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Further results on static linear anti-windup design for control systems subject to magnitude and rate saturation

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Abstract— **In this paper we report on further developments on a novel scheme for the anti-windup compensation of rate and magnitude saturated systems. This scheme, originally proposed in [10] allows to rely on tools for anti-windup compensation for magnitude saturated systems also for the case of rate saturation, by way of a suitable model of the magnitude and rate saturation. The novelty in this paper is to show that, for performance improvement, the compensation scheme of [10] can be modified using a saturated integrator for the magnitude saturation part and only retaining the anti-windup gain for the rate saturation part. This modified strategy is shown mathematically to ensure stability and, on a simulation example, it is shown to induce more desirable closed-loop responses.**

I. Introduction

Anti-windup designs denote augmentations to control systems aimed at addressing input saturation phenomena that weren't taken into account when designing the original (the so-called unconstrained) control system. This design methodology has been long studied, especially in the context of magnitude saturation and several design methodologies are nowadays available for many types of applications, including nonlinear saturated plants and linear plants with exponentially unstable modes. Differently from the magnitude saturation case, not much has been done in the anti-windup context for systems subject to both magnitude and rate saturation (MRS). These systems become nowadays increasingly important due to the relevance of the aerospace applications where they have been experienced (see, e.g., [6], [7], [13], [20], [31]).

Anti-windup is well seen as a different approach from direct design where the control system stabilizing (and possibly inducing some level of performance on) a rate and magnitude saturated plant is designed from scratch. Several approaches achieving this different goal are available in the literature, especially relying on modern nonlinear control techniques (see, e.g., [9], [19], [21], [22], [29], [26], [27]). Some other ones, more similar to the tools used here, rely on the use of convex computational methods (such as LMIs) [3], [4], [16], [18]. The difference between anti-windup and direct design is that anti-windup enforces a strong constraint on the small signal behavior of the system, where the response has to coincide with the prescribed response induced by the "unconstrained" controller, as long as saturation is not active.

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Some recent results addressing anti-windup design for rate and magnitude saturated systems are [1], [24], [28], where application oriented studies are reported. Approaches of more general applicability have been given in [32] and [30] where a plant-order and a static compensation scheme, respectively are proposed. A non-constructive plant-order solution to the problem was also given in [2], but key stabilizing feedbacks need to be designed for the special plant under consideration in that scheme. These feedbacks are not always easy to determine. Finally, the so-called reference governor (or command-governor) approaches which rely on receding horizon optimal control ideas (see, e.g., [5], [11], [25]) can be formulated by incorporating rate saturation in the control design problem.

Recently, in [10], we proposed a model for the rate and magnitude saturation nonlinearity by which the anti-windup augmentation problem for rate and magnitude saturated plants can be reduced to an anti-windup augmentation problem for a magnitude saturated plant, which in turn can be efficiently dealt with by means of recent results [15]. In the present paper, we propose a modified scheme which is easier to implement (as the matrices involved in the anti-windup augmentation are half of the size of the ones in [10]), and, due to its modified structure, has more potential of leading to desirable responses for the anti-windup closed-loop. This fact is illustrated by a simulation study where the new scheme outperforms the one in [10].

The paper is organized as follows. After some notation is presented, in Section II we review some preliminary results in order to present the main contribution of the paper. In particular, in Section II-A, we recall the representation of the magnitude and rate saturation nonlinearity, in Section II-B the anti-windup closed loop setting is presented and in Section II-C the LMI solution of the anti-windup problem is given. In Section III we start presenting the modified MRS model and the main Theorem is established in Section III-B. Simulations based on a physical example that confirm the theoretical results are in Section IV.

Notation Given a vector $w = [w_1, \ldots, w_p]$, $\mathrm{diag}(w)$ is the diagonal $p \times p$ matrix with the vector elements on the diagonal.
The scalar saturation function of level $a \in \mathbb{R}_{>0}$ is defined as

$$\mathrm{sat}_a(v) := \begin{cases} a\,\mathrm{sign}(v), & \text{if } |v| > a; \\ v, & \text{if } |v| \le a; \end{cases}$$

where $\mathrm{sign}(\cdot)$ is the sign function; the (vector, decentralized) saturation function of level $w \in \mathbb{R}^p_{>0}$ is defined by saying that its i-th component is $[\mathrm{sat}_w(v)]_i = \mathrm{sat}_{w_i}(v_i)$, $i = 1, \ldots, p$ where w_i and v_i are the i-th components of w and v, respectively. The decentralized deadzone function of level $w \in \mathbb{R}^p_{>0}$, is defined as

$$\mathrm{dz}_w(v) := v - \mathrm{sat}_w(v).$$

Given saturation levels $a, b \in \mathbb{R}_{>0}$, $v, w \in \mathbb{R}$ and $w \in [-b, +b]$, the scalar function $\varphi_{a,b}(v, w)$ is defined as

$$\varphi_{a,b}(v, w) := \begin{cases} 0, & \text{if } |w| \geq b \text{ and } w \cdot v > 0; \\ \mathrm{sat}_a(v), & \text{otherwise;} \end{cases}$$

the (vector, decentralized) $\varphi_{a,b}(v, w)$ function of levels $a, b \in \mathbb{R}^p_{>0}$ is defined by saying that its i-th component is $[\varphi_{a,b}(v, w)]_i = \varphi_{a_i,b_i}(v_i, w_i)$, $i = 1, \ldots, p$ where w_i and v_i are the i-th components of $v, w \in \mathbb{R}^p$ respectively, and $w \in [-b, +b]^p$. These functions allow to describe the dynamics of a rate limited (with limit a), state saturated (with limit b) integrator as $\dot{x} = \varphi_{a,b}(u, x)$.

A signal $q(\cdot)$ belongs to $\mathcal{L}_2$ ($q(\cdot) \in \mathcal{L}_2$) if its $\mathcal{L}_2$ norm is bounded, i.e., $\|q\|_2^2 := \lim_{t\to\infty} \int_0^t |q(\tau)|^2 d\tau < \infty$. Given a matrix $P = P^T > 0$, $\mathcal{E}(P) := \{x : x^T P x \leq 1\}$. Given a square matrix X, $\mathrm{He}X := X + X^T$.

II. Review of previous results

In this section, the problem setting and the main results from [10] are briefly reviewed. Since in the anti-windup problem it is required that the linear closed loop response is never modified as long as control signals do not exceed the saturation bounds, and since the anti-windup construction is based on incorporating in the modified controller a model of the saturation affecting the plant input[1], it can be shown that a necessary condition for the well-posedness of the problem is that the saturation nonlinearity behaves like an identity as long as its input does not exceed the saturation bounds; for this reason, in the following a MRS model will be called *faithful* if it behaves as an identity for signals whose amplitude and rate do not exceed the magnitude and rate saturation bounds.[2]

A. Magnitude and rate saturation (MRS) representation

A *faithful* MRS model is given by the following dynamical system with discontinuous right hand side:

$$\dot{\xi} = \mathrm{diag}(r)\mathrm{sign}(\mathrm{sat}_m(u_{mrs}) - \xi), \quad y_{mrs} = \xi, \qquad (1)$$

where u_{mrs}, ξ, y_{mrs}, are, respectively, the input, the state and the output, of the MRS, and $m := [m_1, \ldots, m_p]$, $r := [r_1, \ldots, r_p]$, are vectors whose strictly positive components specify the magnitude and rate limits, respectively. Model (1) requires special care due to its discontinuous right hand side; hence, is often approximated by a (non faithful) high gain model where the sign$(\cdot)$ function is replaced by a high gain followed by a saturation (see, e.g., [4], [16], [32], [30]).

[1]In this way, the output of the modified controller never actually cause the real input to the plant to saturate, thus ensuring that the "real" saturation at the input of the plant never activates and then preventing possible actuator damages (due to the actuator reaching its limits).

[2]In this sense, many models of MRS currently used both in the industry and in the academia are not faithful, since they include some additional actuator dynamics, which (strictly speaking) should be considered in the linear closed loop if the anti-windup problem is defined as specified above.

In order to avoid the difficulties related to the discontinuities of (1), the following faithful model (represented in Figure 1) was used in [10]:

$$\dot{\delta} = \mathrm{sat}_r(\dot{u}_{mrs} + K(u_{mrs} - \delta) + v_2), \quad y_{mrs} = \mathrm{sat}_m(\delta), \; (2)$$

where the diagonal matrix $K > 0$ is a free parameter and u_{mrs} is a signal whose derivative, $\dot{u}_{mrs}$, is supposed to exist, almost everywhere, and v_2 is an external signal (used in the anti-windup design). As can be easily seen in Figure 1, the parameter K is introduced in order to avoid an unstable cancellation between the ideal derivative operator s and the integrator even in linear operation.

The models (1) and (2) are not equivalent: in fact, the state of (2) can exceed the magnitude saturation limits m, whereas the state of (1) never does. Nevertheless, (2) can be exploited to solve an anti-windup problem for systems subject to MRS, since (as shown in [10]) it satisfies the following properties:

- y_{mrs} always satisfies the magnitude and rate limits;
- if (2) is properly initialized and v_2 is identically 0, then y_{mrs} coincides with u_{mrs} as long as $(u_{mrs}, \dot{u}_{mrs})$ never exceed the magnitude and rate limits;
- if $v_2 \in \mathcal{L}_2$, and $\exists \varepsilon > 0$: $\mathrm{dz}_m((1 + \varepsilon)u_{mrs}) \in \mathcal{L}_2$ and $\mathrm{dz}_r((1 + \varepsilon)\dot{u}_{mrs}) \in \mathcal{L}_2$, then also $(y_{mrs} - u_{mrs}) \in \mathcal{L}_2$.

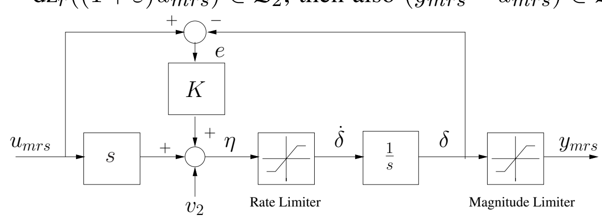

Fig. 1. The proposed MRS block structure.

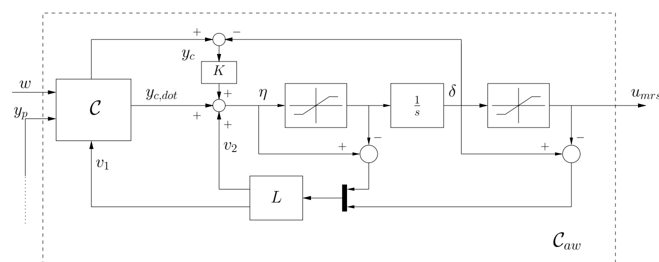

Fig. 2. The structure of the modified controller for anti-windup.

B. Problem statement

Consider a linear plant given by

$$\mathcal{P} \begin{cases} \dot{x}_p = A_p x_p + B_{p,u} u_p + B_{p,w} w \\ y_p = C_{p,y} x_p + D_{p,yu} u_p + D_{p,yw} w \\ z_p = C_{p,z} x_p + D_{p,zu} u_p + D_{p,zw} w \end{cases} \qquad (3)$$

with plant state $x_p \in \mathbb{R}^{n_p}$, control input $u_p \in \mathbb{R}^{n_u}$, exogenous input $w \in \mathbb{R}^{n_w}$ (possibly containing disturbances, references and measurement noise), measurement output $y \in \mathbb{R}^{n_y}$ and performance output $z \in \mathbb{R}^{n_z}$.

Assume that an unconstrained strictly proper controller has been designed to induce desirable performance when interconnected to the plant without saturation:

$$\mathcal{C} \begin{cases} \dot{x}_c &= A_c x_c + B_{c,y} y_p + B_{c,w} w + v_1 \\ y_c &= C_c x_c \end{cases} \qquad (4)$$

where $x_c \in \mathbb{R}^{n_c}$ is the controller state and $y_c \in \mathbb{R}^{n_u}$ is the controller output and the external signal v_1 will be used for the anti-windup augmentation. Since the controller is strictly proper, it is possible to calculate the derivative of the output $\dot{y}_c = y_{c,dot}$ in a closed form, as

$$y_{c,dot} = C_c\big(A_c x_c + B_{c,y} y_p + B_{c,w} w\big) \tag{5}$$

In the case without MRS, we call *unconstrained closed-loop system* the direct feedback interconnection between the controller (4) and the plant (3) via the equations

$$u_p = y_c, \qquad v_1 = 0 \tag{6}$$

We will assume that the unconstrained closed-loop system, (3), (4), (6), satisfies the following assumption:

Assumption 1: The unconstrained closed loop system is well posed and internally stable.

The so-called *saturated closed-loop system* corresponds to the interconnection between (3), (4) through the MRS nonlinearity. In the proposed anti-windup architecture the MRS model (2) will be placed in front of the constrained plant, so that (as described right before Section II-A) the plant will always behave as its linear model (3).

Based on the model (2) and on the controller structure in (4) we have two signals v_1 and v_2 available to choose for anti-windup purposes. Therefore, the scheme that we propose incorporates the selection of a static anti-windup gain L which determines the selection of these signals, based on the excess of saturation on both the saturators present in the MRS model (2) (also shown in Figure 2) as follows:

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = L \begin{bmatrix} \delta - \mathrm{sat}_m(\delta) \\ \eta - \mathrm{sat}_r(\eta) \end{bmatrix}. \tag{7}$$

where, according to Fig. 2, $\eta = \dot{y}_c + K(y_c - \delta) + v_2$. The arising modified controller is represented in Figure 2 and corresponds to equations (4), (5), (2), (7). The closed-loop of this modified controller with the saturated plant (3), (1) will be called *anti-windup closed-loop system* henceforth.

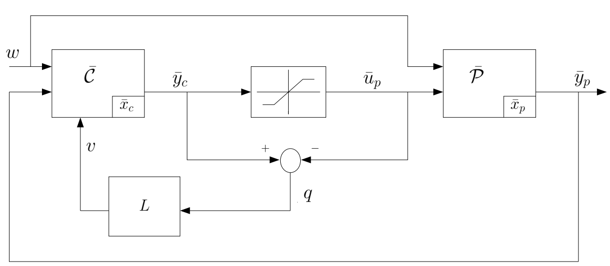

Fig. 3. Equivalent representation for the closed-loop with MRS.

The aim of this paper is to solve the following problem.

Problem 1: Given the plant (3), the controller (4) and MRS limits m and r, augment the control system so that

1) if $\mathrm{sat}_m(y_c(\cdot)) \equiv y_c(\cdot)$, $\mathrm{sat}_r(\dot{y}_c(\cdot)) \equiv \dot{y}_c(\cdot)$ for the unconstrained closed-loop system, then the performance output z_p of the modified control system (for suitable initial conditions) coincides with the performance output of the unconstrained closed-loop (*i.e.*, if the saturation limits are not exceeded, the unconstrained closed-loop response is preserved);
2) for $w = 0$, the origin of the modified closed-loop is regionally exponentially stable (on a region Ω) with the minimum decay rate $\bar{\lambda} < 0$.

C. LMI-based design

The solutions to Problem 1 reported next rely on a suitable transformation of the overall closed loop system that leads to the control structure in Figure 3. In turn, following the approach in [15], the anti-windup closed-loop system of Figure 3 can be represented as (details can be found in [10]):

$$\mathcal{H}\begin{cases} \dot{x} &= Ax + B_q q + B_w w + B_L L q \\ \bar{y}_c &= C_y x + D_{yq} q + D_{yw} w + D_{yL} L q \\ z_p &= C_z x + D_{zq} q + D_{zw} w + D_{zL} L q \end{cases} \tag{12}$$

$$q \;=\; \mathrm{dz}_b(\bar{y}_c) \tag{13}$$

where $b := [\bar{b}_1, \ldots, \bar{b}_{2n_u}] = [m_1, \ldots, m_{n_u}, r_1, \ldots, r_{n_u}]$, $x = [x_p' \;\delta'\; x_c']'$ is the overall state, and, by Assumption 1, the matrices appearing in (12) are uniquely defined based on the plant $\mathcal{P}$, the controller $\mathcal{C}$, the anti-windup matrix L, and the MRS model, as reported in equations (10)-(11) (top of the next page) (which, in turn, depend on the matrices in (8)-(9)), where $\bar{\Delta}_u := (I - \bar{D}_{c,y}\bar{D}_{p,yu})^{-1}$ and $\bar{\Delta}_y := (I - \bar{D}_{p,yu}\bar{D}_{c,y})^{-1}$.

The deadzone function can be dealt with by the following *generalized sector property* [14], [8]:

$$\mathrm{dz}_b^T(\bar{y}_c) M \big(\mathrm{dz}_b(\bar{y}_c) - \bar{y}_c + Hx\big) \le 0, \quad \forall\, x \in \mathcal{L}(H) \tag{14}$$

where $\mathcal{L}(H) := \{x : \mathrm{sat}_b(Hx) = Hx\}$, the matrix H is a free parameter and $M > 0$ is a diagonal scaling matrix.

We report here one of the two main results of [10] from which we will further develop in this paper.

Theorem 1: Given any diagonal $K > 0$, and any solution to the following generalized eigenvalue problem [3]

$$\min_{Q,U,Y,X,\lambda} \lambda \text{ subject to} \tag{15}$$

$Q = Q^T, Q > 0$, $U > 0$ diagonal,

$$\mathrm{He}\begin{bmatrix} AQ & B_q U + B_L X \\ C_y Q - Y & D_{yq} U - U + D_{yL} X \end{bmatrix} < \begin{bmatrix} \lambda Q & 0 \\ 0 & 0 \end{bmatrix} \tag{16}$$

$$\begin{bmatrix} \bar{b}_i^2 & Y_i \\ Y_i^T & Q \end{bmatrix} \ge 0 \quad i = 1, \ldots, 2n_u. \tag{17}$$

then the modified control system (2), (4), (5), (7) with $L = XU^{-1}$ solves Problem 1 for $\Omega := \mathcal{E}(Q^{-1})$ and $\bar{\lambda} = \frac{\lambda}{2}$.

Remark 1: Note that by the results of [15], Theorem 1 guarantees regional closed-loop stability and provides an estimate of the exponential stability domain $\mathcal{E}(Q^{-1})$.

Remark 2: (Well-posedness) Although well-posedness of the nonlinear closed-loop arising from the construction of Theorems 1 is guaranteed, sometimes the solutions arising from the LMI-based numerical optimization solvers may lead to degenerate cases wherein the algebraic loop induced by the anti-windup block is very close to being ill-posed. This may correspond to weak closed-loop robustness and

[3]The constraints in (15) don't actually describe a gevp, but can be easily transformed into one by using an extra variable $Z = Z^T < \lambda Q$.

$$\left[\begin{array}{c|c|c} \bar{A}_p & \bar{B}_{p,u} & \bar{B}_{p,w} \\ \hline \bar{C}_{p,y} & \bar{D}_{p,yu} & \bar{D}_{p,yw} \\ \hline \bar{C}_{p,z} & \bar{D}_{p,zu} & \bar{D}_{p,zw} \end{array}\right] = \left[\begin{array}{cc|cc|c} A_p & 0 & B_p & 0 & B_{p,w} \\ 0 & 0 & 0 & I & 0 \\ \hline C_{p,y} & 0 & D_{p,yu} & 0 & D_{p,yw} \\ 0 & I & 0 & 0 & 0 \\ \hline C_{p,z} & 0 & D_{p,zu} & 0 & D_{p,zw} \end{array}\right] \tag{8}$$

$$\left[\begin{array}{c|c|c|c} \bar{A}_c & \bar{B}_{c,y} & \bar{B}_{c,w} & \\ \hline \bar{C}_c & \bar{D}_{c,y} & \bar{D}_{c,w} & \bar{D}_{c,v_2} \end{array}\right] = \left[\begin{array}{c|cc|c|c} A_c & B_{c,y} & 0 & B_{c,w} & \\ \hline 0 & 0 & I & 0 & 0 \\ KC_c + C_cA_c & C_cB_{c,y} & -K & C_cB_{c,w} & I \end{array}\right] \tag{9}$$

$$\left[\begin{array}{c} A \\ \hline C_y \\ \hline C_z \end{array}\right] = \left[\begin{array}{cc} \bar{A}_p + \bar{B}_{p,u}\bar{\Delta}_u\bar{D}_{c,y}\bar{C}_{p,y} & \bar{B}_{p,u}\bar{\Delta}_u\bar{C}_c \\ \bar{B}_{c,y}\bar{\Delta}_y\bar{C}_{p,y} & \bar{A}_c + \bar{B}_{c,y}\bar{D}_{p,yu}\bar{\Delta}_u\bar{C}_c \\ \hline \bar{\Delta}_u\bar{D}_{c,y}\bar{C}_{p,y} & \bar{\Delta}_u\bar{C}_c \\ \hline \bar{C}_{p,z} + \bar{D}_{p,zu}\bar{\Delta}_u\bar{D}_{c,y}\bar{C}_{p,y} & \bar{D}_{p,zu}\bar{\Delta}_u\bar{C}_c \end{array}\right], \tag{10}$$

$$\left[\begin{array}{c|c|c} B_q & B_w & B_L \\ \hline D_{yq} & D_{yw} & D_{yL} \\ \hline D_{zq} & D_{zw} & D_{zL} \end{array}\right] = \left[\begin{array}{c|c|cc} -\bar{B}_{p,u}\bar{\Delta}_u & \bar{B}_{p,w} + \bar{B}_{p,u}\bar{\Delta}_u(\bar{D}_{c,w} + \bar{D}_{c,y}\bar{D}_{p,yw}) & 0 & \bar{B}_{p,u}\bar{\Delta}_u\bar{D}_{c,v_2} \\ -\bar{B}_{c,y}\bar{\Delta}_y\bar{D}_{p,yu} & \bar{B}_{c,w} + \bar{B}_{c,y}\bar{\Delta}_y(\bar{D}_{p,yw} + \bar{D}_{p,yu}\bar{D}_{c,w}) & I & \bar{B}_{c,y}\bar{\Delta}_y\bar{D}_{p,yu}\bar{D}_{c,v_2} \\ \hline -\bar{\Delta}_u\bar{D}_{c,y}\bar{D}_{p,yu} & \bar{\Delta}_u(\bar{D}_{c,w} + \bar{D}_{c,y}\bar{D}_{p,yw}) & 0 & \bar{\Delta}_u\bar{D}_{c,v_2} \\ \hline -\bar{D}_{p,zu}\bar{\Delta}_u & \bar{D}_{p,zw} + \bar{D}_{p,zu}\bar{\Delta}_u(\bar{D}_{c,w} + \bar{D}_{c,y}\bar{D}_{p,yw}) & 0 & \bar{D}_{p,zu}\bar{\Delta}_u\bar{D}_{c,v_2} \end{array}\right] \tag{11}$$

heavy computational load (even just for closed-loop simulation). Augmenting the optimization problem with the extra condition proposed in [12, equation (5)] typically solves possible numerical problems. That condition, for our closed-loop system (12), is expressed by the following LMI:

$$\text{He}\begin{bmatrix} (\rho-1)U - \rho(UD_{yq} + XD_{yL}) & \rho D_{yL}X \\ \rho U D_{yq} & \frac{\mu-\rho}{4}U \end{bmatrix} < 0$$

where the positive scalars μ and ρ are suitably selected as a trade-off between making the Lipschitz constant of the right hand side of the *anti-windup closed loop system* small and preserving the feasibility of the LMI constraints.

III. Saturating the state of the integrator

The main contribution of the present paper consists in the study of the properties of an anti-windup control structure which can be related to the one in [10] when a suitably modified MRS model is used, and its asymptotic behavior when a certain parameter converges to zero is considered.

A. A modified MRS model

As pointed out in Section II-A, the state of (2) can grow past the magnitude saturation limits m, whereas the state of (1) never does, as well as the state of the following discontinuous faithful model:

$$\dot{\delta} = \varphi_{r,m}(\eta, \delta), \quad y_{mrs} = \delta, \tag{18}$$

where $\varphi_{a,b}(\cdot,\cdot)$ is defined in the notation section and $\eta := \dot{u}_{mrs} + K(u_{mrs} - \delta) + v_2$. Moreover, by generalizing the proof of Lemma 1 of [10] it is immediate to prove that (18) satisfies the same properties informally stated at the end of Section II-A for the model (2), which are a key ingredient in the proof of anti-windup results.

Lemma 1: Given any signal $u_{mrs}(\cdot)$ such that $\dot{u}_{mrs}$ is well defined for almost all t, for any diagonal $K > 0$, the MRS model (18) satisfies the following:

1) for any measurable $v_2(\cdot)$, $\text{sat}_m(y_{mrs}(t)) = y_{mrs}(t)$ and $\text{sat}_r(\dot{y}_{mrs}(t)) = \dot{y}_{mrs}(t)$, for almost all $t \geq 0$;
2) if $\delta(0) = u_{mrs}(0)$, $\text{sat}_m(u_{mrs}(t)) = u_{mrs}(t)$, $\text{sat}_r(\dot{u}_{mrs}(t)) = \dot{u}_{mrs}(t)$ and $v_2(t) = 0$, $\forall\ t \geq 0$, then $y_{mrs}(t) = u_{mrs}(t)$, $\forall t \geq 0$;
3) for any $\delta(0)$, if $\|v_2\|_2 < \infty$ and $\exists \epsilon > 0$ such that $\|u_{mrs} - \text{sat}_{m(1-\epsilon)}(u_{mrs})\|_2 < \infty$ and $\|\dot{u}_{mrs} - \text{sat}_{r(1-\epsilon)}(\dot{u}_{mrs})\|_2 < \infty$, then $\|y_{mrs} - u_{mrs}\|_2 < \infty$.

Simulations have shown that replacing (2) with (18) in the anti-windup compensation schemes described in the previous sections can bear significant advantages in terms of anti-windup performance. Clearly, (18) can be easily implemented in an anti-windup compensation scheme; however, it has the drawback of being discontinuous, and then the analysis of the overall discontinuous dynamics thus arising is far from trivial. Hence, in order to prove interesting properties of the anti-windup schemes involving (18), we will now exploit the characterization of the behaviour of (18) as the limit behaviour of the subsequent continuous model (19); in turn, the fact that (19) can be seen as a special case of (2) allow us to use the tools originally developed for the model (2) to state some properties of the anti-windup closed loop containing the model (18).

The model (18) can be approximated as ε goes to zero by the following modification of the model (2) in Figure 1:

$$\dot{\delta} = \text{sat}_r(\eta - \frac{1}{\varepsilon}\text{dz}_m(\delta)), \quad y_{mrs} = \text{sat}_m(\delta), \tag{19}$$

where $\eta := \dot{u}_{mrs} + K(u_{mrs} - \delta) + v_2$, the symbols have the same meaning as in (2) and $\varepsilon > 0$ is an additional parameter.

The following Lemma 2 shows that, for any given bound on the inputs and any desired accuracy γ, it is possible to choose ε small enough to guarantee that the state response of (19) never exceeds the magnitude bound m by more than γ. Then Lemma 3 shows that when ε tends to zero then the response of (19) is arbitrarily close to the response of (18). In the two lemmas, δ_0 is any initial condition not exceeding the magnitude saturation limits, v_2 and u_{mrs} are external signals bounded almost everywhere, with $\dot{u}_{mrs}$ bounded almost everywhere; $\hat{\delta}(t) := \delta(t; \delta_0, u_{mrs})$ denotes the solution of (19) and $\bar{\delta}(t) := \delta(t; \delta_0, u_{mrs})$ denotes the solution of (18).

Lemma 2: $\forall M > 0$, $\forall \gamma > 0$, $\exists \varepsilon^* > 0$: if $\varepsilon \in (0, \varepsilon^*)$, $\|u_{mrs}\|_\infty < M$, $\|\dot{u}_{mrs}\|_\infty < M$, $\|v_2\|_\infty < M$, $\mathrm{sat}_m(\delta_0) = \delta_0$ then $\left\|\hat{\delta} - \mathrm{sat}_m(\hat{\delta})\right\|_\infty < \gamma$.

Lemma 3: $\forall M > 0$, $\forall \gamma > 0$, $\exists \varepsilon^* > 0$: if $\varepsilon \in (0, \varepsilon^*)$, $\|u_{mrs}\|_\infty < M$, $\|\dot{u}_{mrs}\|_\infty < M$, $\|v_2\|_\infty < M$, $\mathrm{sat}_m(\delta_0) = \delta_0$ then $\left\|\bar{\delta} - \hat{\delta}\right\|_\infty < \gamma$.

B. LMI-based design

To start with, notice that since $y_{mrs} = \delta$ in (18) always satisfies the magnitude saturation limits, the signal $\delta - \mathrm{sat}_m(\delta) = \delta - y_{mrs}$ used for anti-windup compensation in (7) is always zero, so that (7) can be replaced by

$$v = \bar{L}(\eta - \mathrm{sat}_r(\eta)), \quad \bar{L} = \begin{bmatrix} \bar{L}_1 \\ \bar{L}_2 \end{bmatrix} \in \mathbb{R}^{(n_u+n_c)\times nu}. \qquad (20)$$

As shown by the following theorem, the gain $\bar{L}$ can be designed using only the part of the conditions in Theorem 1 relative to the rate saturation anti-windup gain; moreover, in Remark 3 it is pointed out that this reduction does not come at the price of a performance loss. In order to compactly state the theorem, let $C_y = \begin{bmatrix} C_y^1 \\ C_y^2 \end{bmatrix}$ (where each C_y^i for $i = 1, 2$ has n_u rows), and notice that, due to the definitions (10), $C_y^1 = \begin{bmatrix} 0_{n_u \times n_p} & I_{n_u} & 0_{n_u \times n_c} \end{bmatrix}$.

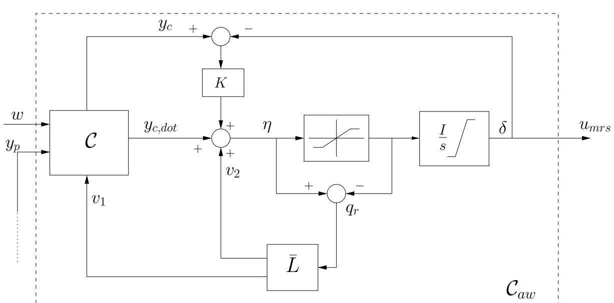

Fig. 4. The structure of the modified controller for anti-windup designed by Theorem 2.

Theorem 2: Given any diagonal $K > 0$, and any solution to the following generalized eigenvalue problem [4]

$$\min_{\bar{U}, \bar{X}_1, \bar{X}_2, \bar{Y}, Q, \lambda} \lambda \quad \text{subject to} \qquad (21)$$

$$Q = Q^T > 0, \ \bar{U} > 0 \text{ diagonal},$$

$$\mathrm{He}\left[\begin{array}{c|c} \begin{matrix}\bar{X}_2 - \bar{U} \\ 0\end{matrix} & C_y^2 Q - \bar{Y} \\ \hline \begin{matrix}\bar{X}_2 - \bar{U} \\ \bar{X}_1\end{matrix} & AQ - \frac{\lambda}{2} Q \end{array}\right] < 0, \qquad (22)$$

$$\begin{bmatrix} m_i^2 & e_i^T Q \\ Q e_i & Q \end{bmatrix} \geq 0, \quad i = n_p + 1, \ldots, n_p + n_u, \qquad (23)$$

$$\begin{bmatrix} r_i^2 & \bar{Y}_i \\ \bar{Y}_i^T & Q \end{bmatrix} \geq 0, \quad i = 1, \ldots, n_u. \qquad (24)$$

[4] The constraints in (21) don't actually describe a gevp, but can be easily transformed into one by using an extra variable $Z = Z^T < \lambda Q$.

where $\bar{X}_1 = \bar{L}_1 \bar{U}$, $\bar{X}_2 = \bar{L}_2 \bar{U}$, $\bar{Y}_i$ is the $i-$th row of $\bar{Y}$ and $e_i = [0, \ldots, 0, 1, 0, \ldots, 0]$, with 1 in position ith, then Problem 1 is solved by the modified control system (4), (5), (18), (20) (see Figure 4) with $\bar{L} = \bar{X}\bar{U}^{-1}$, $\Omega := \{x = [x_p' \ \delta' \ x_c']' : x \in \mathcal{E}(Q^{-1}), \delta = \mathrm{sat}_m(\delta)\}$ and $\bar{\lambda} = \frac{\lambda}{2}$.

Remark 3: Comparing Theorems 1 and 2, since (for the same plant, controller and constraints) the LMIs to be solved in Theorem 2 have a lower dimension than those in Theorem 1 (from [10]), easier implementation, better solution accuracy and the possibility to deal with larger problems has to be expected. Moreover, simulations show that saturating the state of the integrator in the MRS model can lead to better closed loop responses with respect to the results obtained by Theorem 1 (from [10]); on the other hand, in other cases better results can be obtained by the scheme in [10] since the sector conditions used in Theorem 1 are more general than those used in Theorem 2. A precise characterization of the class of problems where one solution outperforms the other is a current research topic. ◦

IV. SIMULATION EXAMPLE

In this section, we apply our anti-windup design technique to the longitudinal dynamics of an $F8$ Aircraft: a fourth order linear model for which an eighth order linear unconstrained controller was introduced in [17] (see also [23]). The plant outputs are the pitch angle and the flight path angle and the two plant inputs are the elevator and the flaperon angles and we assume that they are both subject to MRS. The magnitude limits are selected as ± 25 degrees (as in [17]). As for the rate limits, based on the parameters of other aircraft, we use ± 70 deg/sec. The controller input is the difference between the plant output and the reference input.

The static anti-windup compensator designed as in [10] with $K = \mathrm{diag}\{500, 30000\}$ and $\rho = 7$ and $\eta = 1000$ is

$$L = \left[\begin{array}{cccc} -0.0020271 & 0.0010889 & -0.009791 & -0.0045521 \\ 0.0010176 & -0.0099505 & -0.018225 & -0.069755 \\ -7.2611e-6 & -0.00011695 & 0.009728 & 0.00082988 \\ 3.0596e-6 & -9.8141e-6 & 0.00032413 & 0.00026903 \\ 6.7824e-5 & -0.00014008 & 0.001295 & -0.0020681 \\ -3.5034e-5 & 0.00011248 & -0.00016959 & 0.0014173 \\ -0.002015 & 0.0010642 & -0.0095936 & -0.0049479 \\ 0.0012286 & -0.010403 & -0.014877 & -0.076931 \\ \hline -0.0014969 & 0.0031551 & -0.27799 & 0.034974 \\ -0.00048234 & 0.0099287 & 0.16257 & -27.099 \end{array}\right]$$

while the anti-windup filter using the technique presented in this paper, with the value for K, ρ, η, as chosen before, is

$$L = \left[\begin{array}{cc} 5.1745e-5 & 0.00039197 \\ -7.5156e-8 & 0.00012272 \\ 1.3014e-5 & 0.00044524 \\ 3.855e-7 & 8.1802e-5 \\ -2.3892e-7 & 3.2313e-5 \\ -1.0378e-6 & 2.3811e-5 \\ 5.1786e-5 & 0.00038465 \\ 8.4195e-7 & -1.2283e-5 \\ \hline -0.00031065 & 0.0021484 \\ 0.0065148 & -5.2494 \end{array}\right]$$

The closed loop system responses are shown in Figure 5 (outputs) and 6 (control inputs). Clearly, the anti-windup responses obtained by using the design proposed in this paper (*i.e.* saturating the state of the integrator) are highly desirable and improve over the ones obtained by the scheme in [10].

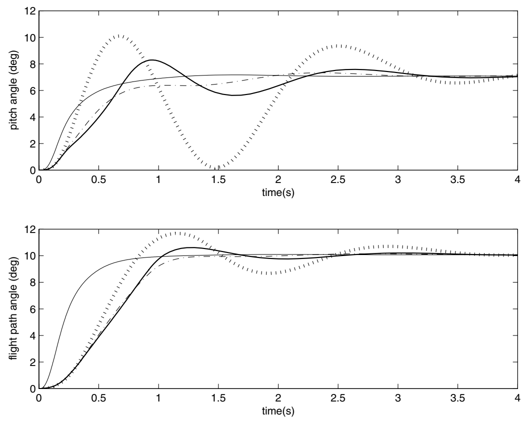

Fig. 5. Pitch angle trajectory and Flight path angle trajectory : unconstrained response (thin solid), saturated response (dotted) and anti-windup response (bold), anti-windup with saturated integrator (dash-dotted).

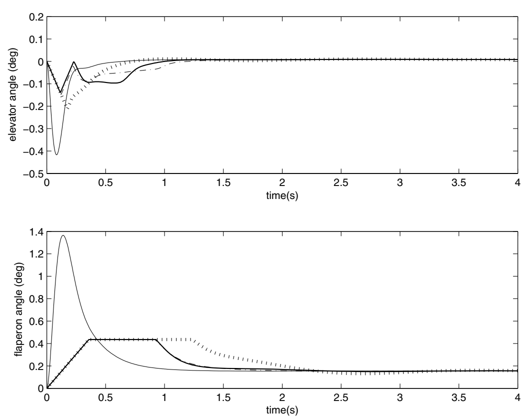

Fig. 6. Control inputs: unconstrained response (thin solid), saturated response (dotted) and anti-windup response (bold), anti-windup with saturated integrator (dash-dotted).

V. Conclusions

This paper develops on the static anti-windup design approach for magnitude and rate saturated systems in [10]. The gains of the new scheme are half of the size of the ones in [10], and are proven to induce regional closed loop stability. The potential of the new scheme to induce very nice closed-loop responses is shown on a simulation example.

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