QUATERNIONS TO MODEL NON-ISOTHERMAL POROELASTIC PHENOMENA

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ABSTRACT

Quaternions are hypercomplex quantities in four dimensions \((q_0, q_1, q_2, q_3)\); they are a generalization of the classical two-dimensional complex numbers \((x, y) = x + iy\), where \(i^2 = -1\). Quaternions satisfy all the algebraic properties of the traditional complex numbers except that their multiplication law is not commutative. They have important applications in aeronautics, space flight dynamics, robotics, control theory, signal processing, and in computer graphics because of their stability and ability to represent three-dimensional rotations. In this paper we present for the first time, an original application of quaternions to model non-isothermal poroelasticity.

To illustrate the practical use of this formulation some applications are outlined: full deduction of the classical Biot’s theory coupled to thermal stresses. Construction of a general, well posed mathematical model to represent the physical behavior of geothermal rocks. Deformations of a non-isothermal reservoir, solved with finite elements, are included. These results can be applied to geothermal and hydrocarbon reservoirs, and to deep aquifers.

INTRODUCTION

All crustal rocks forming natural reservoirs are poroelastic. The fluid inside the pores and any temperature change affect their geomechanical properties. This elasticity is evidenced by the compression resulting from the pore pressure decline, which can shorten the pore volume. This reduction of the pore volume can be the principal source of fluid released from storage. On the other hand, any variation of temperature induces a thermo-poroelastic behavior that influences the elastic response of rocks. The expansivity of rocks is relatively small, but its effects can produce an increase in permeability and severe structural damages in rocks subjected to strong temperature gradients (Bundschuh and Suárez, 2010), as happens during the injection of cold fluids. This mechanism is of great importance in enhanced oil reservoirs and geothermal systems, when the injected cold fluid circulates in the underground. This transport can change both the permeability and the rock thermal conductivity. To develop our proposed four-dimensional model, we need to introduce first the main properties of quaternions, which will be a new mathematical tool to represent thermoporoelastic phenomena.

PROPERTIES OF QUATERNIONS

Algebraic definitions

We can define the quaternions as hypercomplex numbers in a four-dimensional space \(q \in \mathbb{H} \cong \mathbb{R}^4\) (this symbol \(\cong\) means that there is an isomorphism between both spaces, the first one is denoted by \(\mathbb{H}\) in honor of Sir William Rowan Hamilton); they satisfy...
two basic operations, addition and multiplication, defined as follows. Let \( \mathbf{q} \) and \( \mathbf{p} \) be two quaternions such that: \( \mathbf{q} = (q_0, q_1, q_2, q_3) \) and \( \mathbf{p} = (p_0, p_1, p_2, p_3) \), where all their individual components are real numbers \( (p_n, q_n \in \mathbb{R}^1, \ n = 0, 1, 2, 3) \).

**Addition:**
\[
\mathbf{p} + \mathbf{q} = (p_0 + q_0, p_1 + q_1, p_2 + q_2, p_3 + q_3)
\]

**Product:**
\[
\mathbf{p} \cdot \mathbf{q} = (p_0 q_0 - p_1 q_1 - p_2 q_2 - p_3 q_3, \\
p_0 q_1 + p_1 q_0 + p_2 q_3 - p_3 q_2, \\
p_0 q_2 + p_2 q_0 + p_3 q_1 - p_1 q_3, \\
p_0 q_3 + p_3 q_0 + p_1 q_2 - p_2 q_1)
\]

Two quaternions \( \mathbf{p} \) and \( \mathbf{q} \) are equal if and only if each one of their four components are equal:
\[
\mathbf{p} = \mathbf{q} \iff p_n = q_n, \ \forall n = 0, 1, 2, 3
\]

To obtain a vectorial representation of \( \mathbf{q} \), we define the following unit vectors of \( \mathbb{R}^4 \cong \mathbb{H} \): \( \mathbf{h} = (1, 0, 0, 0), \mathbf{i} = (0, 1, 0, 0), \mathbf{j} = (0, 0, 1, 0) \) and \( \mathbf{k} = (0, 0, 0, 1) \). In this way, every quaternion can be written as a linear combination of four-dimensional basis vectors:
\[
\mathbf{q} = q_0 \mathbf{h} + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k} \tag{2}
\]

In this context, \( \mathbf{q} \) is a thermoporoeelastic function, which will be defined later in this paper. The unit vectors satisfy the following multiplication table which is built on from definition (1):

<table>
<thead>
<tr>
<th>*</th>
<th>( \mathbf{h} )</th>
<th>( \mathbf{i} )</th>
<th>( \mathbf{j} )</th>
<th>( \mathbf{k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbf{h} )</td>
<td>( \mathbf{h} )</td>
<td>( \mathbf{i} )</td>
<td>( \mathbf{j} )</td>
<td>( \mathbf{k} )</td>
</tr>
<tr>
<td>( \mathbf{i} )</td>
<td>( \mathbf{i} )</td>
<td>( \mathbf{i} )</td>
<td>( \mathbf{k} )</td>
<td>( \mathbf{j} )</td>
</tr>
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<td>( \mathbf{j} )</td>
<td>( \mathbf{j} )</td>
<td>( \mathbf{k} )</td>
<td>( \mathbf{j} )</td>
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<tr>
<td>( \mathbf{k} )</td>
<td>( \mathbf{k} )</td>
<td>( \mathbf{j} )</td>
<td>( \mathbf{i} )</td>
<td>( \mathbf{k} )</td>
</tr>
</tbody>
</table>

Using this table, we deduce another hypercomplex form of any quaternion in terms of simple complex numbers:
\[
\mathbf{q} = q_0 \mathbf{h} + q_1 \mathbf{i} + q_2 \mathbf{j} + q_3 \mathbf{k} = (q_0 + q_1 i) + (q_2 + q_3 i) j = z_1 + z_2 j \tag{3}
\]

Where \( z_1 = q_0 + q_1 i \) and \( z_2 = q_2 + q_3 i \) are traditional complex quantities, \( z_1 \) and \( z_2 \in \mathbb{C} \equiv \mathbb{R}^2 \); and \( i \neq j = k \).

Therefore, the hypercomplex polar form of \( \mathbf{q} \) is:
\[
\mathbf{q} = z_1 + z_2 j = r_1 e^{i\theta_1} + r_2 e^{i\theta_2} \tag{4}
\]

where: \( e^{i\theta} = \cos \theta + i \sin \theta, (n = 1, 2) \)

Each individual complex part of the quaternion \( \mathbf{q} \) has a very simple geometric representation illustrated by the following two-dimensional graph.

![Figure 1: Geometric representation of each complex part of \( \mathbf{q} \), where \( \xi = r (\cos \theta + i \sin \theta) \).](image)

**Matrix representations in \( \mathbb{H} \)**
We can build quaternions into complex matrices and also into real matrices. The first one uses the hypercomplex form of equation (3):
\[
\mathbf{q} = q_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + q_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + q_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + q_3 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}
\]
\[
= \begin{pmatrix} q_0 + q_1 i & q_2 + q_3 i \\ -q_2 + q_3 i & q_0 - q_1 i \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ -\overline{z}_2 & \overline{z}_1 \end{pmatrix}
\]

where: \( \overline{z}_1 = q_0 - q_1 i, \overline{z}_2 = q_2 - q_3 i \) \tag{5}

The symbols in the last line are called the complex conjugates of \( z_1 \) and \( z_2 \) respectively. The second matrix form of quaternions uses the following four-dimensional objects:

\[
\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

\[
\mathbf{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}
\] \tag{6}
In terms of these unit and anti-symmetrical matrices, any quaternion can be expressed as a linear combination of arrays, which have the following property:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} = -\mathbf{1}
\]

\[
q = q_0 \mathbf{1} + q_i \mathbf{i} + q_j \mathbf{j} + q_k \mathbf{k}
\]

Note that the square matrix property \((\cdot)^2 = -\mathbf{1}\) is also satisfied by the complex \((2 \times 2)\) matrices of equation (5). The matrix representation (7) allows to define, in an alternative way, the quaternion multiplication (1) as a four-dimensional, skew-symmetric matrix-vector product:

\[
q \ast p = \begin{pmatrix}
q_0 & -q_i & -q_j & -q_k \\
qu_i & q_0 & -q_k & q_j \\
qu_j & q_k & q_0 & -q_i \\
qu_k & -q_j & q_i & q_0
\end{pmatrix} \cdot \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} q_0 p_0 - q_i p_1 - q_j p_2 - q_k p_3 \\ q_i p_0 + q_0 p_1 - q_k p_2 + q_j p_3 \\ q_j p_0 + q_k p_1 + q_0 p_2 - q_i p_3 \\ q_k p_0 - q_j p_1 + q_i p_2 + q_0 p_3
\end{pmatrix} \neq p \ast q \quad (8)
\]

This result compared to equation (1) demonstrates the non-commutative nature of quaternion multiplication. From equation (2) a third matrix representation is deduced in the form of a diagonal \((4 \times 4)\) array for every \(q \in \mathbb{H}\):

\[
q = \begin{pmatrix}
q_0 & 0 & 0 & 0 \\
0 & q_0 & 0 & 0 \\
0 & 0 & q_2 & 0 \\
0 & 0 & 0 & q_3
\end{pmatrix} \quad (9)
\]

It is well known that any anti-symmetrical matrix represents a pure rotation where \((q_1, q_2, q_3)\) are the corresponding components of the rotation vector.

**Hybrid scalar-vector representation in \(\mathbb{H}\)**

The result (10) suggests the following hybrid scalar-vectorial definition of a quaternion which clearly separates its two parts:

\[
q = q_0 + Q = q_0 + Q
\]

The term \(q_0\) is the real scalar part of \(q\), and \(Q\) is its imaginary vector part. Replacing definition (11a) into equation (8) and after doing some algebra we arrive to the following practical result which is useful to compute quaternion products in terms of classic vector operations:

\[
p \ast q = (p_0 q_0 - p \cdot Q) h + q_0 p + p_0 Q + P \times Q
\]

where \(P \cdot Q\) represents the scalar product and \(P \times Q\) is the vectorial product of \(P\) and \(Q\) respectively. Because of the fact that \(P \times Q \neq Q \times P\) we confirm again that the quaternion multiplication is a non-commutative operation.

**Arithmetic properties**

Quaternions satisfy all the algebraic and arithmetic properties of the classical two-dimensional complex numbers, except the multiplication commutative law (Hamilton, 1853; Kuipers, 1999). The main properties of \(q \in \mathbb{H}\) are summarized as follows:

1. **Additive inverse**: \(- q = -q_0 h - Q\), \(q \ast (-q) = 0\)
2. **Hypercomplex Conjugate**: \(\overline{q} = q_0 h - Q\)
3. **Norm of \(q\)**: \(|q| = \sqrt{q \ast \overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}\)
4. **Multiplicative Inverse**: \(q^{-1} = \overline{q} / q \ast \overline{q}, q \ast q^{-1} = 1\)
5. **Division of quaternions**: \(p / q = p \ast q^{-1} = p \ast \overline{q} / q \ast \overline{q}\)
6. **Normalized quaternion**: if \(q \ast \overline{q} = 1 \Rightarrow q^{-1} = \overline{q}\)

The addition is commutative and associative; the multiplication is associative but non-commutative; both operations obey the distribution law. The set \(\mathbb{H}\)
of quaternions, along with the addition, product and properties already defined, has an algebraic structure called a non-commutative division ring, or simply an anti-commutative algebra. The future usefulness of quaternions was clearly envisioned by its creator:

"Time is said to have only one dimension, and space to have three dimensions. ... The mathematical quaternion partakes of both these elements; in technical language it may be said to be time plus space, or space plus time; and in this sense it has, or at least involves a reference to, four dimensions. And how the One of Time, of Space the Three, Might in the Chain of Symbols girdled be. ... quaternions would be found to have a powerful influence as an instrument of research."

(William Rowan Hamilton; quoted in Graves, 1882).

**The triple product of quaternions conjugate**

Let \( q = q_0 + Q \) be a non-zero quaternion and \( v \in \mathbb{R}^3 \) be a vector or pure imaginary quaternion \((v_0 = 0)\). The application of equation (11b) to the triple product of \( q \), its conjugate \( q_0 - Q \) and \( v \) gives the following result:

\[
q \ast v \ast \bar{q} = (q_0^2 - Q \cdot Q) v + 2(Q \cdot v) Q + 2q_0 Q \times v \quad (12)
\]

If \( q \) is a normalized quaternion of unit norm:

\[
q \ast v \ast \bar{q} = (2q_0^2 - 1) v + 2(Q \cdot v) Q + 2q_0 Q \times v \quad (13)
\]

**Rotations in three dimensions using \( q \in \mathbb{H} \)**

We can perform easily three-dimensional rotations using quaternions and unit vectors to define an axis. A rotation in \( \mathbb{R}^3 \) about a unit vector \( u \) by an angle \( \theta \) can be computed using the following unit \( q \) and the triple product of quaternions:

\[
q = q_0 + Q = \cos \frac{\theta}{2} + u \sin \frac{\theta}{2} \quad \Rightarrow
\]

\[
w = q \ast v \ast \bar{q} = \left(2 \cos^2 \frac{\theta}{2} - 1\right)v + 2(u \cdot v) u \sin^2 \frac{\theta}{2} + 2 \cos \frac{\theta}{2} \sin \frac{\theta}{2} u \times v \quad (14)
\]

and therefore \( w \in \mathbb{R}^3 \). A series of two connected successive rotations, first \( q \) and then \( p \), is computed using the following formula:

\[
q = \cos \frac{\theta}{2} + u_1 \sin \frac{\theta}{2}, \quad p = \cos \frac{\varphi}{2} + u_2 \sin \frac{\varphi}{2}
\]

\[
\Rightarrow w = p \ast (q \ast v \ast \bar{q}) \ast \bar{p} = (p \ast q) \ast v \ast (\bar{q} \ast \bar{p}) \quad (15)
\]

**LINEAR POROELASTICITY IN TERMS OF QUATERNIONS \( q \in \mathbb{H} \)**

A key question that arises when developing the very basic poroelastic equations is: why do we need a four-dimensional space in poroelasticity? If we perform an experiment in a simple, non-porous solid under an uniaxial stress state, the resulting relationship between the stress \( \sigma_x \) and the strain \( \varepsilon_x \) is a one-dimensional equation:

\[
\sigma_x = E \varepsilon_x \quad (16a)
\]

Where \( E \) is the Young's modulus. But if we carry out the same one-dimensional experiment in a porous rock, because of the fluid contained in the pores, we obtain two simultaneous linear relationships that are equivalent to a \((2 \times 2)\) matrix equation (Bundschuh and Suárez, 2010), which corresponds to a two-dimensional formula:

\[
\begin{bmatrix}
\sigma_x \\
\rho_f 
\end{bmatrix}
= 
\begin{bmatrix}
K_U & -C \\
-C & M
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\zeta
\end{bmatrix}
\]

(16b)

Where \( K_U \) is the undrained bulk modulus (Appendix), \( C \) is a coefficient representing deformations coupling between the solid grains and the fluid, \( M \) is a parameter characterizing the fluid elastic properties, \( p_f \) is the fluid-pore pressure, and \( \zeta \) is a special strain representing the variation of the fluid mass content in the pores. Using only the parameter \( K_U \) we obtain similar 2D relationships:

\[
\begin{bmatrix}
\sigma_x \\
\rho_f 
\end{bmatrix}
= 
\begin{bmatrix}
K_U & -K_U B \\
-K_U B & K_U B / b
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\zeta
\end{bmatrix}
\]

(16c)

Where \( b = C/M \) is a quotient called the Biot-Willis coefficient (Biot & Willis, 1957); \( B \) is the Skempton coefficient ( \( B = C/K_U \)). The inverse matrix equation of (16b,c) is:

\[
\begin{bmatrix}
\varepsilon_x \\
\zeta
\end{bmatrix}
= 
\begin{bmatrix}
K_B^{-1} \ H^{-1} \\
H^{-1} \ R^{-1}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\rho_f 
\end{bmatrix}
\]

(16d)

Where \( K_B^{-1} = C_B \) is the bulk compressibility, \( K_B \) is called the drained bulk modulus; \( H^1 \) represents an expansion coefficient and \( R^1 \) is an unconstrained specific storage modulus (Wang, 2000). In terms of these moduli, the Biot and Skempton coefficients are defined as: \( b = K_B/H \) and \( B = R/H \), respectively. These coefficients represent properties that are not encountered in standard elasticity of simple solids. The experimental definitions of these poroelastic moduli are in the Appendix.
Equation (16d) reveals that changes in $\sigma_x$ and $p_f$ produce changes in $\varepsilon_x$, which require fluid to be added or removed from storage because $\zeta$ also changes as a function of $\sigma_x$ and $p_f$. After this observational result, we conclude that the fluid contained in the pores of the rock acts inside a second hidden spatial dimension, although the experiment was done in only one dimension. This result is immediately generalized to two- and three-dimensional experiments: an extra third or fourth spatial dimension will always appear because of the fluid contained in the pores. At this point, a natural question arises: what are the quaternions properties that are useful in poroelasticity?

The fundamental poroelastic relationship in $\mathbb{H}$

In order to introduce our model, we define first the following poroelastic quaternions for homogeneous, isotropic, porous rocks:

\[
q_a = p_f + h + \sigma_x i + \sigma_y j + \sigma_z k
\]

\[
q_e = 0 h + \varepsilon_x i + \varepsilon_y j + \varepsilon_z k
\]

\[
q_B = -\frac{C}{\lambda} e_B h + \varepsilon_B i + \varepsilon_B j + \varepsilon_B k
\]

\[
q_p = \frac{M}{b} \zeta h - p_f i - p_f j - p_f k
\]

Because of the physical symbols involved, we name the quaternions: $q_a$ the stress, $q_e$ the pure strain, $q_B$ the volumetric strain, and $q_p$ the fluid pressure. These expressions do not contain shear strains ($\varepsilon_{ij}$), because changes in the fluid pressure or in compressions are supposed not to induce shears (Wang, 2000). The fundamental linear relationship among the quaternions in equation (17) is very simple:

\[
q_a = 2G q_e + \lambda q_B + b q_p \quad \Leftrightarrow \quad p_f = -C e_B + M \zeta
\]

\[
\sigma_a = 2G \varepsilon_a + \lambda \varepsilon_B - b p_f, \quad (n = x, y, z)
\]

The symbols in equations (17) and (18) are of common use: $\sigma_n$ ($n = x, y, z$) holds for the three normal stresses acting in the rock; $\varepsilon_n$ are the normal strains describing the elastic response of the porous rock; $\varepsilon_B = \varepsilon_x + \varepsilon_y + \varepsilon_z$ is the volumetric strain, $G$ and $\lambda$ are poroelastic coefficients. The rigidity modulus $G$ represents the rock response to shear; the Lamé module $\lambda = K_B - 2G/3$ measures the rock response to compression under drained conditions. In terms of quaternions it is straightforward to invert the stress-strain relationship (18). For example:

\[
q_e = \frac{1}{2G} \left( q_a - \lambda q_B - b q_p \right)
\]
Inclusion of the shear stresses $\sigma_{ij}$ and Biot’s theory

Note that there are no shear stresses nor shear strains involved in quaternions (17) and (22). They are going to be introduced in a different way. We define two symmetric matrices $s$ and $\varepsilon$. The first one contains the shear stresses and the second one the shear strains. Equations (23) and (24) are diagonal matrices of type (9) containing only principal stresses and strains. Both influence the bulk rock deformation through tension or compression in the principal axes.

Since there are no shear tensions in the fluid, the pore fluid pressure affects only the three normal strains $\varepsilon_n$ ($n = x, y, z$), as shown in equation (20). Therefore the shear stresses can only work on the shear strains:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \sigma_{xy} & \sigma_{xz} \\
0 & \sigma_{xy} & 0 & \sigma_{yz} \\
0 & \sigma_{xz} & \sigma_{yz} & 0
\end{pmatrix}
= 2G_0 \cdot
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & \varepsilon_{xy} & \varepsilon_{xz} \\
0 & \varepsilon_{xy} & 0 & \varepsilon_{yz} \\
0 & \varepsilon_{xz} & \varepsilon_{yz} & 0
\end{pmatrix}
$$

$G_0$ is the shear matrix

$$
\sigma_G = 2G_0 \cdot \varepsilon_G \quad \Leftrightarrow \quad \sigma_{nm} = 2G_{nm} \varepsilon_{nm} \quad (n, m = x, y, z)
$$

Where the full shear matrix $G_0$ holds for anisotropic rocks. Note that the shear strains and corresponding stresses cannot be represented by quaternions. Formula (25) completes the set of linear poroelastic equations when added to Eq. (24) in matrix form:

$$
q_a + \sigma_G = 2G \cdot q_a + 2G_0 \cdot \varepsilon_G + \Lambda \cdot q_B + b \cdot q_p
$$

Equation (26) corresponds to the formulation of the theory for anisotropic rocks in terms of quaternions and shear matrices. The connections among the individual components of equation (27) are known as Biot’s linear poroelastic theory for isotropic rocks (Biot, 1941, 1955). The components of equation (27) can be written in compact form ($\delta_{nm}$ is the unit tensor):

$$
\sigma_{nm} = (\lambda \varepsilon_B \delta_{nm} + 2G \varepsilon_{nm}) - b p_f \delta_{nm}
$$

The term $\tau_{nm}$ is called the Terzaghi effective stress, which acts only in the solid matrix:

$$
\tau_{nm} = \lambda \varepsilon_B \delta_{nm} + 2G \varepsilon_{nm}
$$

Another important fact concerning the inclusion of the shear stress-strains is that the shear matrices connected with linear poroelasticity are symmetric arrays. A special case of the spectral theorem in linear algebra (Lang, 1969) establishes that any real symmetric $(4 \times 4)$ matrix $M$ can be transformed into a diagonal matrix $D$ in the following way:

$$
M \rightarrow U^T \cdot M \cdot U = D, \text{ where } U = U^{-1}
$$

The matrix $U$ is called an orthogonal matrix and is built on the eigenvectors of $M$. The corresponding basis of eigenvectors for the four-dimensional space $\mathbb{R}^4$ is orthogonal. The matrices representing the quaternions in the fundamental equations (18), (24) and (26) are all symmetric. Therefore, any poroelastic matrix equation, which includes shear stresses and shear strains, can be transformed into a diagonal matrix and, consequently it can be represented by a quaternion of type (9) which is valid for every $q \in \mathbb{H}$.

THERMOPOROELASTICITY IN TERMS OF QUATERNIONS $q \in \mathbb{H}$

The basic thermodynamic potentials in porous rocks are the internal energy and the specific enthalpy for the fluid phase. The free energy describes the solid matrix while for the skeleton the appropriate potential is the free enthalpy (Bundschuh and Suárez, 2010). In this section we focus only on isotropic, homogeneous geothermal rocks.

The Gibbs free enthalpy for the porous rock

The volumetric Gibbs free internal enthalpy $g_S$ is defined as the algebraic difference of the enthalpy $h_S$ minus the product of the absolute temperature $T$ and the skeleton entropy $S_S$ per unit volume of rock:

$$
g_S(\varepsilon, p_f, T) = h_S - T S_S \quad [\text{J/m}^3 = \text{Pa}]
$$

If the process is reversible and there is no energy dissipation in the rock, the total differential of this thermodynamic potential is useful to describe the skeleton thermal behavior with shear strains and porosity \( \phi \) included (Coussy, 1991; Bundschuh and Suárez, 2010):

\[
d g_S = \sigma_{ij} d e_{ij} - \varphi d p_f - S_S d T
\]  

(32)

These differential equations are integrated between an initial state \( g_S (e_{ij} = 0, p_f = p_0, T = T_0) \) at zero strain, and a final state \( g_S (e_{ij}, p_f, T) \). Note that an initial temperature \( T_0 \) and an initial pore pressure \( p_0 \) are needed because both thermodynamic variables \( T \) and \( p_f \) are changing in non-isothermal processes:

\[
\int_{g_S^0}^{g_S} d g_S = \int_0^{e_{ij}} \sigma_{ij} \partial e_{ij} - \int_{p_0}^{p_f} \varphi \partial p_f - \int_{T_0}^{T} S_S \partial T
\]  

(33)

After integrating equations (33), an exact, analytic expression for the Gibbs potential of the skeleton is obtained:

Initial State: \( g_S^0 = g_S (e_{ij} = 0, p_f = p_0, T = T_0) \);

Final State: for \( (i, j = x, y, z) \) and for \( t > 0 \):

\[
g_S (e_{ij}, p_f, T) = \sigma_M^0 e_B + \left( \sigma_M^0 - \sigma_M^0 \delta_j \right) \left( e_{ij} - \frac{e_B}{3} \delta_j \right) \varphi_0 p_f - S_S T
\]  

\[
+ \frac{K_B}{2} (e_B)^2 - \varphi \gamma_B \left( \frac{p_f}{p_0} - 1 \right) (T - T_0)
\]  

\[
- \left( \frac{p_f}{p_0} - 1 \right)^2 K_f \frac{M}{T_0} \varphi_0 - \frac{C_B (T - T_0)^2}{2 T_0}
\]  

\[
+ G \left( e_{ij} - \frac{e_B}{3} \delta_j \right)^2
\]  

(34)

Where \( K_f \) is the bulk modulus of the fluid. The parameters \( \gamma_B = 1/[K] \) are volumetric thermal dilatation coefficients. The first one \( \gamma_B \) measures the bulk dilatation of the skeleton, while \( \gamma_\varphi \) is related to the variation of the porosity when the temperature changes:

\[
\gamma_B = \frac{V_B}{V_B} \left( \partial V_B \partial T \right)_\sigma, \quad \gamma_\varphi = \frac{1}{V_f} \left( \partial V_f \partial T \right)_{\phi_f}
\]  

(35)

\( C_V \) is the skeleton volumetric heat capacity under constant deformations. The thermodynamic behavior of the porous rock is obtained by computing the following partial derivatives:

\[
\sigma_{ij} = -\frac{\partial g_S}{\partial e_{ij}}, \quad \varphi = -\frac{\partial g_S}{\partial p_f}, \quad S_S = -\frac{\partial g_S}{\partial T}
\]  

(36)

**Thermodynamics of the fluid mass content**

The symbol \( \zeta \) represents the variation of the fluid content in the pores defined as follows (Biot, 1941):

\[
m_f = \rho_f \varphi \varphi \zeta = \frac{\partial m_f}{\partial \varphi} \rho_f + \frac{\partial \rho_f}{\partial \varphi} d \varphi
\]  

(37a)

Where \( m_f \) is the fluid mass content per unit rock volume, \( \rho_f \) is the fluid density, \( \rho_0 \) is a reference density, \( \varphi \) is the effective porosity and \( \Delta \) represents a change in the corresponding variable. An explicit thermodynamic expression for \( \zeta \) can be obtained by differentiating \( m_f \) in equation (37):

\[
d \zeta = \frac{d m_f}{\rho_f} \rho_f + d \varphi
\]  

(37b)

The fluid density is a function of both, the fluid pressure and the temperature \( \varphi_f (p_f, T) \), thus:

\[
\frac{d \rho_f}{\rho_f} = \frac{1}{\rho_f} \left( \partial \rho_f \partial p_f \right) d p_f + \frac{1}{\rho_f} \left( \partial \rho_f \partial T \right) d T
\]  

(38)

The fluid compressibility \( C_f \) and the fluid thermal expansivity \( \gamma_f \) are defined as:

\[
C_f = \frac{1}{K_f} = \frac{1}{\rho_f} \left( \partial \rho_f \partial p_f \right)_T
\]  

(39)

\[
\gamma_f = \frac{1}{V_f} \left( \partial V_f \partial T \right)_{\rho_f} = \frac{1}{\rho_f} \left( \partial \rho_f \partial T \right)_{\rho_f}
\]  

We define another thermal expansion coefficient \( \gamma_m \), which measures the changes in the fluid mass content when the average stress \( \sigma_M \) and \( p_f \) are held constant:

\[
\gamma_m = \frac{1}{m_f} \left( \partial m_f \partial T \right)_{\sigma_M, p_f}
\]  

(40)

\[
\frac{1}{\rho_f} \left( \partial \rho_f \partial T \right)_{\rho_f} + \frac{1}{\rho_f} \left( \partial \rho_f \partial T \right)_{p_f} = \gamma_f - \gamma_m
\]  

Substituting the coefficients (39) into equation (37b):

\[
d \zeta = \frac{d \varphi}{K_f} \rho_f d p_f - \varphi \gamma_f d T
\]  

(41)
Integrating this expression, between the initial and final states, and replacing the result into equation (36) for the porosity we obtain:

\[
\zeta = b \phi B + \frac{P - P_0}{M} + \phi \gamma_m (T - T_0)
\]  \hspace{1cm} (42)

Equation (42) is a useful expression for the variation of the fluid content in a linear non-isothermal process.

**The thermoporoelastic relationship in \[1\]**

With the previous results, we can define the thermal stress quaternion \(q_T\) for thermoporoelastic rocks:

\[
q_T = \Delta T (\varphi h + i + j + k)
\]  \hspace{1cm} (43)

Where \(\Delta T = T - T_0\). The corresponding matrix of thermal expansivity coefficients is the following diagonal array:

\[
\Gamma = \begin{pmatrix}
M \gamma_m & 0 & 0 & 0 \\
0 & K_B \gamma_B & 0 & 0 \\
0 & 0 & K_B \gamma_B & 0 \\
0 & 0 & 0 & K_B \gamma_B
\end{pmatrix}
\]  \hspace{1cm} (44a)

The linear relationship among the quaternions in non-isothermal processes is:

\[
q_v = 2G \cdot q_T + A \cdot q_B + b \cdot q_p - \Gamma \cdot q_T
\]  \hspace{1cm} (44b)

In terms of four-dimensional matrices, the equation relating stresses and strains in isotropic rocks is deduced directly:

\[
\sigma = \mathcal{E}_B \begin{pmatrix}
-C & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 \\
0 & 0 & \lambda & 0 \\
0 & 0 & 0 & \lambda
\end{pmatrix} + 2G \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \epsilon_x & 0 & 0 \\
0 & 0 & \epsilon_y & 0 \\
0 & 0 & 0 & \epsilon_z
\end{pmatrix} + A_T \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \gamma_B & 0 & 0 \\
0 & 0 & \gamma_B & 0 \\
0 & 0 & 0 & \gamma_B
\end{pmatrix}
\]  \hspace{1cm} (44c)

Where:

\[
\Delta p = p - P_0, \quad A_p = -\frac{M \zeta}{p - P_0}, \quad A_T = \frac{M \phi}{K_B} \gamma_m
\]

Note that every quaternion is equivalent to a diagonal matrix. The fact that linear thermoporoelasticity theory can be represented by quaternions is very important because all quaternions properties, that are useful in this context, can be applied to geothermal rocks deformations. This allows to the development and discovery of other unknown poroelastic processes.

**Some numerical values of the poroelastic moduli and the thermal expansivity coefficients**

To construct the following table of poroelastic coefficients the basic experimental data set used was \{\(E, G, \varphi, K_S\) and \(K_f\)\} (see Appendix).

<table>
<thead>
<tr>
<th>Rock Type [0]</th>
<th>(\varphi) (%)</th>
<th>(E) (GPa)</th>
<th>(v)</th>
<th>(r_V)</th>
<th>(G) (GPa)</th>
<th>(\lambda)</th>
<th>(A_V) (GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clay ([0])</td>
<td>65.0</td>
<td>0.18</td>
<td>0.01</td>
<td>0.49</td>
<td>0.09</td>
<td>0.001</td>
<td>6.1</td>
</tr>
<tr>
<td>Boise sandstone ([1])</td>
<td>26.0</td>
<td>9.7</td>
<td>0.15</td>
<td>0.31</td>
<td>4.2</td>
<td>1.8</td>
<td>7.3</td>
</tr>
<tr>
<td>Berea sandstone ([2])</td>
<td>13.0</td>
<td>30.5</td>
<td>0.26</td>
<td>0.32</td>
<td>12.1</td>
<td>13.1</td>
<td>23.2</td>
</tr>
<tr>
<td>Indiana limestone ([3])</td>
<td>8.5</td>
<td>35.8</td>
<td>0.31</td>
<td>0.32</td>
<td>13.7</td>
<td>21.7</td>
<td>24.8</td>
</tr>
<tr>
<td>Andesite ([4])</td>
<td>2.0</td>
<td>60.0</td>
<td>0.25</td>
<td>0.27</td>
<td>24.0</td>
<td>24.0</td>
<td>28.3</td>
</tr>
<tr>
<td>Tennessee marble ([5])</td>
<td>0.5</td>
<td>42.2</td>
<td>0.42</td>
<td>0.42</td>
<td>14.9</td>
<td>73.9</td>
<td>74.0</td>
</tr>
<tr>
<td>Rock with celestite ([6])</td>
<td>8.5</td>
<td>35.8</td>
<td>0.31</td>
<td>0.32</td>
<td>13.7</td>
<td>21.7</td>
<td>24.8</td>
</tr>
</tbody>
</table>

The rock types are:

\[0\].- Clay, \((K_f = 3.9\,\text{GPa})\).
\[1\].- Boise sandstone, \((K_f = 2.0\,\text{GPa})\).
\[2\].- Berea sandstone \((K_f = 3.5\,\text{GPa})\); 
\[3\].- Indiana limestone, \((K_f = 3.1\,\text{GPa})\). 
\[4\].- Andesite, \((K_f = 2.1\,\text{GPa}, p_k = 250\,\text{bar, } T = 25^\circ\text{C}})\). 
\[5\].- Tennessee marble, \((K_f = 3.5\,\text{GPa})\). 
\[6\].- Rock with celestite, \((K_f = 2.0\,\text{GPa})\). 

References used: Rocks \[0\], \[1\], \[2\] and \[3\], Wang (2000); \[4\], \[5\] and \[6\], Bundschuh and Suárez (2010). The figures in italics were estimated using poroelastic formulas. Rocks of type \[0\] and \[6\] are representative of the two limit cases of linear poroelastic theory, for \(b \approx 1\) and \(b \approx 0\), respectively. In this section, we also give some experimental values of the volumetric thermal dilatation coefficients \(\gamma_B\), \(\gamma_F\), \(\gamma_u\), \(\gamma_m\) and \(\gamma_u\) [1/K]. As a general trend, thermal expansion increases when temperature...
The volumetric thermal expansivity $\gamma_b$ is about $10^3 \text{K}^{-1}$ for solids, $10^4 \text{K}^{-1}$ for liquids and $10^5 \text{K}^{-1}$ for gases. For the expansivity of the pores in granite we estimated $\gamma_b \approx 1.7 \times 10^4 \text{K}^{-1}$. For water, at 80°C and 100 bar, $\gamma_f = 6.3 \times 10^4 \text{K}^{-1}$. For low porosity hard volcanic rocks, having a porosity $\varphi = 0.0018$, $b = 0.3$, and $B = 0.9$, Guéguen & Boutéca (2004) published the values $\gamma_B = 5.0 \times 10^5 \text{K}^{-1}$, $\gamma_m = - 46.0 \times 10^5 \text{K}^{-1}$ and $\gamma_U = 5.075 \times 10^5 \text{K}^{-1}$ and the corresponding bulk moduli $K_B = 27 \text{GPa}$ and $K_U = 38 \text{GPa}$. Using these data, the corresponding variations of the fluid mass content and of the fluid pressure $p$ for drained and undrained conditions are:

$$\left(\frac{\partial \zeta}{\partial t}\right)_{u, p} = \varphi \gamma_m - b \gamma_B = -1.58 \times 10^{-5} \left[\frac{1}{K}\right]$$

$$\left(\frac{\partial p}{\partial t}\right)_{u, \varphi} = - \frac{B K_U}{b} \left(\varphi \gamma_m - b \gamma_B\right) = 18.04 \times 10^5$$

$$\left(\frac{\partial p_U}{\partial t}\right)_{p, \varphi} = - \frac{B K_U}{b} \varphi \gamma_m = 0.67 \times 10^5$$

These numerical values confirm that in low porosity and low permeability rocks, the thermal variation of the fluid content is very small because of the short hydraulic parameters. For the same reason, a significant fluid pressure increase occurs in the presence of thermal stresses, specifically in undrained conditions. This is a common case when injection of cold water takes place in hot dry rock reservoirs. The values presented in Table 2 were obtained under isothermal conditions. But there are evidences suggesting a great sensitivity of the poroelastic parameters with regard to temperature changes in non-isothermal processes (Suárez, 2010). For example, the bulk modulus of cold water at 25°C is equal to 2.1 GPa (Table 2); whereas the bulk modulus of hot water is lower under deep geothermal conditions ($K_0 = 0.45 \text{GPa}$ at 250°C), producing larger variations of the fluid content and inducing higher thermoporoelastic deformations. Therefore, the poroelastic deformations can be much higher in geothermal reservoirs than in cold isothermal aquifers. This effect is more important in high porosity sedimentary rocks. Figure 2 shows the relationship between $b$, the Biot-Willis modulus and the bulk modulus of water $K_f$ (equation 45), which depends on both, the temperature and the fluid pressure (Bundschuh and Suárez, 2010):

$$2K_S b = (K_U - K_B) + \sqrt{(K_U - K_B)^2 - 4\left(1 - K_S / K_f\right)(K_U - K_B)K_S \varphi} \quad(45)$$

Figure 2: The Biot-Willis coefficient as a function of the bulk modulus of water.

**A GENERAL POROELASTIC MODEL IN 3D**

The fundamental law of dynamic poroelasticity in two-dimensions is (Bundschuh and Suárez, 2010):

$$\rho \left\{ \frac{\partial^2 u_x}{\partial t^2} + 2G \frac{\partial^2 u_x}{\partial y^2} + C_0 \frac{\partial^2 u_y}{\partial x \partial y} + F_x \right\} = C_0 \left( \frac{\partial^2 u_y}{\partial x^2} + 2G \frac{\partial^2 u_y}{\partial y^2} + C_0 \frac{\partial^2 u_x}{\partial x \partial y} + F_y \right)$$

$$\ddot{u}(t) = \left(\frac{u_x}{L_0}\right) \frac{u_x^2 + u_y^2}{L_0} \leq 1, \quad (46)$$

Where $(u_x, u_y)$ are the components of the displacement vector; $(F_x, F_y)$ are the components of the external body force acting on the porous rock; the left term in equation (46) is the vectorial acceleration and $\rho$ is the average density, when $\rho_f$ and $\rho_s$ are the fluid and solid phase densities respectively; $L_0$ is a representative dimension of the porous skeleton. This general model is valid only for small, reversible deformations of Hookian poroelastic rocks. We consider only the important case of static poroelastic equations when the external body force is the vertical gravitational force per unit bulk volume and the acceleration in equation (46) is zero. The governing poroelastostatic equations in condensed tensorial notation is equivalent to the following formula:

$$G \nabla^2 u_i + (\lambda + G) \frac{\partial^2 u_i}{\partial i \partial j} - b \frac{\partial p_f}{\partial i} + F_i = 0 \quad (47)$$

for $\{i, j = x, y, z\}$

Equation (47) is coupling the poroelastic mechanism between the pore pressure and the displacement of the porous rock particles. Two drained coefficients $\lambda$ and $G$ emerge because the coupling term is the fluid pressure.
The mathematical model of a poroelastic rock, fully saturated with a moving fluid, is a coupled group of partial differential equations governing the fluid flow inside the pores of the deformable skeleton. All the coefficients, concepts and relationships described in previous sections are sufficient to develop this model. We only need to define the unknowns and methodically arrange the appropriate equations.

There are twelve fundamental unknown variables in three dimensional thermoporoelasticity: the six stresses \((\sigma_{ij})\) of equation (27); the three coordinates \((u_x, u_y, u_z)\) of the relative fluid-solid displacement; the variation of the fluid content \(\zeta\) (Eq. 20), the pore pressure \(p_f\) (Eq. 18), and the temperature \(T\) (Eq. 43).

The six strain components \((\varepsilon_{ij})\) can be computed using either the quaternion equation (19) or their definition in terms of the vector displacement. The corresponding equations of the previously mentioned unknowns are the seven components relating stresses, strains and the pore pressure (Eq. 26) or (Eq. 27), the three partial differential equations obtained from the equilibrium conditions in the skeleton (Eq. 47), the eleventh equation comes from the conservation of the fluid mass relating \(\zeta\) with the Darcy’s law through the fluid pressure (Eq. 20); while the final twelfth comes from the general heat flow equation in geothermal reservoirs, which includes conduction and convection through an advective term related to the transport of energy by the moving fluid in the pores:

\[
c_p \rho \frac{\partial T}{\partial t} - \nabla \cdot (k_T \nabla T) + \nabla \cdot (c_p \rho T \nabla T) = Q_H \tag{48}
\]

Where vector \(\mathbf{v}\) represents the Darcy’s fluid velocity, \(c_p\) is the specific heat coefficient at constant pressure, \(k_T\) is the thermal conductivity tensor, and \(Q_H\) [W/m³] is the volumetric thermal energy production. This group of equations provides an equilibrated system that can be numerically solved using different numerical techniques.

**SOLUTION OF THE POROELASTIC MODEL USING FINITE ELEMENTS: A 2D EXAMPLE**

Equation (47) includes Biot’s poroelastic theory. It can be formulated and numerically solved using the Finite Element Method (FEM). Let \(\Omega\) be the bulk volume of the porous rock, and let \(\partial \Omega\) be its boundary, \(\mathbf{u}\) is the set of admissible displacements in equation (46); there is a volumetric force and a force acting on the surface \(\partial \Omega\). After doing some algebra (Liu and Quek, 2003) we arrive to a FEM fundamental equation for every element \(V_e\) in the discretization of \(\Omega\) composed of \(N_e\) finite elements:

\[
\mathbf{K}^e \cdot \mathbf{d}^e + \mathbf{M}^e \cdot \frac{\partial^2 \mathbf{d}^e}{\partial t^2} = \mathbf{F}^e; \quad e = 1, N_e \tag{49}
\]

\(\mathbf{d}^e\) is a vector containing the displacements of the nodes in each \(V_e\). Equation (49) approximates the displacement \(\mathbf{u}\) of the poroelastic rock. \(\mathbf{F}^e\) is the vector of total nodal forces. \(\mathbf{K}^e\) and \(\mathbf{M}^e\) are the stiffness and equivalent mass matrices for each finite element \(V_e\). The mathematical definitions of both matrices are:

\[
\mathbf{K}^e = \int_{V_e} \mathbf{B}^T \cdot \mathbf{C}_B \cdot \mathbf{B} \; dV; \quad \mathbf{B} = \mathbf{L} \cdot \mathbf{N};
\]

\[
\mathbf{M}^e = \int_{V_e} \rho \mathbf{N}^T \cdot \mathbf{N} \; dV; \quad e = 1, N_e
\]

Where \(\mathbf{C}_B\) is an array of poroelastic coefficients, \(\mathbf{N}\) is the matrix of shape functions that interpolate the displacements, \(\mathbf{L}\) is a matrix differential operator and array \(\mathbf{B}\) is called the strain poroelastic matrix. To illustrate the model (49), two-dimensional strains of an elastic aquifer are presented, including the form in which a temperature change can affect its poroelastic deformation. In the first example, we assume that the aquifer contains cold water at 20°C (1000 kg/m³). In a second example, we consider an uniform aquifer temperature of 250°C. In both cases water obeys Darcy’s law for head \(h\). Three sedimentary layers overlay an impermeable bedrock in a 5000 m long basin where faulting creates a bedrock step (BS) near the mountain front (Figure 3). The sediment stack totals 420 m at the deepest point of the basin \((x = 0)\) but thins to 120 m above the step \((x > 4000)\). The top two layers of the sequence are each 20 m thick. The first and third layers are aquifers; the middle layer is relatively impermeable to flow.

![Figure 3: Initial state and aquifer’s geometry with the impermeable bedrock in the basin.](image)
deformation near a bedrock step in a sedimentary basin described in a previous publication (Leake & Hsieh, 1997).

The problem concerns the impact of pumping for a basin filled with sediments draping an impervious fault block. The flow field is initially at steady state, but pumping from the lower aquifer reduces hydraulic head by 6 m per year at the basin center (under isothermal conditions). The head drop moves fluid away from the step. The fluid supply in the upper reservoir is limitless. The period of interest is 10 years. Streamlines represent the fluid to porous rock coupling. The second simulation was performed for a poroelastic coupled deformation when the water in the aquifer corresponds to geothermal conditions (fluid density of 800.4 kg/m³, temperature of 250 °C, and pressure of 50 bar). The rock is poroelastic, homogeneous and obeys quaternion equation (18). Water obeys Darcy’s law for head

\[
\frac{\partial}{\partial x}(K_x \frac{\partial h}{\partial x}) + \frac{\partial}{\partial y}(K_y \frac{\partial h}{\partial y}) + q_V = S_S \frac{\partial h}{\partial t} \tag{51}
\]

For the computations, data of Table 3 were used. In the first example for the Biot-Willis coefficient we assume that \( b = 0.3 \); in the second example \( b = 0.9 \) because it was affected by the higher temperature through the water bulk modulus (Fig. 2).

**Table 3: Parameters used in the simulations.**

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydraulic conductivity, upper &amp; lower aquifers</td>
<td>( K_x = 25 ) m/day</td>
</tr>
<tr>
<td>Poroelastic storage coeff. upper aquifer</td>
<td>( S_S = 10^6 )</td>
</tr>
<tr>
<td>Hydraulic conductivity confining layer</td>
<td>( K_y = 0.01 ) m/day</td>
</tr>
<tr>
<td>Poroelastic storage coeff., lower aquifer</td>
<td>( b = 0.3 )</td>
</tr>
<tr>
<td>Biot-Willis coefficient (cold water at 20°C)</td>
<td>( E = 8 \times 10^8 ) Pa</td>
</tr>
<tr>
<td>Poisson’s ratio</td>
<td>( \nu = 0.25 )</td>
</tr>
</tbody>
</table>

The corresponding FE mesh has 2967 elements excluding the bedrock step (Fig. 3). Figures (4) and (5) show the horizontal strains in the aquifer for cold and hot water respectively. Figures (6) and (7) show the aquifer vertical deformation after 10 years of pumping with cold and hot water respectively.

**CONCLUSIONS**

- A general thermoporoelastic linear model based on quaternions was introduced. This new formulation is equivalent to a 4D-tensorial model, which contains the classic Biot’s theory. The model takes into account, in a simple way, both the fluid pressure and the temperature effects in porous rock deformations.

- The examples presented suggest the influence of temperature changes on the poroelastic strains and on the coefficients. The poroelastic deformations are much higher in geothermal reservoirs than in isothermal aquifers. For cold water, the estimated value of \( \varepsilon_z \) is about \(-1.5 \cdot 10^{-4}\), while for hot water \( \varepsilon_z \) is about \(-7.5 \cdot 10^{-4}\).
REFERENCES


Hamilton, W. R. (1853), "Lectures on Quaternions", a set of lectures delivered by Hamilton between 1848 and 1853 in the Trinity College of Dublin. Hodges and Smith, Grafton-Street. The original text can be read at this web site:

http://dlxs2.library.cornell.edu/cgi/t/text/pageviewer-idx?c=math&cc=math&idno=05230001&q1=Le ctures+on+Quaternions&frm=frameset&view=image&seq=9


APPENDIX - NOMENCLATURE

Poroelastic coefficients: experimental definitions
In experimental poroelasticity, there are four different types of bulk moduli and two types of deformations, drained and undrained. Both modes represent limiting responses of the rock (Biot, 1941; Wang, 2000):

Drained conditions: The rock is confined and subjected to support an external hydrostatic pressure \( \sigma_H \). The fluid in the pores is allowed to escape and the total stress is entirely supported by the rock skeleton. The deformations are achieved at constant pore fluid pressure \( p_f \). Biot (1962) called this state, “an open system”.

Undrained conditions: Deformations are measured at constant fluid mass content \( m_f \) (\( \Delta \zeta = 0 \)). The rock is entirely submerged in a fluid in such a way that the external hydrostatic pressure is balanced by the pore pressure (\( \sigma_H = - p_f \)). The fluid in the pores remains constant; no fluid is allowed to move into or out of the control volume. The fluid remains trapped in the skeleton. For Biot (1962) this state was characteristic of a “closed system”.

Bulk elastic modulus: The drained bulk coefficient \( K_B \) is one of the central concepts in poroelasticity; it is experimentally defined as the ratio of the change in confining lithostatic pressure \( p_k \) relative to the volumetric strains when the fluid pressure remains constant, under isothermal conditions:

\[
\frac{1}{C_B} = K_B = \left( \frac{\partial p_k}{\partial \varepsilon_B} \right)_{p_f} \tag{A1}
\]

Undrained bulk compressibility: The undrained bulk modulus \( K_U \) measures the resistance of the bulk rock against deformations produced by the confining lithostatic pressure when the fluid content \( \zeta \) remains constant:

\[
\frac{1}{C_U} = K_U = \left( \frac{\partial p_k}{\partial \varepsilon_B} \right)_{\zeta} \tag{A2}
\]

Compressibility of the solid phase: The elastic modulus \( K_S \) measures the resistance of the solid rock against deformations produced by changes on hydrostatic compression (\( \sigma_H = - p_k \)) when the differential pressure \( p_d = p_k - p_f \) remains constant:

\[
\frac{1}{C_S} = K_S = \left( \frac{\partial p_f}{\partial \varepsilon_S} \right)_{p_d} \tag{A3}
\]

Compressibility of the fluid phase: The isothermal compressibility of the fluid \( C_f \) is defined as the change of fluid volume with respect to the effective pressure change per unit volume:

\[
\frac{1}{K_f} = C_f = - \frac{1}{V_f} \left( \frac{\partial V_f}{\partial p_f} \right)_T = \frac{1}{\rho_f} \left( \frac{\partial \rho_f}{\partial p_f} \right)_T \tag{A4}
\]

Compressibility of the pore volume: The unjacketed compressibility of the pore volume \( C_p \), is defined as the change of pore volume with respect to the pore pressure change per unit volume when \( p_d \) remains constant:

\[
\frac{1}{K_p} = C_p = - \frac{1}{V_p} \left( \frac{\partial V_p}{\partial p_f} \right)_{p_d} = \frac{1}{\rho_p} \left( \frac{\partial \rho_p}{\partial p_f} \right)_{p_d} \tag{A5}
\]

The Biot-Willis Coefficient: This fundamental parameter is equal to the change of pressure \( p_k \) with respect to the fluid pressure change when the total volumetric strain remains constant:

\[
b = \left( \frac{\partial p_k}{\partial p_f} \right)_\zeta = 1 - \frac{K_B}{K_S} = \varphi \frac{K_B}{K_p} \left( \frac{\partial V_p}{\partial V_B} \right)_{p_f} \tag{A6}
\]

The last expression in (A6) gives another definition of \( b \) as the ratio of pore volume change to total bulk volume change under drained conditions. This equation also provides two ways of computing the drained bulk modulus of the pore space \( K_B \). These formulae are obtained from experiments in which the pore fluid can flow in or out of the pores to maintain the pore pressure constant.

The poroelastic expansion coefficient: \( H^{-1} = 1/H \) describes how much \( \Delta V_B \) changes when \( p_f \) changes while keeping the applied stress \( \sigma \) constant; \( 1/H \) also measures the changes of \( \zeta \) when \( \sigma \) changes and \( p_f \) remains constant:

\[
\frac{1}{H} = \left( \frac{\partial \varepsilon_B}{\partial p_f} \right)_\sigma = \left( \frac{\partial \zeta}{\partial \sigma} \right)_{p_f} \tag{A7}
\]

The unconstrained specific storage coefficient: \( 1/R \) measures the changes of \( \zeta \) when \( p_f \) changes when the applied stress \( \sigma \) remains constant. Its numerical value is determined by the compressibility of the frame, the pores, the fluid and the solid phase:

\[
\frac{1}{R} = \left( \frac{\partial \zeta}{\partial p_f} \right)_\sigma \tag{A8}
\]
**The constrained specific storage coefficient:** $1/M$ is equal to the change of $\zeta$ when $p_f$ changes, and it is measured at constant strain. This parameter can be expressed in terms of the three fundamental ones previously defined:

$$\frac{1}{M} = \left( \frac{\partial \zeta}{\partial p_f} \right)_{s_s} = \frac{1}{R} \frac{K_b}{H^2} \quad \text{(A9)}$$

**Deformations coupling coefficient:** The parameter $C$ represents the coupling of deformations between the solid grains and the fluid:

$$C = \frac{K_b}{H} M \Rightarrow b = \frac{K_b}{H} = \frac{C}{M} \quad \text{(A10)}$$

**The Skempton coefficient:** $B$ represents the change in pore pressure when the applied stress changes for undrained conditions:

$$B = \left( \frac{\Delta p_f}{\Delta p_h} \right)_{n_r} = -\left( \frac{\Delta p_f}{\Delta \sigma} \right)_{\zeta} = \frac{R}{H} \quad \text{(A11)}$$