

A magnitude and rate saturation model and its use in the solution of a static anti-windup problem

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Abstract

In this paper we address the anti-windup design problem for linear control systems with strictly proper controllers in the presence of input magnitude and rate saturation. Using generalized sector condition, we provide an LMI-based procedure for the construction of a linear anti-windup gain acting on the controller state equation such that regional closed-loop stability is guaranteed and suitable performance measures are optimized. The approach is successfully illustrated on a simulation example.

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1. Introduction

Magnitude saturation of the control input is typically a nonlinear phenomenon that the control system designer must address since it may lead to instability or unacceptable performance degradation. In many control applications, magnitude saturation is coupled with rate saturation, characterizing a limit on the control input variation, in addition to its magnitude. As an example, catastrophic effects arising from the joint action of magnitude and rate saturation have been long experienced in flight control systems (see, e.g., [6,31]). Additional examples include jet engine compressors (see, e.g., [12,21,36]) and general reaction processes with slow actuators [7].

Control of systems with rate and magnitude saturation has been studied in several areas of nonlinear control. Much work

has been done in the field relying on modern nonlinear control techniques (see, e.g., [9,20,22,23,34,30,32,28,25]). More recently, several approaches relied more directly on the use of convex computational methods (such as LMIs) [3,4,16,19]. Magnitude and rate saturation (MRS) has also been addressed in the so-called anti-windup context (which is the approach that we take here). Anti-windup design addresses the saturation problem after a controller inducing desirable performance on the plant without (rate and magnitude) saturation has been designed. The anti-windup action is then aimed at (1) preserving the prescribed closed-loop behavior before saturation is activated and (2) guaranteeing enlarged stability regions and graceful performance degradation for larger signals that interest the saturation effect in a deeper way. Although a broad literature is available on anti-windup for magnitude saturated plants (we avoid mentioning here the extensive list of references in the field), very little has been done for systems with both magnitude and rate saturation. Some application oriented results are reported in [1,26,33]. Non-constructive theoretical approaches can be found in [2] where the proposed compensator is a plant-order filter. A constructive LMI-based technique consisting of a plant-order anti-windup compensator is also reported in [37] whereas [35] propose a static compensator but with a different rate saturation model from our approach. Finally, several

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magnitude saturation oriented anti-windup schemes based on receding horizon techniques (see, e.g., [5,10,29]) can be easily adapted to also address rate saturation by suitably adjusting the underlying optimization constraints.

In this paper, we address the static anti-windup problem for plants with magnitude and rate saturation. To this aim, we assume that a linear controller has been specified, which constrains the desired small signal behavior of the closed-loop. We further assume that this controller is strictly proper, so that we can easily compute the derivative of its output. The main idea that we apply for our design is to transform the closed-loop in such a way that a modified version of the strictly proper controller is interconnected to an augmented plant via the derivative of the original controller output. This equivalent scheme allows to cast the anti-windup design for MRS in a similar way to that used in the recent work [13,27], when dealing with only magnitude saturation. Note that the idea of generating the derivative of the control signal to characterize rate saturation was already used in the work of [18–20] which addresses direct design rather than anti-windup. The representation of rate saturation that we use here resembles one of these papers, except for a key feedback loop that we need to introduce to avoid unstable cancellations. The design approach that we take here resembles in some way what has been previously done in [35,37]. However, our approach relies on a different rate saturation model and relies on a generalized sector condition, to obtain improved closed-loop performance from the anti-windup design. The paper is organized as follows. In Section 2 we propose a new model to represent the magnitude and rate saturation. In Section 3 we give the problem statement. In Section 4 we introduce the anti-windup scheme, a useful closed-loop transformation and some key matrices. In Section 5 we give our main results and in Section 6 we discuss a simulation example.

Notation. Given a vector $w = [w_1, \dots, w_p]$, $\text{diag}(w)$ is the diagonal $p \times p$ matrix with the vector elements on the diagonal.

The scalar saturation function of level $a \in \mathbb{R}_{>0}$ is defined as

$$\text{sat}_a(v) := \begin{cases} a \text{ sign}(v) & \text{if } |v| > a; \\ v & \text{if } |v| \leq a; \end{cases}$$

where $\text{sign}(\cdot)$ is the sign function; the (vector, decentralized) saturation function of level $w \in \mathbb{R}_{>0}^p$ is defined by saying that its i th component is $[\text{sat}_w(v)]_i = \text{sat}_{w_i}(v_i)$, $i = 1, \dots, p$ where w_i and v_i are the i th components of w and v , respectively. The decentralized deadzone function of level $w \in \mathbb{R}_{>0}^p$, is defined as

$$\text{dz}_w(v) := v - \text{sat}_w(v).$$

A signal $q(\cdot)$ belongs to \mathcal{L}_2 ($q(\cdot) \in \mathcal{L}_2$) if its \mathcal{L}_2 norm is bounded, i.e., $\|q\|_2^2 := \lim_{t \rightarrow \infty} \int_0^t |q(\tau)|^2 d\tau < \infty$. Given a matrix $P = P^T > 0$, $\mathcal{E}(P) := \{x : x^T P x \leq 1\}$ and $\sigma_M(P)$ and $\sigma_m(P)$ indicate the maximum and minimum eigenvalues of P , respectively. Given a square matrix X , $\text{He}X := X + X^T$.

2. Rate saturation representation

An approach used to model an MRS is to introduce the following dynamical system with discontinuous right-hand side:

$$\begin{aligned} \dot{\xi} &= \text{diag}(r) \text{sign}(\text{sat}_m(u_{\text{mrs}}) - \xi), \\ y_{\text{mrs}} &= \xi, \end{aligned} \quad (1)$$

where u_{mrs} , ξ , y_{mrs} , are, respectively, the input, the state and the output, of the MRS, and

$$m := [m_1, \dots, m_p], \quad r := [r_1, \dots, r_p] \quad (2)$$

are vectors whose strictly positive components specify the magnitude and rate limits, respectively.

The discontinuous model (1) (which exactly describes the MRS effects) has been used in [2,33] and requires special care due to its discontinuous right-hand side. However, the model is often approximated by a high gain model where the $\text{sign}(\cdot)$ function is replaced by a high gain followed by a saturation (see, e.g., [4,16,37,35]). Other discrete-time and continuous-time, possibly hybrid, models can be found in the literature (see, e.g., [28]); also these alternative models turn out to be not easy to handle from an analysis and synthesis point of view, especially when their dynamics contain discontinuities.

The model for MRS that we propose in this paper is described by the equations

$$\begin{cases} \dot{\delta} = \text{sat}_r(\dot{u}_{\text{mrs}} + K(u_{\text{mrs}} - \delta) + v_2), \\ y_{\text{mrs}} = \text{sat}_m(\delta), \end{cases} \quad (3)$$

where the diagonal matrix $K > 0$ is a free parameter and u_{mrs} is a signal whose derivative, \dot{u}_{mrs} , is supposed to exist, and v_2 is an external signal. (As can be easily seen in Fig. 1, the parameter K is introduced in order to avoid an unstable cancellation between the ideal derivative operator s and the integrator during linear operation.)

The rate saturation model in (3) is represented in Fig. 1, where an ideal derivative operator is also included. Notice that the additional input v_2 is not required in order to model the MRS, but will be instrumental in the design of our anti-windup controller; for this reason, the properties of interest for (3) will be given in the following Lemma 1 already considering (3) with the additional input v_2 .

Remark 1. It is worth noting that a main difference between working directly with (1) and introducing (3) is that in (1) the magnitude of the input to the nonlinearity (i.e., u_{mrs}) is

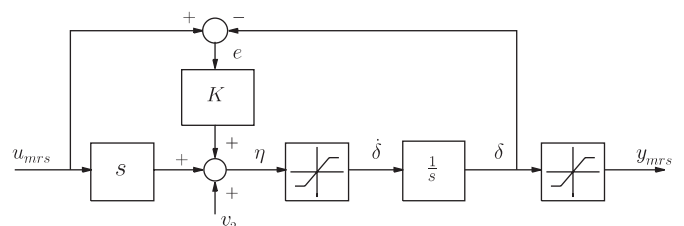


Fig. 1. The proposed MRS block structure.

limited before entering the rate limiter, whereas in Fig. 1, the rate is limited first and the magnitude next. Indeed, in Fig. 1, the rate of y_{mrs} is limited by the first saturation block, whereas its magnitude is limited by the second saturation block. As a consequence of this difference, while the state ζ of (1) never exceeds the magnitude bound m , the state of (3) can exceed the bound m , although its output y_{mrs} never does. Despite these discrepancies, the model (3) can be effectively exploited to solve an anti-windup problem for systems subject to MRS.

The following lemma guarantees some properties of (3) that will be used in our main results. In particular,

- item 1 guarantees that the output of (3) always satisfies the magnitude and rate limits;
- item 2 guarantees that if (3) is properly initialized and v_2 is identically 0, then the output y_{mrs} of (3) coincides with its input u_{mrs} as long as $(u_{\text{mrs}}, \dot{u}_{\text{mrs}})$ never exceed the magnitude and rate limits;
- item 3 then complements item 2, by specifying that $(y_{\text{mrs}} - u_{\text{mrs}})$ will remain an \mathcal{L}_2 signal despite the occurrence of either a wrong initialization, and/or a non zero $v_2 \in \mathcal{L}_2$, and/or a choice of u_{mrs} which exceeds the magnitude or rate limits (restricted by an arbitrarily small amount) in such a way that the excess of saturation is an \mathcal{L}_2 signal.

Notice that, based on item 1, the cascade of (3), (1) and a plant behaves exactly like the cascade of (3) and the plant, i.e., the actual MRS never activates. In Sections 4 and 5, this will allow the anti-windup compensator to be designed based on the cascade of (3) and the plant.

Lemma 1. *Given any signal $u_{\text{mrs}}(\cdot)$ such that \dot{u}_{mrs} is well defined for almost all t , for any diagonal $K > 0$, the MRS model (3) satisfies the following:*

- (i) for any measurable $v_2(\cdot)$, $\text{sat}_m(y_{\text{mrs}}(t)) = y_{\text{mrs}}(t)$ and $\text{sat}_r(\dot{y}_{\text{mrs}}(t)) = \dot{y}_{\text{mrs}}(t)$, for almost all $t \geq 0$;
- (ii) if $\delta(0) = u_{\text{mrs}}(0)$, $\text{sat}_m(u_{\text{mrs}}(t)) = u_{\text{mrs}}(t)$, $\text{sat}_r(\dot{u}_{\text{mrs}}(t)) = \dot{u}_{\text{mrs}}(t)$ and $v_2(t) = 0$, $\forall t \geq 0$, then $y_{\text{mrs}}(t) = u_{\text{mrs}}(t)$, $\forall t \geq 0$;
- (iii) for any $\delta(0)$ if $\|v_2\|_2 < \infty$ and $\exists \varepsilon > 0$ such that $\|u_{\text{mrs}} - \text{sat}_{m(1-\varepsilon)}(u_{\text{mrs}})\|_2 < \infty$ and $\|\dot{u}_{\text{mrs}} - \text{sat}_{r(1-\varepsilon)}(\dot{u}_{\text{mrs}})\|_2 < \infty$, then $\|y_{\text{mrs}} - u_{\text{mrs}}\|_2 < \infty$.

Proof. For conciseness and clarity of presentation we will carry out the proof assuming only scalar signals and unit levels for both magnitude and rate saturation; due to the decentralized nature of the vector saturation considered in this paper, the extension to the vector case with nonunit saturation levels is trivial.

Item 1: Being the output of a unit saturation function, $|y_{\text{mrs}}(t)| \leq 1$, $\forall t \geq 0$, for any signal $v_2(\cdot)$. As far as \dot{y}_{mrs} is concerned, since $|\dot{\delta}(t)| \leq 1$ for all $t \geq 0$, then trivially $|\dot{y}_{\text{mrs}}(\cdot)| \leq 1$ whenever $\dot{y}_{\text{mrs}}(t)$ is defined.

Item 2: If $\delta(0) = u_{\text{mrs}}(0)$, $v_2(\cdot) = 0$, and also the saturation limits are never exceeded by $u_{\text{mrs}}(\cdot)$ and $\dot{u}_{\text{mrs}}(\cdot)$, then it is easy to verify that $y_{\text{mrs}}(t) = u_{\text{mrs}}(t) = \delta(t)$, $e(t) = 0$, $\forall t \geq 0$ is a

solution of (3); moreover, such a solution is unique because the right-hand side of (3) is Lipschitz.

Item 3: As a preliminary step, by the global Lipschitz property of the saturation function, and since $y_{\text{mrs}} = \text{sat}_1(\delta)$, we have for all t (for brevity the dependence of t will be omitted).

$$\begin{aligned} |u_{\text{mrs}} - y_{\text{mrs}}| &\leq |u_{\text{mrs}} - \text{sat}_{1-\varepsilon}(u_{\text{mrs}})| \\ &\quad + |\text{sat}_1(u_{\text{mrs}}) - \text{sat}_1(\delta)| \\ &\quad + |\text{sat}_1(\text{sat}_{1-\varepsilon}(u_{\text{mrs}})) - \text{sat}_1(u_{\text{mrs}})| \\ &\leq 2 \cdot |u_{\text{mrs}} - \text{sat}_{1-\varepsilon}(u_{\text{mrs}})| + |u_{\text{mrs}} - \delta|, \end{aligned}$$

therefore it is only necessary to prove that $\|u_{\text{mrs}} - \delta\|_2 < \infty$, because $\|u_{\text{mrs}} - \text{sat}_{1-\varepsilon}(u_{\text{mrs}})\|_2 < \infty$ by hypothesis.

Consider the positive definite and radially unbounded Lyapunov function $V(e) = \int_0^e \text{sat}_\varepsilon(Ks) ds$, where $e = u_{\text{mrs}} - \delta$, for which it is possible to find functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that $\alpha_1(|e|) \leq V(e) \leq \alpha_2(|e|)$, $\forall e \in \mathbb{R}$.³

Defining the signal d as

$$\begin{aligned} d := &\dot{u}_{\text{mrs}} - \text{sat}_{1-\varepsilon}(\dot{u}_{\text{mrs}} + v_2) - \text{sat}_1(\dot{u}_{\text{mrs}} + Ke + v_2) \\ &+ \text{sat}_1(\text{sat}_{1-\varepsilon}(\dot{u}_{\text{mrs}} + v_2) + Ke) \end{aligned}$$

the derivative of $V(e)$ along the trajectories of (3) is

$$\begin{aligned} \dot{V}(e) &= \text{sat}_\varepsilon(Ke)\dot{e} \\ &= \text{sat}_\varepsilon(Ke)(\dot{u}_{\text{mrs}} - \text{sat}_1(\dot{u}_{\text{mrs}} + Ke + v_2)) \\ &= \text{sat}_\varepsilon(Ke)(\text{sat}_{1-\varepsilon}(\dot{u}_{\text{mrs}} + v_2) \\ &\quad - \text{sat}_1(\text{sat}_{1-\varepsilon}(\dot{u}_{\text{mrs}} + v_2) + Ke) + d) \\ &\leq -\text{sat}_\varepsilon^2(Ke) + |\text{sat}_\varepsilon(Ke)||d|, \end{aligned} \quad (4)$$

where the signal d can be shown to be \mathcal{L}_2 -bounded by using the globally Lipschitz property of saturation as follows:

$$\begin{aligned} |d| &\leq |\dot{u}_{\text{mrs}} - \text{sat}_{1-\varepsilon}(\dot{u}_{\text{mrs}})| \\ &\quad + |\text{sat}_{1-\varepsilon}(\dot{u}_{\text{mrs}}) - \text{sat}_{1-\varepsilon}(\dot{u}_{\text{mrs}} + v_2)| \\ &\quad + |\dot{u}_{\text{mrs}} + v_2 - \text{sat}_{1-\varepsilon}(\dot{u}_{\text{mrs}} + v_2)| \\ &\leq |\dot{u}_{\text{mrs}} - \text{sat}_{1-\varepsilon}(\dot{u}_{\text{mrs}} + v_2)| + |v_2| + |v_2| \\ &\quad + |\dot{u}_{\text{mrs}} - \text{sat}_{1-\varepsilon}(\dot{u}_{\text{mrs}}) - \text{sat}_{1-\varepsilon}(\dot{u}_{\text{mrs}} + v_2)| \\ &\leq 2|\dot{u}_{\text{mrs}} - \text{sat}_{1-\varepsilon}(\dot{u}_{\text{mrs}})| + 3|v_2|. \end{aligned}$$

Completing the square in (4) yields

$$\dot{V}(e) \leq -0.5 \text{sat}_\varepsilon^2(Ke) + 0.5d^2. \quad (5)$$

In order to show that $e \in \mathcal{L}_\infty$, ignore the term $-0.5 \text{sat}_\varepsilon^2(Ke)$ in (5) and integrate to get

$$\begin{aligned} V(e(t)) &\leq V(e(0)) + \int_0^t 0.5d^2(s) ds \leq V(e(0)) + 0.5\|d\|_2^2 \\ &\leq \alpha_2(|e(0)|) + 0.5\|d\|_2^2, \quad \forall t \geq 0 \end{aligned}$$

so that, $\forall t \geq 0$,

$$|e(t)| \leq \alpha_1^{-1}(V(e(t))) \leq \alpha_1^{-1}(\alpha_2(|e(0)|) + 0.5\|d\|_2^2)$$

³ For example, a possible choice is $\alpha_1(|e|) = \alpha_2(|e|) = V(|e|)$.

- (ii) for $w=0$ find the minimum $\lambda < 0$ such that there is a region Ω of the initial conditions where the origin of the modified closed-loop is regionally exponentially stable with decay rate λ .

The solutions to Problems 1 and 2 reported next will rely on a suitable transformation of the unconstrained interconnection (7), (8), (10) that will lead to the same framework used in [15] for the regional analysis and design of control system involving magnitude saturation. As a byproduct of the regional nature of the approaches proposed therein we will be able to give a quantitative estimate of the domain of attraction (with zero input) and of the reachability set from bounded inputs of the closed-loop.

4. Anti-windup scheme and an equivalent closed-loop representation

To suitably cast the anti-windup design problem that we address here, we will introduce a new controller and a new plant whose interconnection is equivalent to that of the anti-windup closed-loop system (7), (8), (9), (3), (11) with the interconnection equations $(u_{\text{mrs}}, \dot{u}_{\text{mrs}}) = (y_c, y_{c,\text{dot}})$ but for which it is possible to easily isolate the saturation elements. First note that by item (i) in Lemma 1, the discontinuous dynamics (1) can be disregarded because (1) will always act like an identity. Then, we incorporate the states δ of (3) into an augmented linear plant that we call $\bar{\mathcal{P}}$, so that the closed-loop between the plant \mathcal{P} in (7) and the controller \mathcal{C} in (8) via the extra dynamics (9), (3) can be represented in a compact way as in Fig. 3, where the augmented plant equations are

$$\begin{aligned}\dot{\bar{x}}_p &= \bar{A}_p \bar{x}_p + \bar{B}_{p,u} \bar{u}_p + \bar{B}_{p,w} w, \\ \bar{y}_p &= \bar{C}_{p,y} \bar{x}_p + \bar{D}_{p,yu} \bar{u}_p + \bar{D}_{p,yw} w, \\ z_p &= \bar{C}_{p,z} \bar{x}_p + \bar{D}_{p,zu} \bar{u}_p + \bar{D}_{p,zw} w,\end{aligned}\quad (12)$$

where the matrices in (12) are reported in (14) and $\bar{y}_p = \begin{bmatrix} y_p \\ \delta \end{bmatrix}$ and $\bar{x}_p = \begin{bmatrix} x_p \\ \delta \end{bmatrix}$.

Moreover, based on the controller \mathcal{C} in (8) we design a new controller $\bar{\mathcal{C}}$ (also represented in Fig. 3) which has a larger output also including the derivative of y_c as in (9):

$$\begin{aligned}\dot{\bar{x}}_c &= \bar{A}_c \bar{x}_c + \bar{B}_{c,y} \bar{y}_p + \bar{B}_{c,w} w + v_1, \\ \bar{y}_c &= \bar{C}_c \bar{x}_c + \bar{D}_{c,y} \bar{y}_p + \bar{D}_{c,w} w + \bar{D}_{c,v_2} v_2\end{aligned}\quad (13)$$

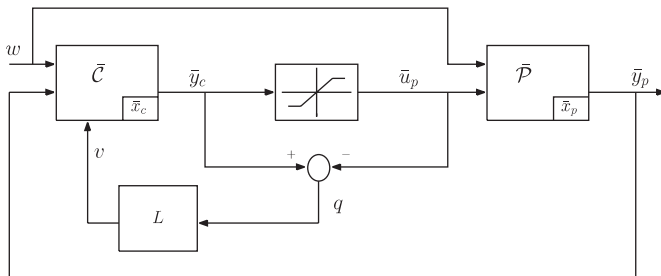


Fig. 3. Equivalent closed-loop representation for the closed-loop with MRS.

with the matrix selections in (15) and $\bar{y}_c = \begin{bmatrix} \delta \\ \eta \end{bmatrix}$.

$$\begin{bmatrix} \bar{A}_p & \bar{B}_{p,u} & \bar{B}_{p,w} \\ \bar{C}_{p,y} & \bar{D}_{p,yu} & \bar{D}_{p,yw} \\ \bar{C}_{p,z} & \bar{D}_{p,zu} & \bar{D}_{p,zw} \end{bmatrix} = \begin{bmatrix} A_p & 0 & B_p & 0 & B_{p,w} \\ 0 & 0 & 0 & I & 0 \\ C_{p,y} & 0 & D_{p,yu} & 0 & D_{p,yw} \\ 0 & I & 0 & 0 & 0 \\ C_{p,z} & 0 & D_{p,zu} & 0 & D_{p,zw} \end{bmatrix}\quad (14)$$

$$\begin{bmatrix} \bar{A}_c & \bar{B}_{c,y} & \bar{B}_{c,w} \\ \bar{C}_c & \bar{D}_{c,y} & \bar{D}_{c,w} & \bar{D}_{c,v_2} \end{bmatrix} = \begin{bmatrix} A_c & B_{c,y} & 0 & B_{c,w} \\ 0 & 0 & I & 0 & 0 \\ KC_c + C_c A_c & C_c B_{c,y} & -K & C_c B_{c,w} & I \end{bmatrix}\quad (15)$$

Then, the anti-windup gain (11) will act on the equivalent closed-loop representation similarly to the classical static anti-windup compensation schemes for control systems with magnitude saturation (see, Fig. 3), namely

$$v = L(\bar{y}_c - \text{sat}_b(\bar{y}_c)),\quad (16)$$

where $v = [v_1^T \ v_2^T]^T$. Therefore, by relying on known anti-windup design techniques, the anti-windup gain L will be selected using convex optimization tools (LMIs). In this framework, given the vector b that incorporates the MRS bounds as follows:

$$\begin{aligned}b &:= [\bar{b}_1, \dots, \bar{b}_{2n_u}] \\ &= [m_1, \dots, m_{n_u}, r_1, \dots, r_{n_u}].\end{aligned}\quad (17)$$

The anti-windup interconnection is given by

$$\bar{u}_p = \text{sat}_b(\bar{y}_c).\quad (18)$$

The resulting nonlinear closed-loop, represented in Fig. 3, will be characterized by the state $x = [\bar{x}_p^T \ \bar{x}_c^T]^T$. By doing the aforementioned rearrangement, the problem of dealing with the MRS for the closed-loop (7), (8), has been recast as a magnitude only saturation problem for (12), (13). This allows us to deal with the simpler problem of input magnitude saturation for (non-exponentially stable because of the δ dynamics) linear systems, for which the results developed in [15] are available. We will only use the corresponding static anti-windup construction for simplicity (and we will avoid applying the projection lemma), but we emphasize that the same approach can be used for the convex design of the augmented plant-order anti-windup.

5. LMI-based design

Similar to the approach taken in [15] (which, in turns, exploits the generalized sector condition first used in [13,8]) the anti-windup closed-loop system of Fig. 3 can be represented in the following compact form:

$$\mathcal{H} \begin{cases} \dot{x} = Ax + B_q q + B_w w + B_L Lq, \\ \bar{y}_c = C_y x + D_{yq} q + D_{yw} w + D_{yL} Lq, \\ z_p = C_z x + D_{zq} q + D_{zw} w + D_{zL} Lq, \end{cases}\quad (19)$$

$$q = \text{dz}_b(\bar{y}_c),\quad (20)$$

where $x = [\bar{x}_p^T \ \bar{x}_c^T]^T$ is the overall state, and, by Assumption 1, the matrices appearing in (19) are uniquely defined based on the plant \mathcal{P} , the controller \mathcal{C} , the anti-windup matrix L , and the MRS model, as reported in Eq. (21), where $\bar{A}_u := (I - \bar{D}_{c,y}\bar{D}_{p,yu})^{-1}$ and $\bar{A}_y := (I - \bar{D}_{p,yu}\bar{D}_{c,y})^{-1}$.

$$\begin{bmatrix} A \\ C_y \\ C_z \end{bmatrix} = \begin{bmatrix} \bar{A}_p + \bar{B}_{p,u}\bar{\Delta}_u\bar{D}_{c,y}\bar{C}_{p,y} & \bar{B}_{p,u}\bar{\Delta}_u\bar{C}_c \\ \bar{B}_{c,y}\bar{\Delta}_y\bar{C}_{p,y} & \bar{A}_c + \bar{B}_{c,y}\bar{D}_{p,yu}\bar{\Delta}_u\bar{C}_c \\ \bar{\Delta}_u\bar{D}_{c,y}\bar{C}_{p,y} & \bar{\Delta}_u\bar{C}_c \\ \bar{C}_{p,z} + \bar{D}_{p,zu}\bar{\Delta}_u\bar{D}_{c,y}\bar{C}_{p,y} & \bar{D}_{p,zu}\bar{\Delta}_u\bar{C}_c \end{bmatrix},$$

$$\begin{bmatrix} B_q & B_w & B_L \\ D_{yq} & D_{yw} & D_{yL} \\ D_{zq} & D_{zw} & D_{zL} \end{bmatrix} = \begin{bmatrix} -\bar{B}_{p,u}\bar{\Delta}_u & \bar{B}_{p,w} + \bar{B}_{p,u}\bar{\Delta}_u(\bar{D}_{c,w} + \bar{D}_{c,y}\bar{D}_{p,yw}) & 0 & \bar{B}_{p,u}\bar{\Delta}_u\bar{D}_{c,v_2} \\ -\bar{B}_{c,y}\bar{\Delta}_y\bar{D}_{p,yu} & \bar{B}_{c,w} + \bar{B}_{c,y}\bar{\Delta}_y(\bar{D}_{p,yw} + \bar{D}_{p,yu}\bar{D}_{c,w}) & I & \bar{B}_{c,y}\bar{\Delta}_y\bar{D}_{p,yu}\bar{D}_{c,v_2} \\ -\bar{\Delta}_u\bar{D}_{c,y}\bar{D}_{p,yu} & \bar{\Delta}_u(\bar{D}_{c,w} + \bar{D}_{c,y}\bar{D}_{p,yw}) & 0 & \bar{\Delta}_u\bar{D}_{c,v_2} \\ -\bar{D}_{p,zu}\bar{\Delta}_u & \bar{D}_{p,zw} + \bar{D}_{p,zu}\bar{\Delta}_u(\bar{D}_{c,w} + \bar{D}_{c,y}\bar{D}_{p,yw}) & 0 & \bar{D}_{p,zu}\bar{\Delta}_u\bar{D}_{c,v_2} \end{bmatrix}. \quad (21)$$

To deal with the deadzone function, the proofs of both the following theorems are carried out by using the approach proposed in [14], exploiting the following *generalized sector property*:

$$dz_b^T(\bar{y}_c)M(dz_b(\bar{y}_c) - \bar{y}_c + Hx) \leq 0, \quad \forall x \in \mathcal{L}(H), \quad (22)$$

where

$$\mathcal{L}(H) := \{x : \text{sat}_b(Hx) = Hx\},$$

the matrix H is a free parameter and the matrix $M > 0$ is a diagonal scaling matrix.

Theorem 1. Consider a plant–controller pair (7), (8) and MRS limits m and r . Consider the equivalent closed-loop (12), (13), (16), (18). Given any diagonal $K > 0$, any $s > 0$ and any solution to the following LMI optimization problem:

$$\begin{aligned} & \min_{Q,U,Y,X,\gamma^2} \gamma^2 \quad \text{subject to} \\ & Q = Q^T > 0, U > 0 \quad \text{diagonal}, \end{aligned} \quad (23)$$

$$\text{He} \begin{bmatrix} AQ & B_q U + B_L X & B_w & 0 \\ C_y Q - Y & D_{yq} U - U + D_{yL} X & D_{yw} & 0 \\ 0 & 0 & -\frac{I}{2} & 0 \\ C_z Q & D_{zq} U + D_{zL} X & D_{zw} & -\frac{\gamma^2}{2} I \end{bmatrix} < 0,$$

$$\begin{bmatrix} \frac{\bar{b}_i^2}{s^2} & Y_i \\ Y_i^T & Q \end{bmatrix} \geq 0, \quad i = 1, \dots, 2n_u \quad (24)$$

then a solution to Problem 1 is given by the modified control system (8), (9), (3), (11) with $L = XU^{-1}$, and γ is the optimal \mathcal{L}_2 gain from w to z_p at item (ii).

Proof. It is immediate to see that, with the proposed model for the MRS, the saturated closed-loop system (7), (8), (9) and (3), rearranged according to the augmented plant (12) and the

augmented controller (13) is equivalent to the classical anti-windup scheme shown in Fig. 3. The solution of this equivalent problem has been exhaustively carried out in [15,13], where the proof of the Theorem 2 in [15] can be applied for the augmented

system (12), (15), with the shrewdness to replace the term $B_q U$ in Theorem 2 of [15], in which a dynamical anti-windup filter is found, by the term $B_q U + B_L X$, in the first row of the LMI (24). \square

Theorem 2. Consider a plant–controller pair (7), (8) and MRS limits given by m and r . Consider the equivalent closed-loop (12), (13), (16), (18). Given any diagonal $K > 0$, and any solution to the following generalized eigenvalue problem:⁴

$$\begin{aligned} & \min_{Q,U,Y,X,\lambda} \lambda \quad \text{subject to} \\ & Q = Q^T, \quad Q > 0, \quad U > 0 \quad \text{diagonal}, \end{aligned} \quad (25)$$

$$\text{He} \begin{bmatrix} AQ & B_q U + B_L X \\ C_y Q - Y & D_{yq} U - U + D_{yL} X \end{bmatrix} < \begin{bmatrix} \lambda Q & 0 \\ 0 & 0 \end{bmatrix}, \quad (26)$$

$$\begin{bmatrix} \bar{b}_i^2 & Y_i \\ Y_i^T & Q \end{bmatrix} \geq 0, \quad i = 1, \dots, 2n_u \quad (27)$$

then a solution to Problem 2 is given by the modified control system (8), (9), (3), (11) with $L = XU^{-1}$, and for all $x(0) \in \mathcal{E}(Q^{-1}) = \Omega$, and the closed-loop response satisfies

$$|x(t)| \leq ce^{(\lambda/2)t} |x(0)| \quad (28)$$

with $c = \sqrt{\sigma_M(Q)/\sigma_m(Q)}$.

Proof. Eq. (28) comes from considering a quadratic Lyapunov function $V(x) = x^T P x$ with $P = P^T > 0$, and imposing the inequality $\dot{V}(x) < \lambda V(x)$. We want to minimize the scalar λ , such that

$$\begin{aligned} \dot{V}(x) - \lambda V(x) &= 2x^T P \dot{x} - \lambda x^T P x \\ &= 2x^T P (Ax + B_q q + B_L Lq - \frac{\lambda}{2} x) < 0, \quad \forall x \in \Omega, \end{aligned} \quad (29)$$

⁴ The constraints in (25) do not actually describe a gevp, but can be easily transformed into one by using an extra variable $Z = Z^T$ satisfying $Z < \lambda Q$.

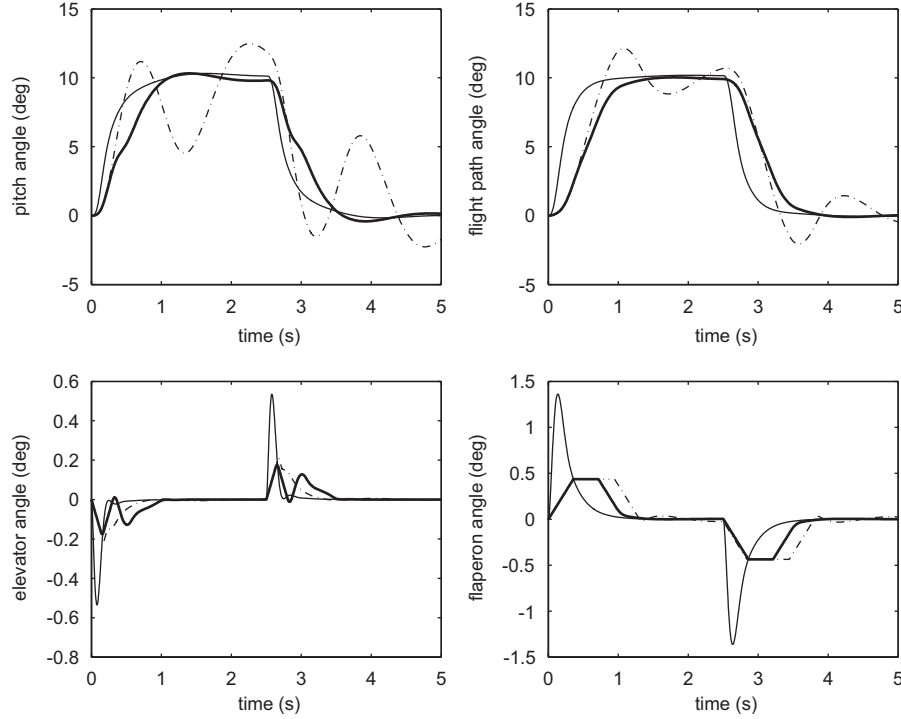


Fig. 4. Solution to Problem 1: unconstrained response (thin solid), saturated response (dash-dotted), and anti-windup response (bold).

where Ω is a suitable region and with $q = dz_b(\bar{y}_c)$. By (22) if $\Omega \subset \mathcal{L}(H)$, a sufficient condition for (29) is

$$2x^T P \left(Ax + B_q q + B_L L q - \frac{\lambda}{2} x \right) + 2q^T M (C_y x + (D_{yq} + D_{yL} L) q - q - Hx) \leq 0$$

that, introducing the variable $\chi = [x^T \ q^T]^T$ and defining

$$\Theta := \text{He} \begin{bmatrix} PA - \frac{\lambda}{2} P & PB_q + PB_L L \\ MC_y - MH & MD_{yq} - M + MD_{yL} \end{bmatrix} \leq 0 \quad (30)$$

can be rewritten as $\chi^T \Theta \chi \leq 0$. Pre- and post-multiplying (30) by $\text{diag}(P^{-1}, M^{-1})$ and letting $U = M^{-1}$, $Q = P^{-1}$, $HQ = Y$ and $L = XU^{-1}$, (30) becomes the LMI (26). To guarantee that $\Omega = \mathcal{E}(P) \subset \mathcal{L}(H)$, it is sufficient to impose that

$$\begin{bmatrix} \bar{b}_i^2 & H_i \\ H_i^T & P \end{bmatrix} \geq 0, \quad i = 1, \dots, 2n_u,$$

which corresponds to (27) pre- and post-multiplied by $\text{diag}(1, Q)$ and with $HQ = Y$ and $Q = P^{-1}$. \square

Remark 2. Note that by the results of [15], both the approaches in Theorems 1 and 2 guarantee regional closed-loop stability, regional \mathcal{L}_2 gain for all \mathcal{L}_2 norm bounded inputs $\|w\|_2 < s$ and can provide an estimate of the exponential stability domain ($\mathcal{E}(Q^{-1}/s^2)$ and $\mathcal{E}(Q^{-1})$, respectively), and of the reachable set under \mathcal{L}_2 norm bounded inputs ($\mathcal{E}(Q^{-1}/s^2)$). The difference between the two approaches stands in the optimization

goal. Although it is not clear which of the two performance measures leads to the best performance, we show on a simulation example in Section 6 that each of the two approaches may be desirable in certain problem settings.

Remark 3 (Well-posedness). Although well-posedness of the nonlinear closed-loop arising from the constructions of Theorems 1 and 2 is guaranteed, sometimes the solutions arising from the LMI-based numerical optimization solvers may lead to degenerate cases wherein the algebraic loop induced by the anti-windup block is very close to being ill-posed. This may correspond to weak closed-loop robustness and heavy computational load (even just for closed-loop simulation). Augmenting the optimization problems with the extra condition proposed in [11, Eq. (5)] typically solves possible numerical problems. That condition, for our closed-loop system (19), is expressed by the following LMI:

$$\text{He} \begin{bmatrix} (\rho - 1)U - \rho(UD_{yq} + XD_{yL}) & \rho D_{yL} X \\ \rho U D_{yq} & \frac{\mu - \rho}{4} U \end{bmatrix} < 0,$$

where the positive scalars μ and ρ are suitably selected as a trade-off between making the Lipschitz constant of the right-hand side of the *anti-windup closed-loop system* small and preserving the feasibility of the LMI constraints.

6. Simulation example

In this section, we apply our construction to the longitudinal dynamics of an F8 Aircraft: a fourth-order linear model

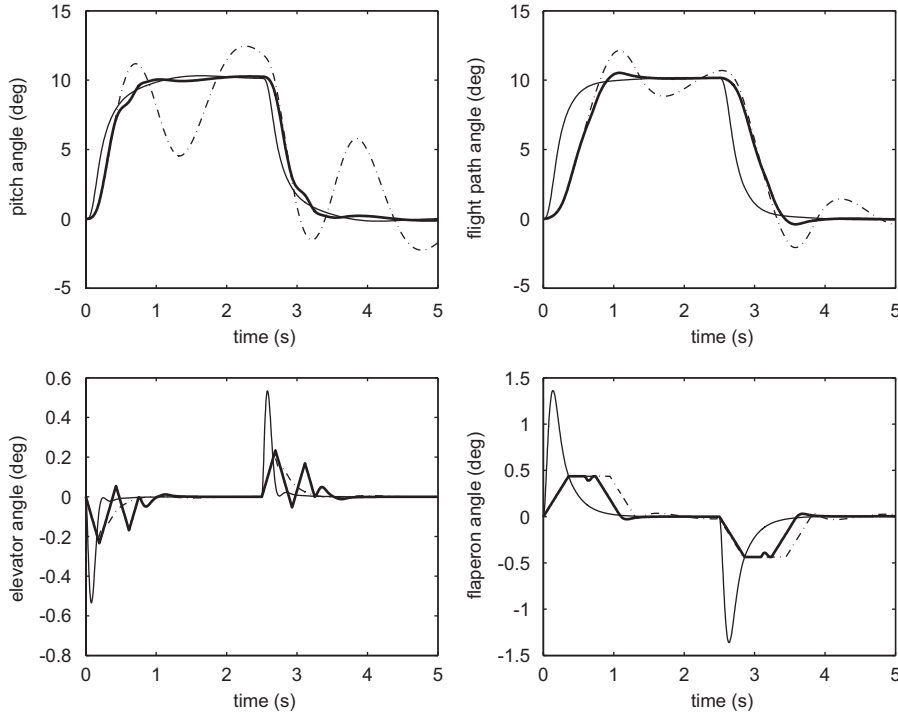


Fig. 5. Solution to Problem 2: unconstrained response (thin solid), saturated response (dash-dotted) and anti-windup response (bold).

for which an eighth-order linear unconstrained controller was introduced in [17] (see also [24]). The two plant inputs are the elevator and the flaperon angles. We assume that they are both subject to MRS. The magnitude limits are selected as $\pm 25^\circ$ (as in [17]). As for the rate limits, reasonable limits (based on the parameters of other aircrafts) are $\pm 70^\circ/\text{s}$. The outputs are the pitch angle and the flight path angle. The controller input is the difference between the plant output and the reference input, while the performance output is defined via the matrices (see [17]) $C_{p,z} = \begin{bmatrix} 0 & 0 & 0 & 3/4 \\ -0.8 & -0.0006 & -12 & 0 \end{bmatrix}$, $D_{p,zw} = \begin{bmatrix} -3/4 & 0 \\ 0 & 0 \end{bmatrix}$ and $D_{p,zu} = 0_{2 \times 2}$.

The static anti-windup compensator, built using the conditions in Theorem 1 (i.e., solving Problem 1) with $K = \text{diag}\{1000, 500\}$ and $s = 1.1$, guarantees an optimal regional performance level of $\gamma = 60.64$. Simulation results are in Fig. 4, from where we can see that the transient behavior of the linear control system is quite well recovered. Note that the control signal is almost permanently saturated in rate, thus indicating that the available control effort is fully exploited by the controller.

For the same plant, the anti-windup construction proposed in Theorem 2 (therefore, the one solving Problem 2) assures very satisfying performance as it is shown in Fig. 5. In this case we selected $K = \text{diag}\{1000, 1000\}$ and obtained the optimal decay rate $\lambda = -0.0147$.

Note that the arising response is slightly more oscillatory but appears to be faster. Indeed (also based on the experience on other examples), in general the approach of Problem 2 showed more satisfactory responses than that of Problem 1. However,

it should be emphasized that the latter approach may in some cases be more desirable when wanting to only optimize the performance seen at a specific output, while the optimality of the former one applies generally to the overall closed-loop state response.

7. Conclusions

In this paper, a static anti-windup control design has been proposed for systems subject to magnitude and rate saturation under the only assumption that the a priori given unconstrained controller is strictly proper and stabilizing. The proposed LMI-based approach exploits a new model of the magnitude and rate saturation. Future studies concern a dynamic anti-windup controller design and the investigation of the robust properties of the proposed compensator.

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