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ANALYTICAL SOLUTION TO THE MINIMUM FUEL CONSUMPTION OPTIMIZATION PROBLEM WITH THE EXISTENCE OF A TRAFFIC LIGHT

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ABSTRACT
Today's driving patterns consume significant amount of useless energy, especially, when the fuel consumptions while braking, idling and re-accelerating at each traffic light are considered for millions of vehicles. This makes a high level management of driving profile crucial. In this paper, an analytical solution to the fuel consumption minimization problem with the existence of a single traffic light is investigated. The analytical solution is important for on-line implementation and sharing the information of the estimated fuel consumption of the road ahead with other vehicles. Pontryagin's minimum principle is used to calculate the optimal velocity profile. Prior to the calculations, it is assumed that we have the knowledge of starting and ending points of the trip, the position and the operation sequence of the traffic light. In order to make the problem solvable, a simplified vehicle model is used. Furthermore, Willans approximation is utilized for fuel consumption calculations with addition of certain amount of idle speed fuel cost. The vehicle is forced to operate between a feasible torque and speed range. The optimization problem is simulated for an SUV vehicle first on a level road, then on a level road with the traffic light and finally on a road with grade. The results have shown that in addition to operating the vehicle close to its optimal point, it is possible to avoid the consumption of useless fuel due to the braking, idling and re-acceleration phases of a traffic light.

1 Introduction
The traffic networks become more complex as the number of vehicles entering to the network increases. For such networks more traffic lights are required to prevent congestion and direct the vehicles to desired destination points in an optimal rate. Unfortunately, the increased number of traffic lights causes more fuel consumption rates with in the network. To illustrate, first the vehicle needs to dissipate all of its kinetic energy by braking until stopping, then it will need more energy to run the engine at idle speed and finally the vehicle will need to accelerate to move on its way. Nevertheless, especially when the traffic density is not very high it might be possible to avoid waiting at the traffic lights. In this paper, an analytical solution is developed in order to minimize the fuel consumption of a vehicle given a traffic light signal information, its location and the total distance to be travelled. In addition the optimal velocity profiles will be developed for roads with known grade information.

The optimization problem is divided into several parts. In the first part the analytical solution is developed for the case without traffic light. Then, using the information obtained, an algorithm is developed for the optimization problem with the existence of a single traffic light. Finally the optimization problem is solved for the case with known road grade profile. All the important points are summarized and the future works are discussed in the conclusion part.

2 Analytical Solution for Fuel Consumption Optimization
In order to find an analytical solution to fuel consumption minimization problem, the vehicle and fuel consumption models are required to be defined. Initially, the vehicle is assumed to move at a constant gear and with no road grade. Then the general vehicle model can be written as

$$C_1 \cdot \frac{dV}{dt} = C_2 \cdot T_e - C_3 \cdot V^2 - C_4 \quad (1)$$

where

$$C_1 = m_v + \left(\frac{J_{wh}}{R_{wh}^2} + \frac{J_e}{R_{wh}^2}\cdot\lambda^2\right) \quad \text{Inertial terms}$$

$$C_2 = \frac{\eta_{tr}}{R_{wh}}\cdot\lambda \quad \text{Engine torque to thrust force}$$

$$C_3 = \frac{1}{2}\cdot\rho_a\cdot A_f\cdot C_d \quad \text{Aerodynamic resistance}$$

$$C_4 = m_v\cdot g\cdot C_r \quad \text{Rolling resistance}$$

In the above equations m_v is the mass of the vehicle, J_{wh} and J_e are the rotational inertia of the wheels and engine parts, R_{wh} is the radius of the tires. λ represents the combination of the gear and the differential ratios. η_{tr} is the transmission efficiency. ρ_a is the air density, A_f is the frontal area and C_d is the drag coefficient of the vehicle. g is the gravitation force and C_r is the rolling coefficient between the tires and the road surface.

Having determined the equation of motion of the vehicle, the fuel consumption is required to be modelled for optimization calculations. The Willians Approximation is used for fuel consumption modelling.

$$\dot{m}_{fuel} = C_5\cdot V(t)\cdot T_e - C_6\cdot V(t) + C_7 \tag{2}$$

where:

$$C_5 = \frac{\lambda}{e(\omega_e)\cdot H_{LHV}\cdot R_{wh}}$$

$$C_6 = \frac{P_{loss}(\omega_e)\cdot V_d\cdot\lambda}{e(\omega_e)\cdot 4\pi\cdot H_{LHV}\cdot R_{wh}}$$

It is assumed that the global efficiency, $e(\omega_e)$, and the loss term, $p_{loss}(\omega_e)$ of the engine are constant, so by choosing an average engine speed the constant values can be determined by interpreting the fuel consumption engine map. Furthermore, to take into consideration the idle speed fuel consumption, a constant term is also added to the model which is denoted as C_7.

2.1 Model Simplification

A number of differential equations are needed to be solved for the analytical solution of the optimization problem. In order to make the problem solvable analytically an additional simplification is done. The equation 1 is simplified by linearizing the model around an unknown velocity. This modification significantly simplifies the solutions of the differential equations. The simplified model which will be used for analytical solution calculations takes the following form

$$C1\cdot\frac{dV}{dt} = C_2\cdot T_e - C_3\cdot V_{lin}\cdot V - C_4 \tag{3}$$

In equation 3, V_{lin} is the speed around which the model is linearized. At this point a new parameter, C_3^* is defined in order to have simpler expressions for the rest of the calculations.

$$C_3^* = C_3\cdot V_{lin} \tag{4}$$

After having determined the simplified model, the total dynamic state equations can be written as

$$\dot{x} = f(x,u) \tag{5}$$

where:

$$\dot{x} = \begin{bmatrix}\dot{X}\\ \dot{V}\end{bmatrix} \quad , \quad f(x,u) = \begin{bmatrix} V \\ \frac{C_2}{C_1}\cdot T_e - \frac{C_3^*}{C_1}\cdot V - \frac{C_4}{C_1}\end{bmatrix} \tag{6}$$

2.2 Analytical Solution: No Constrained Case

For no constraint case the aim is to find an optimal solution to minimize the total fuel consumption during the trip. The total cost function is given in equation 7.

$$J^* = \min_{u\in U}\int_0^{t_f}\dot{m}_{fuel}\cdot dt \tag{7}$$

The Hamiltonian Function is defined as

$$H = \dot{m}_{fuel} + \lambda^T\cdot f(x,u) \tag{8}$$

From the Hamiltonian Function the optimal control is calculated:

$$\frac{\partial H}{\partial T_e} = C_5\cdot V(t) + \lambda_v(t)\cdot\frac{C_2}{C_1} = 0 \tag{9}$$

Then the optimal control is:

$$u^*(x,t) = \begin{cases} Te_{max} & \text{if} \quad C_5\cdot V(t) + \lambda_v(t)\cdot\frac{C_2}{C_1} < 0 \\ Unknown & \text{if} \quad C_5\cdot V(t) + \lambda_v(t)\cdot\frac{C_2}{C_1} = 0 \\ 0 & \text{if} \quad C_5\cdot V(t) + \lambda_v(t)\cdot\frac{C_2}{C_1} > 0 \end{cases} \tag{10}$$

As given in above equation, the optimal control takes the two extreme values if one of the inequality constraints are satisfied. Nevertheless, for the equality of the condition(singular point case) the control input, T_e, can be any value between 0 and the maximum torque.

For $t_0 \longrightarrow t_1$ Since we start at zero position and velocity, initially it will be assumed that:

$$u^*(x,t) = Te_{max} \quad \longrightarrow \quad C_5 \cdot V(t) + \lambda_v(t) \cdot \frac{C_2}{C_1} < 0$$

Then after certain time there is a point such that at $t = t_1$:

$$C_5 \cdot V(t) + \lambda_v(t) \cdot \frac{C_2}{C_1} = 0$$

Only after reaching the above condition (singular point) the control input will be changed. Having the information of $X_0 = 0$ and $V_0 = 0$, the equations of the states and the Lagrange multipliers between t_0 to t_1 are calculated as

$$\begin{aligned}
V^*(t) &= -\tfrac{1}{C_3^*} \cdot (C_2 \cdot Te_{max} - C_4) \cdot e^{-\frac{C_3^*}{C_1} t} + \tfrac{1}{C_3^*} \cdot (C_2 \cdot Te_{max} - C_4) \\
X^*(t) &= \tfrac{C_1}{(C_3^*)^2} \cdot (C_2 \cdot Te_{max} - C_4) \cdot e^{-\frac{C_3^*}{C_1} t} + \tfrac{1}{C_3^*} \cdot (C_2 \cdot Te_{max} - C_4) \cdot t \\
&\quad - \tfrac{C_1}{(C_3^*)^2} \cdot (C_2 \cdot Te_{max} - C_4) \\
\dot{\lambda}_x(t) &= -\tfrac{\partial H}{\partial x} = 0 \quad \longrightarrow \quad \lambda_x(t) = K_1 \\
\dot{\lambda}_v(t) &= -\tfrac{\partial H}{\partial V} = -C_5 \cdot Te_{max} + C_6 + \lambda_v \cdot \tfrac{C_3^*}{C_1} - \lambda_x \\
\lambda_v(t) &= K_2 \cdot e^{\frac{C_3^*}{C_1} t} + \tfrac{C_1}{C_3^*} (C_5 \cdot Te_{max} - C_6 + K_1)
\end{aligned} \tag{11}$$

In the above equations, K_1 and K_2 are the constants that are to be determined by the boundary conditions at $t = t_f$.

For $t_1 \longrightarrow t_2$ At time t_1 the condition, $C_5 \cdot V(t) + \lambda_v(t) \cdot \frac{C_2}{C_1} = 0$ is hit and the control becomes undefined. However, between t_1 and t_2 the equality condition holds.

$$\lambda_v(t) = -C_5 \cdot \frac{C_1}{C_2} \cdot V(t) \tag{12}$$

In addition at the time interval, the time derivative of the condition should also be 0:

$$C_5 \cdot \dot{V}(t) + \dot{\lambda}_v(t) \cdot \frac{C_2}{C_1} = 0 \tag{13}$$

Then using the equations 11, 12 and 13 it is possible to show that the velocity is constant at the value of:

$$V(t) = \frac{C_2 \cdot C_6}{2 \cdot C_3^* \cdot C_5} - \frac{C_4}{2 \cdot C_3^*} - \frac{K_1 \cdot C_2}{2 \cdot C_3^* \cdot C_5} \tag{14}$$

Since the velocity is constant its derivative should be zero then it can be shown that:

$$T_e(t) = \frac{C_6}{2 \cdot C_5} + \frac{C_4}{2 \cdot C_2} - \frac{K_1}{2 \cdot C_5} \tag{15}$$

In addition, at $t = t_1$, the first equation in 11 and 14 must be equal, so it is possible to calculate t_1 as:

$$t_1 = -\frac{C_1}{C_3^*} \cdot ln\left(1 + \frac{K_1 \cdot C_2 + C_4 \cdot C_5 - C_2 \cdot C_6}{2 \cdot C_5 \cdot (C_2 \cdot Te_{max} - C_4)}\right) \tag{16}$$

From equation 12, the Lagrange multiplier is calculated:

$$\lambda_v(t) = -\frac{C_1 \cdot C_6}{2 \cdot C_3^*} + \frac{C_1 \cdot C_4 \cdot C_5}{2 \cdot C_2 \cdot C_3^*} + \frac{K_1 \cdot C_1}{2 \cdot C_3^*} \tag{17}$$

From equations 11 and 17 it is possible to show that:

$$\begin{aligned}
K_2 = \Bigg[&-\frac{C_1 \cdot C_6}{2 \cdot C_3^*} + \frac{C_1 \cdot C_4 \cdot C_5}{2 \cdot C_2 \cdot C_3^*} + \frac{K_1 \cdot C_1}{2 \cdot C_3^*} \\
&-\frac{C_1}{C_3^*} (C_5 \cdot Te_{max} - C_6 + K_1) \Bigg] \cdot e^{-\frac{C_3^*}{C_1} t_1}
\end{aligned} \tag{18}$$

If K_1 is known all the unknowns given above can be calculated. K_1 will be calculated in the following part.

For $t_2 \longrightarrow t_f$ At the end of the constant velocity region , $t = t_2$ the following condition must hold to satisfy terminal boundary conditions:

$$C_5 \cdot V(t) + \lambda_v(t) \cdot \frac{C_2}{C_1} > 0 \tag{19}$$

Then from optimal control, equation 10, the optimal torque value is determined.

$$T_e = 0 \tag{20}$$

By using equation 20 and the boundary conditions at $t = t_2$, the state and the Lagrange multiplier trajectories are calculated between the time interval t_2 to t_3.

$$V^*(t) = \left[\frac{C_2 \cdot C_6}{2 \cdot C_3^* \cdot C_5} + \frac{C_4}{2 \cdot C_3^*} - \frac{K_1 \cdot C_2}{2 \cdot C_3^* \cdot C_5}\right] \cdot e^{-\frac{C_3^*}{C_1} \cdot (t - t_2} - \frac{C_4}{C_3^*} \tag{21}$$

Similarly;

$$\lambda_v(t) = \left[\frac{C_1 \cdot C_6}{2 \cdot C_3^*} + \frac{C_1 \cdot C_4 \cdot C_5}{2 \cdot C_2 \cdot C_3^*} - \frac{K_1 \cdot C_1}{2 \cdot C_3^*}\right] \cdot e^{\frac{C_3^*}{C_1} \cdot (t - t_2)} - \frac{C_1 \cdot C_6}{C_3^*} + \frac{C_1 \cdot K_1}{C_3^*} \tag{22}$$

There are three boundary conditions at $t = t_f$. Obviously the speed of the vehicle at the end of the trip is zero. Furthermore, since the terminal time is free the Hamiltonian Function should be zero at $t = t_f$. Finally, the position of the vehicle should be the desired travel distance at the final time:

$$\begin{aligned} &V(t_f) = 0 \\ &H(t_f) = 0 = C_7 - \lambda_v(t_f) \cdot \tfrac{C_4}{C_1} \quad \longrightarrow \quad \lambda_v(t_f) = \tfrac{C_7 \cdot C_1}{C_4} \\ &x(t_f) = x_{final} \end{aligned} \tag{23}$$

Using the three boundary conditions 23, it is possible to calculate the three unknowns t_2, t_f and K_1. Since the equations get complex, these calculations are performed in a software environment. Note that the parameter V_{lin} is selected such that it coincides with the velocity at the constant speed region($t_1 \rightarrow t_2$). By this way, although the aerodynamic resistance forces are over estimated during the acceleration and the braking phases, it is the exact value for the constant velocity region. This degrades the bad effect of linearization. In fact although the characteristics of the acceleration and deceleration phases would be different, the value of the constant velocity should be the same of the case without linearization.

After the unknown parameters are calculated, the simulation is performed for the vehicle. In figure 1, the simulation results are presented.

Having determined the optimal solution for the case with zero initial velocity, the optimal velocity profiles of the cases with non-zero initial velocities are also point of interest. This information will be helpful for optimization calculations with the existence of a traffic light. In order to do, a set of initial conditions are determined and the simulations are performed using the equations of the analytical solution. The two optimal velocity profiles, one with an initial velocity lower and one with an initial velocity higher than the constant velocity are determined and the results are presented in the figure 2. The simulation results show that whatever the initial speed is, the optimal velocity profile tries to reach a certain speed by initially applying either maximum or minimum torque, Te_{max} or 0, respectively, and remains at the constant speed until the vehicle is a certain distance away from the destination point. As soon as it reaches that certain distance, the applied torque becomes zero, vehicle starts coasting and stops at the destination point.

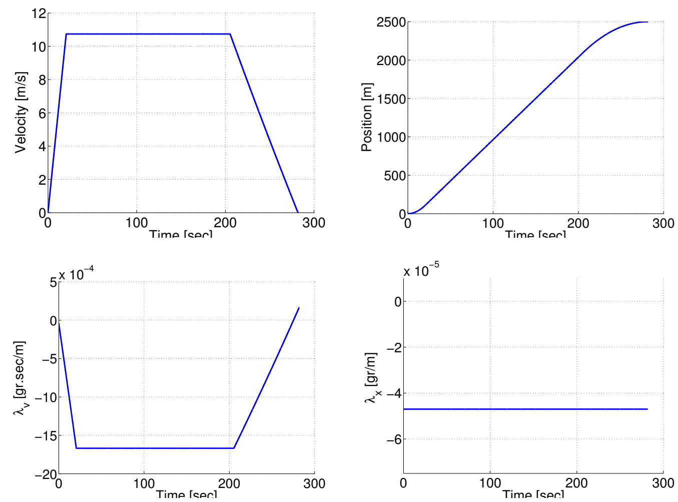

Figure 1. The simulation results of the case without traffic light and grade.

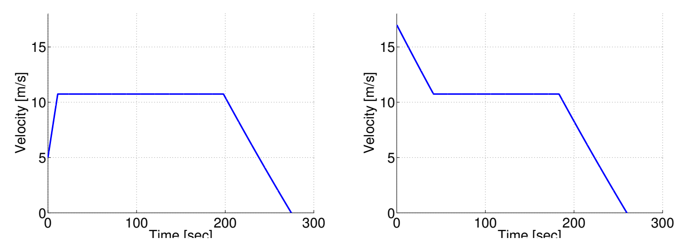

Figure 2. The simulation results of the case without traffic light and grade having different initial velocities.

2.3 Analytical Solution: With Traffic Light

For the case with a traffic light additional considerations are needed for the calculation of the minimum fuel consumption. The problem can be divided into two parts, as shown in figure 3, the first part is the region until the vehicle reaches where traffic light is located. The second part is the region from traffic light to the destination point. The aim is to find the minimum fuel consumption until the destination point. However by benefiting from the results of unconstrained case, the minimum fuel consumption will be optimized for the first part (until traffic light), then the fuel consumption corresponding to the second part will be added as a terminal cost. After further inspection it is possible to say that if there were no traffic light then the vehicle would continue with the constant speed as it is found in unconstrained case (the red line in the figure 3). So it is better idea just to find the deviation of the fuel consumption of actual velocity profile at the second phase from the optimal one, (difference between the blue and red dashed lines), and add this deviation to the optimization problem as the terminal cost. Then the total fuel consumption

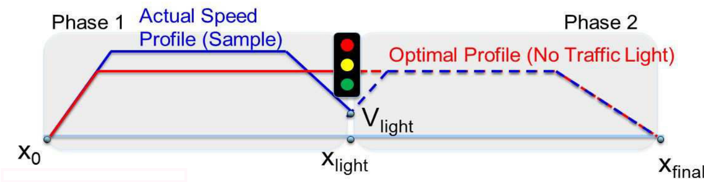

Figure 3. The sketch describing the division of the problem into two phases.

can be shown as:

$$J = h(x(t_l),t_l) + \int_0^{t_l} \dot{m}_{fuel} \cdot dt \tag{24}$$

where t_l is the time when traffic light is reached and $x(t_l)$ is the position of the traffic light. $h(x(t_l),t_l)$ is the deviation of the fuel consumption from the optimal one for the region after traffic light. $h(x(t_l),t_l)$ is defined as:

$$\begin{aligned} h &= \int_{t_l}^{t_{f,1}} \dot{m}_{fuel} \cdot dt - \int_{t_l}^{t_{f,2}} \dot{m}_{fuel,opt.} \cdot dt \\ &= \int_{t_l}^{t_l+t_{1PH2}} (C_5 \cdot Te - C(6)) \cdot V(t) \cdot dt - \int_{t_l}^{t_{f,2}} (C_5 \cdot Te_c - C(6)) \cdot V_c \cdot dt \\ &\quad + C_7 \cdot (t_{1_{PH2}} - t_{f,2}) \\ &= (C_5 \cdot Te - C(6)) \cdot \int_{t_l}^{t_l+t_{1PH2}} V(t) \cdot dt \, (C_5 \cdot Te_c - C(6)) \cdot \int_{t_l}^{t_{f,2}} V_c \cdot dt \\ &\quad + C_7 \cdot (t_{1_{PH2}} - t_{f,2}) \end{aligned} \tag{25}$$

In the equation above, the integration should be performed until the vehicle velocity reaches to the optimal constant velocity calculated in unconstrained case. The time required for the vehicle to reach this speed is denoted as t_{1PH2} and the integration is performed from t_l to $t_l + t_{1PH2}$. On the other hand for the optimal velocity profile, the time to reach the same point will be different since the velocity profiles are different. However the travelled distances should be the same. Because of the fact that the actual and optimal driving profiles operate at constant torque values, we can take all the constant terms out of the integration and we end up with the integration of the speed which gives us the distance travelled. Then, the terminal cost takes the form given in equation 26.

$$h(x(t_l),t_l) = C_5 \cdot (Te - Te_c) \cdot \int_{t_l}^{t_l+t_{1PH2}} V(t) \cdot dt + C_7 \cdot (t_{1_{PH2}} - t_m) \tag{26}$$

For the calculation of the distance travelled the velocity profile after the traffic light is required. Using the vehicle dynamics relation (equation 3) and the boundary condition $V(t_l)$ at $t = t_l$, which is unknown yet, the equation of velocity is calculated and shown in equation 27.

$$V^*(t) = -\frac{1}{C_3^*} \cdot (C_2 \cdot T_e - C_4 - C_3^* \cdot V(t_l)) \cdot e^{-\frac{C_3^*}{C_1} t} + \frac{1}{C_3^*} \cdot (C_2 \cdot Te_{max} - C_4) \tag{27}$$

In the equation T_e can either be 0 or $T_{e,max}$ depending on the speed at the traffic light,$V(t_l)$. Combining the equations 26 and 27, the terminal cost takes the following form:

$$\begin{aligned} h &= C_5 \cdot (T_e - T_{e,c}) \left[\frac{C_1}{(C_3^*)^2} \cdot (C_2 \cdot T_e - C_4 - C_3^* \cdot V(t_l)) \cdot \left(e^{-\frac{C_3^*}{C_1} t_{1PH2}} - 1 \right) \right. \\ &\quad \left. + \frac{1}{C_3^*} \cdot (C_2 \cdot T_e - C_4) \cdot t_{1PH2} \right] \end{aligned} \tag{28}$$

The required time, t_{1PH2}, until the speed reaches the optimal velocity is determined and shown in the equation 29.

$$\begin{aligned} t_{1PH2} &= -\frac{C_1}{C_3^*} \cdot \left[ln \left(\frac{C_6 \cdot C_2}{2 \cdot C_5 \cdot C_3^*} - \frac{C_4}{2 \cdot C_3^*} - \frac{K_1 \cdot C_2}{2 \cdot C_5 \cdot C_3^*} - \frac{1}{C_3^*} (C_2 \cdot Te_{max} - C_4) \right) \right. \\ &\quad \left. - ln \left(-\frac{1}{C_3^*} (C_2 \cdot Te_{max} - C_4) + V(t_l) \right) \right] \end{aligned} \tag{29}$$

Since we know all the constants, V_c, Te_c and since t_{1PH2} is a function of $V(t_l)$, then the terminal cost, h, depends only on $V(t_l)$.

Although we have determined a terminal cost which depends on the speed at the point of traffic light, we still did not ensure how to find the optimal velocity profile to minimize the fuel consumption with the existence of the traffic light. To do so, first we need to model the traffic light.

2.3.1 Modelling of the Traffic Light

The traffic light that will be used for further calculations is modelled with some basic assumptions. First of all it is assumed that the light operates in 50 percent duty cycle. This means that the time for red light on is the same as the time for green light on. In addition the period is assumed to be 60 seconds, so for the first 30 seconds the red light is to be on and for the other 30 seconds the green light is to be on. The operation of the model is represented in the figure 4. For optimal velocity profile it is intuitive that the vehicle should reach to the traffic light when the green light is on. Otherwise, the vehicle needs to stop at the light, wait until green light turns on and then accelerate to reach a reasonable speed. These sequence of actions would consume too much energy compared to the constant speed case. So the best way to avoid these energy losses is to reach the traffic light when it is green. In order to ensure this to happen another terminal cost should be applied. Nevertheless, it is still not clear at which cycle the vehicle should reach to the traffic light. That is why a more comprehensive terminal cost should be selected.

The convex cost function given in the equation 30 is chosen as the terminal cost to ensure the vehicle reaches the traffic light when the green light is on.

$$\phi(t_l) = A \cdot (t_l - 45 - 60 \cdot k)^2 \tag{30}$$

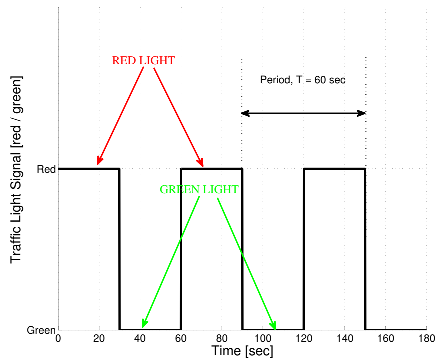

Figure 4. Graphical Representation of the Traffic Light.

The terminal cost function given in equation 30, enforces the vehicle to reach the traffic light when the green light has been on for 15 seconds and is going to be on for another 15 seconds. The parameter, k, determines the number of cycles after which the vehicle should reach to the traffic light. Finally the parameter A, grades the punishment level as the vehicle deviates from the center of the green light on range.

After the addition of the terminal cost for traffic light, the general cost function takes the final form given in equation 31.

$$J = \phi(t_l) + h(x(t_l), t_l) + \int_0^{t_l} \dot{m}_{fuel} \cdot dt \tag{31}$$

In the following sections using the traffic light pattern shown in figure 4 and the given cost function, an optimal driving profile will be investigated to minimize fuel consumption, but first the boundary conditions should be defined.

2.3.2 Boundary Conditions

Five boundary conditions are necessary to find the solution of four differential equations, $\dot{X}$, $\dot{V}$, $\dot{\lambda}_V$, $\dot{\lambda}_X$, and the time, t_l, when the vehicle reaches the traffic light. The boundary conditions are listed below:

$$X(0) = 0$$

$$V(0) = 0$$

$$X(t_l) = X_{light}$$

$$H(t_l) = -\tfrac{\partial \phi(t)}{\partial t}$$

$$\lambda_V(t_l) = \tfrac{dh}{dV}$$

The first three boundary conditions are obvious, the initial conditions for velocity and position are assumed to be zero, in addition the position of the traffic light is assumed to be known.

In the unconstrained case the Hamiltonian function at $t = t_f$ was zero, because there were no fixed terminal time. Although for the case with the traffic light we still do not have a fixed terminal time, we do have a terminal cost as a function of time. This cost function was defined in the equation 30. Then the Hamiltonian Function at $t = t_l$ can be written as 32

$$H(t_l) = -\frac{\partial \phi}{\partial t} = -2 \cdot A \cdot (t_l - 45 - 60 \cdot k) \tag{32}$$

From boundary condition 5, the derivative of the terminal cost,$h(x,t)$ with respect to velocity at $t = t_l$ is required. The equations for terminal cost has been given in 28 and 29. Then the derivative of $h(V(t_l), t_1(V(t_l)))$ with respect to velocity is:

$$\frac{dh}{dV} = \frac{\partial h}{\partial V} + \frac{\partial h}{\partial t_1} \cdot \frac{dt_1}{dV} \tag{33}$$

where;

$$\begin{aligned}
\tfrac{\partial h}{\partial V} &= C_5 \cdot (T_e - T_{e,c}) \cdot \left(-\tfrac{C_1}{C_3^*}\right) \cdot \left(e^{-\frac{C_3^*}{C_1} t_1} - 1\right) \\
\tfrac{\partial h}{\partial t_1} &= C_5 \cdot (T_e - T_{e,c}) \left[\tfrac{-1}{C_3^*} \cdot (C_2 \cdot T_e - C_4 - C_3^* \cdot V(t_l)) \cdot e^{-\frac{C_3^*}{C_1} t_1} \right. \\
&\quad \left. + \tfrac{1}{C_3^*} \cdot (C_2 \cdot T_e - C_4)\right] \\
\tfrac{dt_1}{dV} &= \tfrac{-C(1)}{C(2) \cdot T_e - C(4) - C(3) \cdot V(t_l)}
\end{aligned} \tag{34}$$

2.3.3 Calculation of Optimal Velocity Profile

The analytical solution equations of the minimum fuel consumption optimization problem with traffic light case is exactly the same as the problem with unconstrained case. The only difference between the two is the boundary conditions and the engine torque is not necessarily zero for the time range of t_2 to t_l. This means that the equations from 8 to 18 are also valid for the traffic light case. Due to uncertainty of the torque value between t_2 to t_l, some slight modifications are needed for the equations 21 and 22. The modified equations are given in 35 and 36.

$$\begin{aligned}
V^*(t) = &\left[-\tfrac{C_2 \cdot Te_{tl}}{C_3^*} + \tfrac{C_2 \cdot C_6}{2 \cdot C_3^* \cdot C_5} + \tfrac{C_4}{2 \cdot C_3^*} - \tfrac{K_1 \cdot C_2}{2 \cdot C_3^* \cdot C_5}\right] \cdot e^{-\frac{C_3^*}{C_1} \cdot (t - t_2)} \\
&+ \tfrac{C_2 \cdot Te_{tl} - C_4}{C_3^*}
\end{aligned} \tag{35}$$

Similarly;

$$\lambda_v^*(t) = \left[-\frac{C_1 \cdot C_5 \cdot Te_{tl}}{C_3^*} + \frac{C_1 \cdot C_6}{2 \cdot C_3^*} + \frac{C_1 \cdot C_4 \cdot C_5}{2 \cdot C_2 \cdot C_3^*} - \frac{K_1 \cdot C_1}{2 \cdot C_3^*}\right] \cdot e^{\frac{C_3^*}{C_1} \cdot (t - t_2)} + \frac{C_1}{C_3^*} \cdot (C_5 \cdot Te_{tl} - C_6 + K_1) \tag{36}$$

The analytical solution to the equations 8 to 18 and 35 and 36 with five boundary conditions as defined in previous section, is very hard to find. Instead numerical solution is investigated. For the chosen numerical solution a value for the speed at the traffic light, V_{tl}, is assumed then all the other parameters are calculated. According to the estimation of the position of the traffic light, x_{light}, new update for the V_{tl} is performed. The detailed description of the algorithm is given in the following section.

Numerical Solution Algorithm Before starting to describe the algorithm, the value k used in 30 is first defined. As it was mentioned before the parameter k determines the number of cycle after which the vehicle should reach to the traffic light. Then the numerical calculations follow the steps listed below:

Step 1: First an initial value for V_{t_l} is selected

Step 2: Using the selected V_{t_l}, the torque value after the traffic light is anticipated. For the unconstrained case the constant speed at phase 2 is found to be around 10.74 m/s, so if V_{t_l} is larger than this constant speed value the torque should be zero, otherwise it should be Te_{max}. After determining the torque value using the equation 29, the time needed for actual speed to reach optimum constant speed is calculated. Then using the equations 33 and 34 the derivative of the terminal cost of vehicle speed with respect to velocity, $\frac{dh}{dV}$, is calculated.

Step 3: After calculating $\frac{dh}{dV}$, using the fifth boundary condition, $\lambda_V(t_l) = \frac{dh}{dV}$, and the equations 35 and 36 it is possible to determine the parameter K_1 and the term $e^{\frac{C_3^*}{C_1} \cdot (t_l - t_2)}$ by assuming Te_{t_l} to be 0 (in step 6, the calculations are redone with the assumption $Te_{t_l} = Te_{max}$).

Step 4: Using the calculated parameter K_1 and the term $e^{\frac{C_3^*}{C_1} \cdot (t_l - t_2)}$, and using the equations 8, 14, 16 and 32 it is possible to calculate the parameters t_1, t_l, t_2, and V_c for phase 1 (the profile before traffic light).

Step 5: After determining all the parameters, x_{light} is estimated. If it is the same as the actual distance of the traffic light the calculations are stopped. If not first the selected torque value,Te_{t_l}, at time range t_2 to t_l is checked. If it is 0 then the algorithm goes to step 6, if it is Te_{max} then it goes to step 1.

Step 6: This step is reached when the torque, Te_{t_l}, was assumed to be 0 for the time range t_2 to t_l in last case. Now the torque is assumed to be Te_{max} for the time range t_2 to t_l. Then the parameter K_1 and the term $e^{\frac{C_3^*}{C_1} \cdot (t_l - t_2)}$ are determined and the calculations are re-performed from step 4.

The algorithm continues until the estimated x_{light} exactly equals to the actual distance of the traffic light from the starting position.

This algorithm calculates the optimal velocity profile for the selected k parameter. For example when the parameter k is selected as 2, the optimal velocity profile is calculated by enforcing the vehicle to be at the traffic light at the time 225 seconds. Notice that at t = 225 sec, the green light has been on for 15 seconds and will continue to be on for the next 15 seconds. However, we do not know if the selected k parameter is the optimal one. So all the reasonable k values should be tested in order to find the global optimum velocity profile for minimum fuel consumption with the traffic light case.

2.3.4 Simulation

Having implemented the algorithm, the simulation is performed. For the chosen cycle the total distance is assumed to be 5000 meters. The traffic light is located at the middle of the road. Initially the vehicle is assumed to be at position 0, and it starts from rest. Although for this test simulation a number of k parameter values are tested in order to find the global minimum, it will be discussed later that there is no need to test all the k values. It will be possible to select reasonable k values, so testing only a few of them will be sufficient to find the global optimum.

Table 1. Simulation results for varying k parameter.

k	2	3	4	5	6
$t_{1,PH2}$ [sec]	12.3	0.49	8.3	11.6	13.5
t_1 [sec]	33.4	22.4	17.4	14.2	12.0
t_2 [sec]	133.5	218.9	265.8	325.5	386.2
t_l [sec]	164.9	224.9	284.9	344.9	404.9
V_c [m/s]	17.36	11.70	9.14	7.47	6.32
V_{t_l} [m/s]	12.56	10.81	6.39	4.72	3.70
Te_{t_l} [Nm]	0	0	0	0	0
Te_{PH2} [Nm]	0	0	200	200	200
FC [gr]	134	126	130	131	132
$\#ofIter.$	22	20	17	15	19

In the table 1, the simulation results for different k values, changing from 2 to 6, are presented. As the k parameter increases there is more time for vehicle to reach to the point where traffic light stands. That is why the velocity values tend to decrease. In the unconstrained case the optimal constant speed has been found to be 10.74 m/s. It is also possible to notice that as the average speed of the vehicle differs from the optimal constant speed value the fuel consumption tends to increase. Finally the number of iterations for the algorithm described above are presented. Since only a few analytical calculations are performed in each iteration, the calculation times are not too long.

The variation of the fuel consumption with respect to the parameter k is plotted in the figure 5. From the figure it is clear that the global optimum fuel consumption is obtained at the k value of 3. Here in the simulations in order to show all the steps

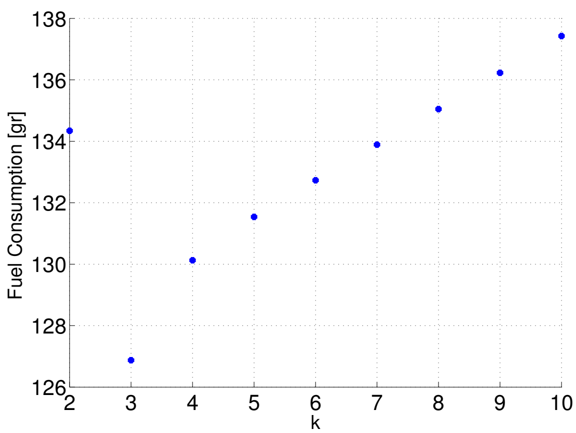

Figure 5. Fuel consumption values with respect to the parameter k.

more clearly the global optimal point is found by calculating fuel consumptions for all k parameters. However, that number of calculations would take significant time. Indeed, it is unnecessary. There is more rigorous way to find the optimal k value. In the unconstrained case we have shown that if there is no traffic light then the vehicle moves at constant speed during most of its trip. This constant speed is found to be 10.74 m/s for selected vehicle and trip parameters. The easiest way of calculating the optimal k parameter is to find the required time (let's denote it by t_{opt}) until the vehicle reaches the traffic light from initial state assuming there is no terminal cost and then we need to find the nearest t_l value obtained by the equation $t_l = 45 + 60 \cdot k$. It is possible to say that the k parameter which leads to minimum error between the t_l and the t_{opt} is in the very close vicinity of the optimal k value. After finding this k parameter we may need to calculate the fuel consumption with +/-1 values of k, to be sure which one is the global optimum. This method can be expressed mathematically as follows:

$$k_{vic} = \left(x_{light}/V_{opt} - 45\right) \cdot \frac{1}{60}; \tag{37}$$

For example for our case $x_{light} = 2500m$ and $V_{opt} = 10.74m/s$, then:

$$\begin{aligned} k_{vic} &= (2500/10.74 - 45) \cdot \frac{1}{60} \\ &= 3.13 \end{aligned} \tag{38}$$

Then by simply trying k=3 and k=4 we could have found the global optimal k value. In fact as it has been shown earlier the global optimal k is found to be k=3.

The simulation results with the optimal parameter, k, are presented in the figure 6.

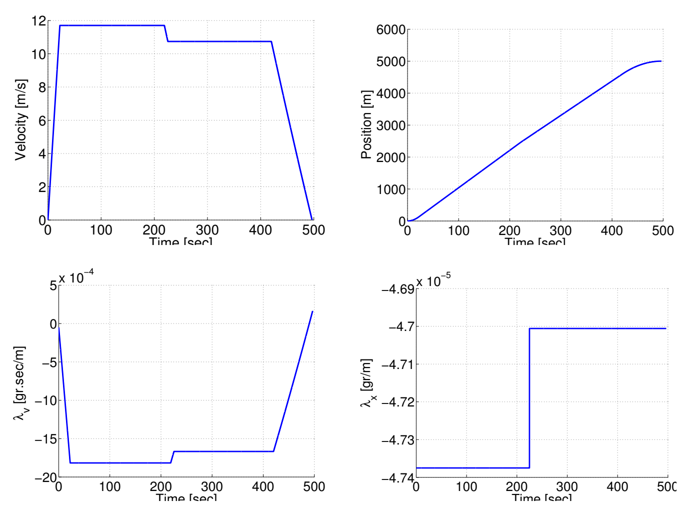

Figure 6. The Simulation Results for Traffic Light Case.

In order to show clearly the state of the vehicle at the traffic light a more detailed plot is presented in the figure 7, where the optimal velocity profile is plotted with respect to the distance travelled. In addition the traffic light signal is presented. As can be seen in the optimal case, the vehicle reaches to the traffic lights when the green light is on, so it avoids unnecessary fuel consumptions due to stopping, idling, re-accelerating phases of the traffic light.

2.4 Analytical Solution: With Grade

In this section, the analytical solution to the optimization problem with known grade information of the road is investigated.

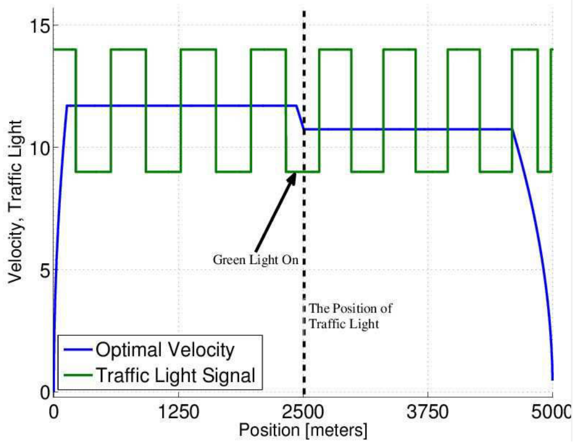

Figure 7. The optimal velocity and the traffic light.

Firstly, a road profile with grade is determined and it is shown in the figure 8. As can be seen in the figure 8, the road profile with a

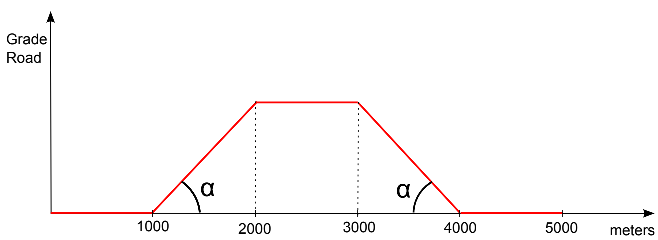

Figure 8. The Road Grade Profile

length of 5 km, is divided into five equidistant regions. The first, third and fifth regions are level roads, while at the second region there is a positive grade and at the fourth region there is a negative grade. Later on the positions of 1000, 2000, 3000, and 4000 meters are referred as x_1, x_2, x_3 and x_4. The dynamical equation of the velocity of the vehicle needs to be slightly modified for the regions with grade. With the modifications the general equation of motion takes the form given in 39.

$$\dot{V} = \frac{C_2}{C_1} \cdot T_e - \frac{C_3^*}{C_1} \cdot V - \frac{C_4}{C_1} \cdot cos(\alpha) - \frac{C_8}{C_1} \cdot sin(\alpha)$$
$$where \begin{cases} \alpha = 0 & for\ x_0 \le x < x_1 \\ \alpha = 1^\circ & for\ x_1 \le x < x_2 \\ \alpha = 0 & for\ x_2 \le x < x_3 \\ \alpha = -1^\circ & for\ x_3 \le x < x_4 \\ \alpha = 0 & for\ x_4 \le x < x_5 \end{cases} \tag{39}$$

The Hamiltonian function also requires to be modified using the equation 39. All the other equations including system dynamics and the boundary conditions are same as the ones of unconstrained optimization which has been discussed in section 2.2.

The only difference for the case with varying road grade is the fact that at the points of change in grade, some optimality conditions [1] should be satisfied. These optimality equations are given as

$$\lambda^T(t_{1^-}) = \lambda^T(t_{1^+}) + \pi^T \cdot \frac{\partial N}{\partial x(t_1)} \tag{40}$$

$$H(t_{1^-}) = H(t_{1^+}) - \pi^T \cdot \frac{\partial N}{\partial t_1} \tag{41}$$

where the function $N(x(t),t)$ is the interior boundary condition. From the optimality equations it is possible to observe that there is discontinuity in $\lambda_x(t)$ but not in $\lambda_v(t)$. Then by finding the variations in $\lambda_x(t)$ at each road grade change point, and using the equations derived in section 2.2, it is possible to combine the optimal solutions of the regions with different grades.

Calculation of $\lambda_x(t)$ at a Road Grade Change Point

Here, the calculations are performed for the point $x = x_1$, and the for the other interior boundary points exactly the same procedure is followed. The interior boundary condition at x_1 is defined as:

$$N(x(t),t) = x(t) - x_1 = 0 \tag{42}$$

Then from equation 41, the λ_x between the region x_1 to x_2 can be written as

$$\begin{aligned} H(t_{1^-}) &= H(t_{1^+}) - \pi^T \cdot \underbrace{\frac{\partial N}{\partial t_1}}_{0} \\ \lambda_x(t_{1^+}) &= \lambda_x(t_{1^-}) + \left(C_5 + \frac{\lambda_v(t_1)}{V(t_1)} \cdot \frac{C_2}{C_1}\right) \cdot [Te(t_{1^-}) - Te(t_{1^+})] \\ &\quad + \frac{\lambda_v(t_1)}{V(t_1)} \cdot \frac{C_8}{C_1} \cdot sin(\alpha) - \frac{\lambda_v(t_1)}{V(t_1)} \cdot \frac{C_4}{C_1} \cdot (1 - cos(\alpha)) \end{aligned} \tag{43}$$

2.4.1 Simulation Results

Having determined all of the variations in $\lambda_x(t)$ and by solving the analytical equations, the simulations are performed for the given road profile. The simulation results are presented in the figure 9.

The figure 9, clearly illustrates that although there are changes in the road grade, this does not effect optimal velocity profile unless the applied torque values exceed the limits. On the other hand, the changes in grade leads the influence function λ_x,

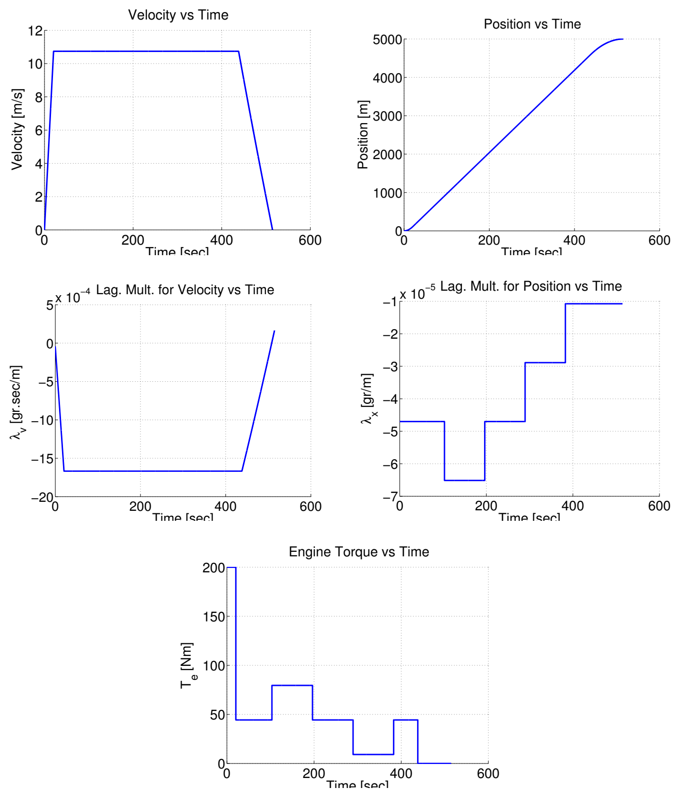

Figure 9. The Simulation Results for the Road Profile with Grade.

the Hamiltonian Function and torque values to be piecewise continuous.

2.4.2 Discussion

The analysis performed so far depends on the assumption that the torque values at singular points do not exceed the torque limits ($0 \leq Te \leq Te_{max}$). If at any of the singular points like at positive grade or negative grade regions the torque limits are exceeded then special considerations are required. In fact, the equations derived above will not be optimal solutions.

3 Conclusion

The aim of this research is to develop an analytical solution to the optimization problem of finding the minimum fuel consumption of a vehicle given the road and possibly the traffic light information. In order to find the analytical solution first the vehicle and the fuel consumption is modelled. However, soon it is realized that the modelled vehicle results in complex equations which makes the calculations very hard to solve. Then the model is simplified in a reasonable way such that it was possible to find the analytical solutions to the optimization problem. First, the solution is performed assuming there is no traffic light and no grade. The solution of the unconstrained problem, helped to solve the same problem but this time with the existence of the traffic light. Since for the constrained problem the analytical equations become more demanding a numerical solution algorithm is developed and presented in detail. To decrease the number of iterations, a logical way of finding the optimal k parameter which determines at which cycle the vehicle should reach to the traffic light is presented. Then by extending the knowledge gained, the optimal solution is determined for the case with known road grade information. Finally the simulation results for all cases are shown and discussed.

The results clearly show that by some certain simplifications it is possible to find the optimal velocity profile given all the information about the vehicle and the trip. Furthermore since the solution is analytical, the calculations can be performed in relatively short time. As a future work the results are to be compared with the results obtained from a numerical algorithm in order to determine the sensitivity of the simplifications made.

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