( C_{V_1V_2} \) is the covariance between \( V_1 \) and \( V_2 \), in the case that \( V_1 \) and \( V_2 \) are the same type of variable. Otherwise, this moment is referred to as a cross-covariance. The weight term, \( f(V_1, V_2) \), is the joint pdf of the random variables \( V_1 \) and \( V_2 \).
In addition to the first and second moments, higher order moments appear in the moment equations. These moment, which can be expressed as $\langle V_1 \ldots V_n \rangle$ for $n \geq 3$, are dropped off in order to obtain low-order approximations.
Appendix B

Probability Distributions and Moment Transformations

In this study, we make use of various probability distribution functions. For convenience, a distribution is assumed to be either normal or log-normal. In addition, when joint distribution or bivariate distribution function is required, it is assumed to be bivariate normal, bivariate log-normal, or of mixed type.

Define $U$ as a random variable, and $u$ as a realization. The distribution is normal with mean of $\mu$ and variance of $\sigma^2$. Define $W = e^U$ as a random variable, $w$ as a realization; it is then a log-normal distribution with mean of $\ln(W)$ and variance of \( Var\{W\} \). The analytical forms and any necessary transformations are listed below.

- **Normal Distribution Function**
  \[
  f(u) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(u-\mu)^2}{2\sigma^2}}, \quad -\infty < u < \infty 
  \]  
  (B.1)

- **Log-Normal Distribution Function**
  \[
  f(w) = \frac{1}{\sigma w \sqrt{2\pi}} e^{-\frac{(\ln(w)-\mu)^2}{2\sigma^2}}, \quad 0 < w < \infty 
  \]  
  (B.2)

It can be seen that the moment information, used in the log-normal distribution function of variable $W$, are the mean and the variance of $U$. A transformation is required to obtain the moment information of $U$ from the moment information of $W$.

- **Mean Transformation**
  \[
  \mu = 2 \ln\langle W \rangle - \frac{1}{2} \ln \left( \langle W \rangle^2 + Var\{W\} \right). 
  \]  
  (B.3)
Variance Transformation

\[ \sigma^2 = \ln \left[ 1 + \frac{\text{Var}\{W\}}{(W)^2} \right]. \]  

(B.4)

The expressions thus far are for univariate distribution. In some case, bivariate distributions are necessary. Those distributions are formulated as follows.

- **Bivariate Normal Distribution Function**

\[
f(u_1, u_2) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \exp \left\{ \frac{-1}{1 - \rho^2} \left[ \frac{(u_1 - \mu_1)^2}{2\sigma_1^2} \right. \right. \\
\left. \left. - \rho \frac{(u_1 - \mu_1)(u_2 - \mu_2)}{\sigma_1 \sigma_2} \right] + \left. \frac{(u_2 - \mu_2)^2}{2\sigma_2^2} \right\}, \]  

(B.5)

where \( \rho = \text{correlation coefficient} \)

- **Correlation Coefficient**

\[
\rho = \frac{\text{Cov}\{U_1, U_2\}}{\sigma_1 \sigma_2}. \]  

(B.6)

- **Bivariate Log-Normal Distribution Function**

\[
f(w_1, w_2) = \frac{1}{2\pi w_1 w_2 \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \exp \left\{ \frac{-1}{1 - \rho^2} \left[ \frac{(\ln w_1 - \mu_1)^2}{2\sigma_1^2} \right. \right. \\
\left. \left. - \rho \frac{(\ln w_1 - \mu_1)(\ln w_2 - \mu_2)}{\sigma_1 \sigma_2} \right] + \left. \frac{(\ln w_2 - \mu_2)^2}{2\sigma_2^2} \right\}. \]  

(B.7)

Again, the bivariate log-normal distribution of \( W \) requires some moment information of \( U \). Some of the necessary transformations have been given above. The additional transformation is for the correlation coefficient.

- **Covariance Transformation**

\[
\text{Cov}(U_1, U_2) = \ln \left[ 1 + \frac{\text{Cov}(W_1, W_2)}{(W_1)(W_2)} \right]. \]  

(B.8)

- **Bivariate Mixed Normal Log-Normal Distribution Function**

\[
f(u_1, w_2) = \frac{1}{2\pi w_2 \sigma_1 \sigma_2 \sqrt{1 - \rho^2}} \exp \left\{ \frac{-1}{1 - \rho^2} \left[ \frac{(u_1 - \mu_1)^2}{2\sigma_1^2} \right. \right. \\
\left. \left. - \rho \frac{(u_1 - \mu_1)(\ln w_2 - \mu_2)}{\sigma_1 \sigma_2} \right] + \left. \frac{(\ln w_2 - \mu_2)^2}{2\sigma_2^2} \right\}. \]  

(B.9)

In this case, the required transformation is between \( \text{Cov}(U_1, U_2) \) and \( \text{Cov}(U_1, W_2) \).
• **Mixed-Type Covariance Transformation**

\[
\text{Cov}(U_1, W_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (u_1 - \mu_1)(e^{u_2} - \langle W_2 \rangle) \frac{1}{\sqrt{2\pi} \sigma_1} \exp\left(-\frac{(u_1 - \mu_1)^2}{2\sigma_1^2}\right) \frac{1}{\sqrt{2\pi} \sigma_2} \exp\left(-\frac{(e^{u_2} - \langle W_2 \rangle)^2}{2\sigma_2^2}\right) \, du_1 \, du_2.
\]  

(B.10)

This equation is basically the definition of covariance. The unknown, \( \text{Cov}(U_1, U_2) \), is embedded in the joint probability density function; the left-hand term is known. The Secant method is implemented as a root-finding algorithm to determine the unknown covariance.