

## **DOUBLE POROSITY MODEL WITH TRANSIENT INTERPOROSITY FLOW FOR THE RESPONSE OF TRACERS IN NATURALLY FRACTURED RESERVOIRS, CONSIDERING CONSTANT MASS FLUX INJECTION**

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### **ABSTRACT**

In tracer test interpretation, fissured reservoirs are commonly referred to as double porosity systems. Different approaches have been proposed for analyzing their well tracer concentration responses. This work presents a double porosity model with *transient* fracture-matrix transfer, considering constant mass flux at the injection well, for the interpretation of the observed tracer response in the producing wells for continuous injection of a tracer. A new parameter is presented which represents the gradient of mass flow that governs the fracture-matrix transfer and it is used to characterize the porous media. The solution for the proposed mathematical model was obtained in Laplace space.

The numerical inversion was carried out with Stehfest's algorithm. In addition, approximate analytical solutions for short and long dimensionless time are obtained, and are compared with the solution obtained by numerical inversion, providing satisfactory results. The values of the numerical inversion were used to generate the "type curve", presented in terms of the dimensionless groups obtained from the approximate analytical solution.

### **INTRODUCTION**

Radioactive tracers are substances that are added to the injected fluid and they are used to study the trajectory of the fluids inside a reservoir, as they advance toward the producing wells. A tracer should meet certain characteristics, among others: not interfere with the fluid flow, require a small concentration and easy detection. Another purpose of introducing a tracer in the injected fluid is to detect the fractures preferential orientation or discontinuities in the reservoir. In any project of fluid injection, the channels of high permeability in the reservoir can quickly establish "short-circuits" for the injected fluid, reducing the efficiency of the process drastically, and in some cases can cause the failure of

the project. The field tests with well to well tracer injection and the data analysis with the correct model, provide the possibility for the improvement of the reservoir characterization.

The information that is possible to obtain from well to well tracer tests is the following: sweeping efficiency, identification of high permeability zones, inadequate injectors wells, preferential flow tendencies, location of barriers, relative velocity of the injected fluids, and in general the trajectories of the fluids injected into the reservoir toward the producing wells. Radioactive tracers that have been successfully used in the industry are: tritiated methane, ammonium nitrate, isopropilic alcohol, thorium, tritium, deuterium, krypton 85, iodide and strontium. These tracers have fulfilled the national and international norms of security.

Radioactive tracers such as tritium and krypton 85 emit beta radiation, and can be detected using a highly sensitive proportional counter, in concentrations as low as of one picocurie/liter. Tritium in water is detected using a Geiger counter coupled with wireline. With respect to chemical tracers, these must be used in high concentrations, are relatively most expensive and easily detectable. The perfluorinated molecules are of great interest as gas tracers. SF<sub>6</sub> has been used with success for many years. The focus has been put on perfluorinated cyclic molecules, like perfluorodimethylcyclobutane (PDMCB) and, perfluoromethylcyclopentane; these are commonly named PFTs. The PFTs are non toxic products, low-cost and with an exceptionally low detection limit with Geiger counters. The first field injection of PFTs was carried out in the Ekofisk field in the North Sea in 1987, on a trial-and error basis. The mass of radioactive tracer injected depends among other factors on the travel distance half life, degree of adsorption in the rock, high permeability channels, (short circuits), formation temperature, magnitude of the inaccessible pore volume, etc.

The injection of a tracer in a naturally fractured reservoirs allows through a solution of the mathematical models that describe these flow problems, the interpretation of the response in producers wells, as a result we can estimate parameters of practical interest.

The purpose of this study is to present a new model with transient fracture-matrix transfer, for the tracer response in the producing wells, considering the continuous injection of a tracer. The main contributions of this work consist in taking into account radioactive decay, the inaccessible matrix pore volume, the definition of a dimensionless radius similar to that previously used in well test analysis (instead of the traditional one that is referred to as the mixing coefficient [dispersivity]), the approximate analytical solutions for short and long dimensionless times, and a type curve that allows the estimation of the radial dispersion coefficient ( $D_r$ ). Models previously presented for Tracer Flow Study of the tracer dispersion in naturally fractured reservoirs is of growing interest for the characterization of a geothermal reservoir, mainly due to the current importance of fluid injection projects to improved the recovery of energy. The dispersion effect controls the success of a tracer injection process. In the past, several papers have discussed the theory related to the flow of tracers in porous media. A complete revision was presented by Perkins and Johnston (1963), as well as Pozzi and Blackwell (1963); Raimondi et al. (1959), presented an approximate solution for the chemical tracer flow under radial flow conditions. Bentsen and Nielsen (1965) presented laboratory data for radial systems, and showed that the tracer dispersion behavior can be appropriately described by means of the solution of Raimondi et al., when the mobility relationship is favorable. Brigham and Smith<sup>8</sup> applied the solution of Raimondi et al. to predict the behavior of a chemical tracer in a five spot pattern. Yuen, Brigham and Cinco (1979), presented a methodology to predict the radial flow behavior of a chemical tracer in a stratified reservoir, where the response showed peaks depending on strata characteristics. Later, Abbaszadeh and Brigham (1984) continued this work to determine stratified characteristics. Moench and Ogata (1981), presented a solution in Laplace space for chemical tracer radial flow, and obtained the numerical inversion with the algorithm of Stehfest (1970), and through a finite difference solution. Tang and Babu (1979) presented a solution for the radial dispersion problem and confirmed the results of Moench and Ogata. Hsieh (1986) presented the solution to the problem of Moench and Ogata, expressed in terms of an integral in the complex plane.

Ramírez S. et al.(1991) obtained a solution in Laplace space for the radial flow of tracers in

naturally fractured reservoirs, and carried out the inversion of their Laplace space solution with Crump's algorithm, coupled with Epsilon's algorithm for acceleration. They considered slabs and cubic models for the matrix-fracture geometry.

## PROPOSED MATHEMATICAL MODEL

With the purpose of allowing a mathematical analysis of the tracer flow problem, it is necessary to model the real system, irregular and complex, composed of matrix and fractures by matrix blocks that have the same size and form (**Fig. 1**). High percent of the connected pore volume in naturally fractured reservoirs is not accessible to injection; thus changes are required to include inaccessible pore volume in the mathematical models, because it affects tracer propagation significantly.

The double porosity model proposed in this work considers the following assumptions:

1. The matrix and fractures are homogeneous and compressible systems.
2. Fluid is injected across the fractures and then flows from the fractures to matrix.
3. There is not resistance to flow between the fracture and matrix.
4. The matrix-fracture geometries considered are slabs and cubic blocks (**Fig. 1**).
5. Injected flow rate is constant and uniformly distributed over the interval.
6. Fracture width is small compared with that of the matrix block.
7. The effective diffusivity coefficient in the fractures is constant, and in the matrix the longitudinal dispersion coefficient is proportional to the radial velocity.
8. Diffusion in the fractures and in the matrix obeys Fick's law.
9. A uniform vertical gradient concentration exists in the fractures in z direction.
10. The porous media are of infinite extent.

Under the previous assumptions, the equation for the radial flow of tracers in naturally fractured reservoirs can be expressed:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r D_r \frac{\partial C_f(r,t)}{\partial r} \right) - v_r \frac{\partial C_f(r,t)}{\partial r} - \lambda [C_f(r,t) - C_i] - \frac{J^*(h_f, t)}{\phi_{fb} \sigma} = \frac{\partial C_f(r,t)}{\partial t} \quad (1)$$

The longitudinal dispersion and radial velocity are expressed by Eqs. (2) and (3):

$$D_r = \alpha v_r \quad (2)$$

$$v_r = \frac{q}{2\pi r h} = \frac{a}{r} \quad (3)$$

Combining Eqs. (2) and (3):

$$rD_r = \alpha a \quad (4)$$

Substituting Eqs. (3) and 4 in Eq. (1), the equation for the radial flow of tracers in naturally fractured reservoirs is obtained:

$$\frac{\alpha a}{r} \frac{\partial^2 C_f(r,t)}{\partial r^2} - \frac{a}{r} \frac{\partial C_f(r,t)}{\partial r} - \lambda [C_f(r,t) - C_i] - \frac{J^*(h_f,t)}{\phi_{fb}\sigma} = \frac{\partial C_f(r,t)}{\partial t} \quad (5)$$

Where the tracer mass transfer by rock volume unit  $J^*(h_f,t)$  is given by Eq. (6):

$$J^*(h_f,t) = \frac{D_m}{V_b} \frac{\partial C_m(h,t)}{\partial z} \quad (6)$$

Substituting Eq. (6) in Eq. (5), the equation for the radial flow of tracers in naturally fractured reservoirs can be expressed as follows:

$$\frac{\alpha a}{r} \frac{\partial^2 C_f(r,t)}{\partial r^2} - \frac{a}{r} \frac{\partial C_f(r,t)}{\partial r} - \lambda [C_f(r,t) - C_i] + \frac{D_m}{\phi_{fb}\sigma V_b} \frac{\partial C_m(h_f,t)}{\partial z} = \frac{\partial C_f(r,t)}{\partial t} \quad (7)$$

It is important to notice that in the left-hand side of Eq. (7) the first term refers to the tracer transfer by *dispersion*, the second to the concentration change of tracer due to the *convection*, the third to the *radioactive decay*, and the fourth term refers to *fracture-matrix transfer*, where the new interporosity flow coefficient governs the tracer flow to the matrix, and therefore controls the time period of the transition between the early tracer flow only through the fractures to the composite by fractures and matrix tracer flow. The right-hand side member of Eq. (7) considers the concentration change with time that represents the cumulative effect.

Equation for linear flow in the matrix taking into account the inaccessible pore volume:

$$D_m \frac{\partial^2 C_m(z,t)}{\partial z^2} - \lambda [C_m(z,t) - C_i] - [1-f] \frac{\partial C_{sm}(z,t)}{\partial t} = f \frac{\partial C_m(z,t)}{\partial t} \quad (8)$$

In Eq. (8) the first term on the left-hand side refers to the tracer transfer by *diffusion*, the second to the *radioactive decay*, and the third term represents the loss of tracer to the inaccessible pore volume; the right member of Eq. (8) considers the concentration change with the time, that represents the cumulative effect.

Transfer between two zones in the matrix by inaccessible pore volume:

$$\frac{\partial C_{sm}(z,t)}{\partial t} = \frac{M_m}{1-f} [C_m(z,t) - C_{sm}(z,t)] \quad (9)$$

The dimensionless model for the radial flow of a tracer in a fractured system, is given by Eqs. (10) to (19).

Tracer flow equation:

$$\frac{\alpha_D}{r_D} \frac{\partial^2 C_{fD}(r_D,t_D)}{\partial r_D^2} - \frac{1}{r_D} \frac{\partial C_{fD}(r_D,t_D)}{\partial r_D} - \lambda_D C_{fD}(r_D,t_D) + \frac{1}{\sigma_D} \frac{\partial C_{mD}(z_{Df},t_D)}{\partial t_D} = \frac{\partial C_{fD}(r_D,t_D)}{\partial t_D} \quad (10)$$

Initial condition:

$$C_{fD}(r_D, 0) = 0 \quad (11)$$

Internal boundary condition: constant mass flux at injection well.

$$\frac{\partial C_{fD}(1,t_D)}{\partial t_D} = -1 \quad (12)$$

External boundary condition: infinite medium:

$$\lim_{r_D \rightarrow \infty} C_{fD}(r_D,t_D) = 0 \quad (13)$$

Dimensionless equation for linear flow in the matrix, taking into account the inaccessible pore volume:

$$D_{mD} \frac{\partial^2 C_{mD}(z_D, t_D)}{\partial z_D^2} - \lambda_D C_{mD}(z_D, t_D) - [1-f] \frac{\partial C_{smD}(z_D, t_D)}{\partial t_D} = f \frac{\partial C_{mD}(z_D, t_D)}{\partial t_D} \quad (14)$$

Initial condition:

$$C_{mD}(z_D, 0) = 0 \quad (15)$$

Internal boundary condition: free interaction between fracture and matrix:

$$C_{mD}(z_{fD}, t_D) = C_{fD}(r_D, t_D) \quad (16)$$

External boundary condition: closed system:

$$\frac{\partial C_{mD}(z_{HD}, t_D)}{\partial z_D} = 0 \quad (17)$$

Transfer between two zones in the matrix by adsorption:

$$\frac{\partial C_{smD}(z_D, t_D)}{\partial t_D} = \frac{M_{mD}}{1-f} [C_{mD}(z_D, t_D) - C_{smD}(z_D, t_D)] \quad (18)$$

Initial condition:

$$C_{smD}(z_D, 0) = 0 \quad (19)$$

### **Dimensionless variables:**

Dimensionless radius:

$$r_D = \frac{r}{r_w} \quad (20)$$

Dimensionless time:

$$t_D = \frac{\alpha_D a_f t}{r_w^2} \quad (21)$$

Dimensionless length in the matrix:

$$z_D = \frac{z}{r_w} \quad (22)$$

Dimensionless fracture half width:

$$z_{fD} = \frac{h_f}{r_w} \quad (23)$$

Dimensionless (symmetrical) maximum distance for tracer flow in the matrix:

$$z_{HD} = \frac{H + h_f}{r_w} \quad (24)$$

Dimensionless tracer concentration in the fractures:

$$C_{fD}(r_D, t_D) = \frac{D_m 2\pi h}{\dot{m}_w} [C_i - C_f(r, t)] \quad (25)$$

where  $\dot{m}_w$  is defined by Eq. (A-1).

Dimensionless tracer concentration in the matrix:

$$C_{mD}(z_D, t_D) = \frac{D_m 2\pi h}{\dot{m}_w} [C_i - C_m(z, t)] \quad (26)$$

Dimensionless tracer concentration in the matrix inaccessible pore volume:

$$C_{smD}(z_D, t_D) = \frac{D_m 2\pi h}{\dot{m}_w} [C_i - C_{sm}(z, t)] \quad (27)$$

Dimensionless mixing coefficient:

$$\alpha_D = \frac{\alpha}{r_w} \quad (28)$$

Dimensionless interporosity coefficient:

$$\sigma_D = \frac{\phi_{fb} V_b a \sigma}{r_w D_m} \quad (29)$$

Dimensionless effective diffusivity in the matrix:

$$D_{mD} = \frac{D_m}{\alpha_D a} \quad (30)$$

Dimensionless adsorption coefficient for the inaccessible pore volume:

$$M_{mD} = \frac{r_w^2 M_m}{\alpha_D a} \quad (31)$$

Dimensionless radioactive decay constant:

$$\lambda_D = \frac{r_w^2 \lambda}{\alpha_D a} \quad (32)$$

The wellbore concentration in Laplace space for the problem considered is (see Appendix A):

$$\bar{C}_{wD}(s) = \frac{-1}{s \left[ 0.5\alpha_D + (\beta(s))^{1/3} Ai'(Y_0)/Ai(Y_0) \right]} \quad (33)$$

where:

$$Y_0 = \frac{\beta(s) + 1/4\alpha_D^2}{(\beta(s))^{2/3}} \quad (34)$$

$$m(s) = \frac{1}{D_{mD}} \left[ \frac{[1-f]M_{mD}}{M_{mD} + s[1-f]} + f + \frac{\lambda_D}{s} \right] \quad (35)$$

For slabs matrix:

$$\beta(s) = \frac{\sqrt{sm(s)}}{\alpha_D \sigma_D} \left[ \frac{h(z_{fD} \sqrt{sm(s)})}{1 - h(z_{fD} \sqrt{sm(s)})} - \frac{h(z_{fD} \sqrt{sm(s)})}{h(z_{fD} \sqrt{sm(s)})} \right] + \frac{\lambda_D}{\alpha_D} + \frac{s}{\alpha_D} \quad (36)$$

For cubic matrix:

$$\beta(s) = \frac{1}{\alpha_D \sigma_D} \left[ \sqrt{sm(s)} \quad h(\sqrt{sm(s)} z_{Df}) - \frac{1}{z_{Df}} \right] + \frac{\lambda_D}{\alpha_D} + \frac{s}{\alpha_D} \quad (36-b)$$

### Approximate analytical solution for short times

During the early injection phase when a small pore volume has been injected, the fracture-matrix mass

transfer is negligible, and the naturally fractured formation behaves as a “homogeneous in fractures media”. Thus for short dimensionless times, it is possible to invert Eq. (33) (see Appendix C):

$$C_{wD}(t_D) = \frac{t_D^{1/3}}{0.2968\Gamma(4/3)} \quad (37)$$

### Approximate analytical solution for long times

For long dimensionless times the solution for the radial flow of a radioactive tracer in naturally fractured reservoirs, Eqs. (10) and (33) (see Appendix C), can be expressed as follows:

$$C_{wD}(t_D) = \frac{t_D^{1/4}}{0.6619 \Gamma(5/4)} \quad (38)$$

These solutions given by Eqs. (37) and (38) were used in the present work to test the validity of the numerical inversion results.

### Derivation of the dimensionless groups

If the analytical solution for short times for the continuous radial flow of a radioactive tracer given by Eq. (38), is derived with respect to time  $t_D$  and multiplied by  $t_D$ , a useful way to present the results of this problem is obtained in terms of:

$$t_D \frac{dC_{wD}(t_D)}{dt_D} \text{ vs. } t_D \quad (39)$$

### Numerical inversion

The comparison between the numerical inversion with Stehfest's algorithm and the approximate analytical solutions for short and long dimensionless times indicates that the numerical solution is correct. (Figs. 2 and 3) were generated through numerical inversion using the algorithm of Stehfest (1970). Fig. 3 shows in addition to the tracer response a graph of its logarithmic derivative.

Numerical inversion has the advantage that the calculation time is smaller than the time simulation using finite differences. In addition, it has been concluded that the numerical inversion is efficient for any boundary condition.

## CONCLUSIONS

The main aim of this work has been to develop an improved model and its solution for the flow of tracers in naturally fractured reservoirs, that considers injection under constant mass flux. This model assumes transient fracture matrix tracer transfer.

From the result of this work, the following conclusions can be established:

1. A solution for the tracer response has been derived.
2. Approximate analytical solutions were presented for short and long dimensionless times, which can be used to interpret the response of the radioactive tracer, and to obtain a representative value of the "in situ" mixing coefficient.
3. A dimensionless "type curve" was developed for the interpretation of the continuous constant mass flux injection of a radioactive tracer.
4. The combination of the interpretation techniques through the "type curve" and of the analytical solutions, allows the improvement of reservoir characterization obtained by the interpretation of a tracer injection test.

## NOMENCLATURE

$a_f$	= injection constant, Eqs. 2 and 3 = $q/2\pi h$ , $L^2/T$ .
$C_o$	= initial tracer concentration, $M/L^3$ .
$C_{sm}(z, t)$	= tracer concentration absorbed in the matrix, $M/L^3$ .
$C_m(z, t)$	= tracer concentration in the matrix, $M/L^3$ .
$C_f(r, t)$	= tracer concentration in the fractures, $M/L^3$ .
$D_r$	= longitudinal dispersion coefficient, $L^2/T$ .
$D_m$	= effective diffusivity coefficient, $L^2/T$ .
$f$	= ratio between accessible and inaccessible porosity
$G(r, t)$	= mass as a function of the radial distance and time.
$h$	= thickness of the porous media, $L$ .
$h_f$	= fracture half width, $L$ .
$H$	= half of the average matrix block size, $L$ .
$J$	= mass flux density, $M/L^2T$ .

$J^*$	= rock volume unit, mass transfer $(M/L^3)/(L^2T)$ .
$M$	= amount solute mass injected
$q_i$	= injection flow rate in the porous media, $L^3/T$ .
$r$	= radial distance, $L$ .
$t$	= time
$U$	= macroscopic velocity, $L/T$ .
$v$	= microscopic velocity, $L/T$ .
$V_u$	= fluid volume in the fracture, transfer area of the matrix block multiply by half of the width fracture, $L^3$ .
$Ai(z)$ and $Bi(z)$	= Airy functions.
$\beta(s)$	= transfer function in the Laplace space, Eq. 10.
$\delta(t)$	= Dirac's $\delta$ -function.

## Greek symbols

$\alpha$	= fracture mixing coefficient $\alpha$ (dispersivity), $L$ .
$\phi$	= effective porosity, dimensionless.
$\lambda$	= radioactive decay constant, $1/T$ .
$\kappa$	= mass flow gradient, $1/T$ .
$\sigma$	= interporosity shape factor, $1/L^2$ .

## Subscripts

$b$	= bulk.
$D$	= dimensionless.
$f$	= fracture.
$m$	= matrix.
$sm$	= inaccessible matrix pore volume.
$w$	= well.

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#### **APPENDIX A. INTERNAL BOUNDARY CONDITION USING FICK'S LAW.**

The mass flux is generated due to a gradient concentration across a perpendicular area, and it is proportional to the molecular diffusion constant:

$$\dot{m}(r,t) = \frac{\partial G(r,t)}{\partial t} = D_m A \frac{\partial C_f(r,t)}{\partial r} \quad (\text{A-1})$$

The mass flux per unit area (mass flux density) can be obtained from the previous Eq. A-1:

$$\frac{\dot{m}(r,t)}{A} = J_f(r,t) = D_m \frac{\partial C_f(r,t)}{\partial r} \quad (\text{A-2})$$

Solving for the derivative:

$$\frac{\partial C_f(r,t)}{\partial r} = \frac{J_f(r,t)}{D_m} \quad (\text{A-3})$$

For the conditions at the well:

$$\frac{\partial C_f(r_w,t)}{\partial r} = \frac{J_f(r_w,t)}{D_m} \quad (\text{A-4})$$

Defining mass flux at the well:

$$\dot{m}(r_w,t) = \dot{m}_w \quad (\text{A-5})$$

Substituting Eq. (A-5) in Eq. (A-4), the internal boundary condition for a continuous constant mass flux tracer injection is obtained:

$$\frac{\partial C_f(r_w,t)}{\partial r} = \frac{\dot{m}_w}{D_m 2\pi r_w h} \quad (\text{A-6})$$

The internal boundary for conditions of a constant mass flux pike injection:

$$\frac{\partial C_f(r_w,t)}{\partial r} = \frac{M\delta(t)}{D_m 2\pi r_w h} \quad (\text{A-7})$$

Initial condition: uniform distribution.

$$C_f(r,0) = C_i \quad (\text{A-8})$$

$$C_{fD}\left(\frac{r}{r_w}, \frac{\alpha_D a[0]}{r_w^2}\right) = \frac{D_m 2\pi h}{\dot{m}_w} [C_i - C_f(r,0)] = \frac{D_m 2\pi h}{\dot{m}_w} [C_i - C_i] = 0$$

$$C_{fD}(r_D, 0) = 0 \quad (\text{A-9})$$

Internal boundary condition: *constant mass flux*:

$$\frac{\partial C_f(r_w,t)}{\partial r} = \frac{\dot{m}_w}{D_m 2\pi r_w h} \quad (\text{A-10})$$

$$\frac{-\dot{m}_w}{D_m 2\pi r_w h} \frac{\partial C_{fD}(1, t_D)}{\partial r_D} = \frac{\dot{m}_w}{D_m 2\pi r_w h} \frac{\partial C_{fD}(1, t_D)}{\partial r_D} = -1 \quad (\text{A-11})$$

Internal boundary condition: *pulse mass flux*:

$$\frac{\partial C_f(r_w,t)}{\partial r} = \frac{M\delta(t)}{D_m 2\pi r_w h} \quad (\text{A-12})$$

$$\frac{-\dot{m}_w}{D_m 2\pi r_w h} \frac{\partial C_{fD}(1, t_D)}{\partial r_D} = \frac{M\delta(t)_D}{D_m 2\pi r_w h}$$

$$\frac{\partial C_{fD}(1, t_D)}{\partial r_D} = -T_0 \delta(t)_D \quad (\text{A-13})$$

where  $T_0$  is given by Eq. (A-14).

$$T_0 = \frac{M}{\dot{m}_w} \quad (\text{A-14})$$

External boundary condition: infinite reservoir:

$$\lim_{r \rightarrow \infty} C_f(r, t) = C_i \quad (\text{A-15})$$

$$\lim_{\substack{r \rightarrow \infty \\ r_m}} C_{fD} \left( \frac{[r]}{r_w}, \frac{\alpha_D a[t]}{r_w^2} \right) = \frac{D_m 2\pi h}{\dot{m}_w} [C_i - \lim_{r \rightarrow \infty} C(r, t)]$$

$$\lim_{r_D \rightarrow \infty} C_{fD}(r_D, t_D) = 0 \quad (\text{A-16})$$

## **APPENDIX B. SOLUTION IN LAPLACE SPACE FOR THE FLOW OF TRACERS IN NATURALLY FRACTURED RESERVOIRS.**

Flow in the matrix taking into account the inaccessible pore volume:

$$D_{mD} \frac{\partial^2 C_{mD}(z_D, t_D)}{\partial z_D^2} - \lambda_D C_{mD}(z_D, t_D) - [1-f] \frac{\partial C_{smD}(z_D, t_D)}{\partial t_D}$$

$$= f \frac{\partial C_{mD}(z_D, t_D)}{\partial t_D} \quad (\text{B-1})$$

Initial condition:

$$C_{mD}(z_D, 0) = 0 \quad (\text{B-2})$$

Internal boundary condition: free fracture matrix interaction:

$$C_{mD}(z_{fD}, t_D) = C_{fD}(r_D, t_D) \quad (\text{B-3})$$

External boundary condition for the matrix block:

$$\frac{\partial C_{mD}(z_{DH}, t_D)}{\partial z_D} = 0 \quad (\text{B-4})$$

Transfer in the matrix between the effective and non-effective (inaccessible) pore volume:

$$\frac{\partial C_{smD}(z_D, t_D)}{\partial t_D} = \frac{M_{mD}}{1-f} [C_{mD}(z_D, t_D) - C_{smD}(z_D, t_D)] \quad (\text{B-5})$$

Initial condition:

$$C_{smD}(z_D, 0) = 0 \quad (\text{B-6})$$

Applying the Laplace transform to equation (B-5), substituting the initial condition given by Eq. (B-6) and solving for the tracer concentration in the inaccessible matrix volume pore:

$$\bar{C}_{smD}(z_D, s) = \left[ \frac{M_{mD}}{(1-f)s + M_{mD}} \right] \bar{C}_{mD}(z_D, s) \quad (\text{B-7})$$

Applying the Laplace transform to the Eq. (B-1) and substituting the initial conditions (B-2) and (B-6):

$$D_{mD} \frac{d^2 \bar{C}_{mD}(z_D, s)}{dz_D^2} - \lambda_D \bar{C}_{mD}(z_D, s) - (1-f) s \bar{C}_{smD}(z_D, s)$$

$$= f s \bar{C}_{mD}(z_D, s) \quad (\text{B-8})$$

Substituting Eq. (B-7) in Eq. (B-8):

$$\frac{d^2 \bar{C}_{mD}(z_D, s)}{dz_D^2} - \frac{1}{D_{mD}} \left[ \frac{\lambda_D}{s} + f + \frac{M_{mD}(1-f)}{(1-f)s + M_{mD}} \right] s \bar{C}_{mD}(z_D, s) = 0 \quad (\text{B-9})$$

The general solution for flow in the matrix can be obtained solving Eq. (B-9):

$$\bar{C}_{mD}(z_D, s) = A \cosh(z_D \sqrt{sm(s)}) + B \sinh(z_D \sqrt{sm(s)}) \quad (\text{B-10})$$

where:

$$m(s) = \frac{1}{D_{mD}} \left[ \frac{(1-f)M_{mD}}{M_{mD} + s(1-f)} + f + \frac{\lambda_D}{s} \right] \quad (\text{B-11})$$

Applying the Laplace transform to Eqs. (B-3) and Eq. (B-4):

$$\bar{C}_{mD}(z_{fD}, s) = \bar{C}_{fD}(r_D, s) \quad (\text{B-12})$$



$$\frac{d\bar{C}_{mD}(z_{DH}, s)}{dz_D} = 0 \quad (\text{B-13})$$

Applying the condition Eq. (B-13) to the general solution for matrix:

$$B = -A \tanh(z_{DH} \sqrt{sm(s)}) \quad (\text{B-14})$$

Substituting Eq. (B-14) in Eq. (B-10):

$$\bar{C}_{mD}(z_D, s) = A \left[ \cosh(z_D \sqrt{sm(s)}) - \tanh(z_{DH} \sqrt{sm(s)}) \sinh(z_D \sqrt{sm(s)}) \right] \quad (\text{B-15})$$

Evaluating Eq. (B-15) in the fracture-matrix interface, and using (B-12):

$$A = \frac{\bar{C}_{fD}(r_D, s)}{\cosh(z_{Df} \sqrt{sm(s)}) - \tanh(z_{DH} \sqrt{sm(s)}) \sinh(z_{Df} \sqrt{sm(s)})} \quad (\text{B-16})$$

Substituting Eq. (B-16) in Eq. (B-15), obtaining the equation that represents the tracer matrix concentration in function of the fracture concentration:

$$\bar{C}_{mD}(z_D, s) = \left[ \frac{(z_D \sqrt{sm(s)}) - (z_{DH} \sqrt{sm(s)})}{(z_{Df} \sqrt{sm(s)}) - (z_{DH} \sqrt{sm(s)})} \frac{(z_D \sqrt{sm(s)})}{(z_{Df} \sqrt{sm(s)})} \right] \bar{C}_{fD}(r_D, s) \quad (\text{B-17})$$

The gradient concentration in the matrix block is obtained deriving Eq. (B-17), and the resulting expression at the fracture-matrix interface is:

$$\frac{d\bar{C}_{mD}(z_{Df}, s)}{dz_D} = \sqrt{sm(s)} \left[ \frac{(z_{Df} \sqrt{sm(s)}) - (z_{DH} \sqrt{sm(s)})}{1 - (z_{DH} \sqrt{sm(s)})} \frac{(z_{Df} \sqrt{sm(s)})}{(z_{Df} \sqrt{sm(s)})} \right] \bar{C}_{fD}(r_D, s) \quad (\text{B-18})$$

Radial tracer flow in the fractures, source term including the that considers the fracture matrix interaction:

$$\frac{\alpha_D}{r_D} \frac{\partial^2 C_{fD}(r_D, t_D)}{\partial r_D^2} - \frac{1}{r_D} \frac{\partial C_{fD}(r_D, t_D)}{\partial r_D} - \lambda_D C_{fD}(r_D, t_D) + \frac{1}{\sigma_D} \frac{\partial C_{mD}(z_{Df}, t_D)}{\partial t_D} = \frac{\partial C_{fD}(r_D, t_D)}{\partial t_D} \quad (\text{B-19})$$

Initial condition: uniform distribution:

$$C_{fD}(r_D, 0) = 0 \quad (\text{B-20})$$

Internal boundary: constant mass flux:

$$\frac{\partial C_{fD}(1, t_D)}{\partial t_D} = -1 \quad (\text{B-21})$$

External boundary: infinite reservoir:

$$\lim_{r_D \rightarrow \infty} C_{fD}(r_D, t_D) = 0 \quad (\text{B-22})$$

Applying the Laplace Transform to Eq. (B-20) and substituting the initial condition given by (B-20):

$$\frac{\alpha_D}{r_D} \frac{d^2 \bar{C}_{fD}(r_D, s)}{dr_D^2} - \frac{1}{r_D} \frac{d\bar{C}_{fD}(r_D, s)}{dr_D} - \lambda_D \bar{C}_{fD}(r_D, s) + \frac{1}{\sigma_D} \frac{d\bar{C}_{mD}(z_{Df}, s)}{dz_D} = s \bar{C}_{fD}(r_D, s) \quad (\text{B-23})$$

Substituting the Eq. (B-18) in Eq. (B-23):

$$\frac{\alpha_D}{r_D} \frac{\partial^2 C_{fD}(r_D, t_D)}{\partial r_D^2} - \frac{1}{r_D} \frac{\partial C_{fD}(r_D, t_D)}{\partial r_D} - \lambda_D C_{fD}(r_D, t_D) + \frac{\sqrt{sm(s)}}{\sigma_D} \left[ \frac{(z_{Df} \sqrt{sm(s)}) - (z_{DH} \sqrt{sm(s)})}{1 - (z_{DH} \sqrt{sm(s)})} \frac{(z_{Df} \sqrt{sm(s)})}{(z_{Df} \sqrt{sm(s)})} \right] \bar{C}_{fD}(r_D, s) = s \bar{C}_{fD}(r_D, s)$$

This may be written:

$$\frac{d^2 \bar{C}_{fD}(r_D, s)}{dr_D^2} - \frac{1}{\alpha_D} \frac{d\bar{C}_{fD}(r_D, s)}{dr_D} - \beta(s) r_D \bar{C}_{fD}(r_D, s) = 0 \quad (\text{B-24})$$

where:

$$\beta(s) = \frac{\sqrt{sm(s)}}{\alpha_D \sigma_D} \left[ \frac{(z_{DH} \sqrt{sm(s)}) - (z_{Df} \sqrt{sm(s)})}{1 - (z_{DH} \sqrt{sm(s)})} \frac{(z_{Df} \sqrt{sm(s)})}{(z_{Df} \sqrt{sm(s)})} \right] + \frac{\lambda_D}{\alpha_D} + \frac{s}{\alpha_D} \quad (\text{B-25})$$

The general solution of the ordinary variables coefficients differential equation is given for Eq. (B-26):

$$\begin{aligned} \bar{C}_{fD}(r_D, s) = e^{\frac{r_D}{2\alpha_D}} & \left[ B_1 Ai \left( \frac{r_D \beta(s) + \frac{1}{4\alpha_D^2}}{(\beta(s))^{2/3}} \right) \right. \\ & \left. + B_2 Bi \left( \frac{r_D \beta(s) + \frac{1}{4\alpha_D^2}}{(\beta(s))^{2/3}} \right) \right] \end{aligned} \quad (\text{B-26})$$

Applying the Laplace transform to the boundary conditions given by Eq. (B-21) and (B-22):

$$\frac{d\bar{C}_D(1, s)}{dr_D} = -\frac{1}{s} \quad (\text{B-27})$$

$$\lim_{r_D \rightarrow \infty} \bar{C}_D(r_D, s) = 0 \quad (\text{B-28})$$

Applying the external boundary condition and considering that the Airy function  $Ai(x)$  for big arguments tend to zero,  $B_2 = 0$  in Eq. (B-26).

Substituting the constant  $B_1$  in Eq. (B-26):

$$\bar{C}_D(r_D, s) = e^{\frac{r_D}{2\alpha_D}} B_1 Ai \left( \frac{r_D \beta(s) + 1/4\alpha_D^2}{\beta(s)^{2/3}} \right) \quad (\text{B-29})$$

Applying the internal boundary condition:

$$\begin{aligned} \frac{d\bar{C}_{fD}(1, s)}{dr_D} &= \frac{1}{2\alpha_D} e^{\frac{1}{2\alpha_D}} B_1 Ai \left( \frac{1\beta(s) + 1/4\alpha_D^2}{(\beta(s))^{2/3}} \right) \\ &+ e^{\frac{1}{2\alpha_D}} (\beta(s))^{1/3} B_1 Ai \left( \frac{1\beta(s) + 1/4\alpha_D^2}{(\beta(s))^{2/3}} \right) = -\frac{1}{s} \end{aligned}$$

From this equation,  $B_1$  can be expressed:

$$B_1 = \frac{-1}{s} \frac{e^{-\frac{1}{2\alpha_D}}}{\left[ Ai(Y_0)/2\alpha_D + (\beta(s))^{1/3} Ai(Y_0) \right]} \quad (\text{B-30})$$

Substituting this constant in the general solution given by Eq. (B-26):

$$\bar{C}_{fD}(r_D, s) = \frac{-e^{\frac{r_D-1}{2\alpha_D}}}{s} \frac{Ai \left( \frac{r_D \beta(s) + 1/4\alpha_D^2}{\beta(s)^{2/3}} \right)}{\left[ Ai \left( \frac{\beta(s) + 1/4\alpha_D^2}{\beta(s)^{2/3}} \right) / 2\alpha_D + (\beta(s))^{1/3} Ai \left( \frac{\beta(s) + 1/4\alpha_D^2}{\beta(s)^{2/3}} \right) \right]} \quad (\text{B-31})$$

The wellbore concentration:

$$\bar{C}_{wD}(s) = \bar{C}_{fD}(1, s) \quad (\text{B-32})$$

The wellbore concentration:

$$\bar{C}_D(1, s) = \frac{-1}{s \left[ 1/2\alpha_D + (\beta(s))^{1/3} Ai(Y_0)/Ai(Y_0) \right]} \quad (\text{B-33})$$

where:

$$Y_0 = \frac{\beta(s) + 1/4\alpha_D^2}{\beta(s)^{2/3}} \quad (\text{B-34})$$

Using the Bender and Orszag method (Bender and Orszag, 1978) we proposed a **new** original approximation for Airy Function (**Fig. 2**).

$$Ai(Y_0) = 0.355 e^{-0.729Y_0 - 0.18Y_0^2}$$

The **original analysis** for the relationship between the Airy function and derivative (**Fig. 3**):

$$\begin{aligned} \frac{Ai(Y_0)}{Ai'(Y_0)} &= \frac{0.355 [-0.729 - 0.36Y_0] e^{-0.729Y_0 - 0.18Y_0^2}}{0.355 e^{-0.729Y_0 - 0.18Y_0^2}} \\ &= -0.729 - 0.36Y_0 \end{aligned} \quad (\text{B-35})$$

Using the ratio between the Airy function and its derivate, we get for  $r = r_w$ :

$$\bar{C}_{wD}(s) = \frac{1}{s \left[ -1/2\alpha_D + (\beta(s))^{1/3} [0.36Y_0 + 0.729] \right]} \quad (\text{B-36})$$

where:

$$\beta(s) = \frac{\sqrt{sm(s)}}{\alpha_D \sigma_D} \left[ \frac{(z_{DH} \sqrt{sm(s)}) - (z_{DF} \sqrt{sm(s)})}{1 - (z_{DH} \sqrt{sm(s)}) (z_{DF} \sqrt{sm(s)})} \right] + \frac{\lambda_D + s}{\alpha_D \alpha_D} \quad (\text{B-37})$$

$$m(s) = \frac{1}{D_{mD}} \left[ \frac{(1-f)M_{mD}}{M_{mD} + s(1-f)} + f + \frac{\lambda_D}{s} \right] \quad (\text{B-38})$$

Appendix D discusses the solution of or a constant mass flux pulse injection.

### **APPENDIX C. APPROXIMATE ANALYTICAL SOLUTIONS**

**For large s values**  $\rightarrow \infty$  :

$$Y_0 \approx (\beta(s))^{1/3} \quad (\text{C-1})$$

When substituting practical values of **Table 1** in Eq. (B-39), the function  $m(s)$  presents an almost constant value (**Fig. 4**):

$$m(s) \approx 2500 \quad (\text{C-2})$$

Then:

$$\sqrt{sm(s)} \approx 50\sqrt{s} \quad (\text{C-3})$$

Then, when  $m(s)$  is substituted in  $\beta(s)$  given in Eq. (B-38), we obtain the behaviour for this function also shown in (**Fig 4**):

$$\beta(s) \approx 0.75\sqrt{s} \quad (\text{C-4})$$

Then:

$$Y_0 \approx (0.75\sqrt{s})^{1/3} \approx 0.908s^{1/6} \quad (\text{C-5})$$

Using the ratio of the Airy function given by Eq. (35):

$$Ai(Y_0)/Ai(Y_0) = -0.36[0.908s^{1/6}] - 0.729 \quad (\text{C-6})$$

Substituting (C-5) and (C-6) in the second term of the denominator of Eq. (B-33):

$$\begin{aligned} (\beta(s))^{1/3} Ai(Y_0)/Ai(Y_0) &= -0.908s^{1/6} \left[ 0.36[0.908s^{1/6}] + 0.729 \right] \\ &= -0.2968s^{1/3} - 0.6619s^{1/6} \end{aligned} \quad (\text{C-7})$$

Substituting Eq. (C-7) in Eq. (B-33):

$$\bar{C}_{wD}(s) = \frac{1}{s \left[ -0.5/\alpha_D - \left[ 0.2968s^{1/3} + 0.6619s^{1/6} \right] \right]} \quad (\text{C-8})$$

For large arguments ( $s \rightarrow \infty$ ), this expression can be written as,

$$\bar{C}_{wD}(s) = \frac{-1}{0.2968s \left[ s^{1/3} + 0.5/0.2968\alpha_D \right]} \quad (\text{C-9})$$

The analytical inversion for **short times** of Eq. (C-9) is given by Eq. (C-10):

$$C_{wD}(t_D) = \frac{t_D^{1/3}}{0.2968 \Gamma(4/3)} \quad (\text{C-10})$$

**For large times (short arguments,**  $\rightarrow$  ),

$$\sqrt{sm(s)} \approx 50\sqrt{s} \quad (\text{C-11})$$

Substituting this value in Eq. (B-37), (**Fig 5**):

$$\beta(s) \approx 0.75s^{3/4} \quad (\text{C-12})$$

Then, from Eq. (C-1):

$$Y_0 \approx (0.75s^{3/4})^{1/3} \approx 0.908s^{1/4} \quad (\text{C-13})$$

Using Eq. (C-13), the second term of the denominator of Eq. (B-33) can be expressed:

$$Ai(Y_0)/Ai(Y_0) = -0.36[0.908s^{1/4}] - 0.729 \quad (\text{C-14})$$

Similarly to the short times previous solution, substituting Eqs. (C-13) and (C-14) in the second term of the denominator of Eq. (B-33).

$$\begin{aligned} (\beta(s))^{1/3} Ai(Y_0)/Ai(Y_0) &= -0.908s^{1/4} \left[ 0.36[0.908s^{1/4}] + 0.729 \right] \\ &= -0.2968s^{1/2} - 0.6619s^{1/4} \end{aligned} \quad (\text{C-15})$$

Substituting Eq. (C-15) in Eq. (B-33):

$$\bar{C}_{wD}(s) = \frac{1}{s \left[ -0.5/\alpha_D - \left[ 0.2968s^{1/2} + 0.6619s^{1/4} \right] \right]} \quad (\text{C-16})$$

For large times arguments ( $s \rightarrow 0$ ),

$$\bar{C}_{wD}(s) = \frac{-1}{0.6619 s \left[ s^{1/4} + 0.5/0.6619 \alpha_D \right]} \quad (C-17)$$

The analytical inversion for **large times** is expressed as follows:

$$C_{wD}(t_D) = \frac{t_D^{1/4}}{0.6619 \Gamma(5/4)} \quad (C-18)$$

#### **APPENDIX D. A SOLUTION FOR CONSTANT MASS FLUX PULSE INJECTION.**

Applying the Laplace transform to the inner boundary condition for a pulse mass flux given by Eq. (A-13):

$$\frac{d\bar{C}_{fD}(1, s)}{dr_D} = -T_0 \quad (D-1)$$

The solution for a constant mass flux pulse:

$$\bar{C}_{fD}(r_D, s)_{pulse} = -T_0 e^{-r_D} \left[ \frac{Ai(Y_w)}{Ai(Y_0)/2\alpha_D + (\beta(s))^{1/3} Ai(Y_0)} \right] \quad (D-2)$$

where:

$$Y_w = \frac{r_D \beta(s) + 1/4\alpha_D^2}{\beta(s)^{2/3}} \quad (D-3)$$

and  $Y_0$  is given by Eq. (B-34).

The wellbore concentration for a constant mass flux pulse:

$$\bar{C}_{wD}(s)_{pulse} = T_0 \left[ \frac{1}{-1/2\alpha_D - (\beta(s))^{1/3} Ai(Y_0)/Ai(Y_0)} \right] \quad (D-4)$$

Considering only one pulse:

$$T_0 = 1 \quad (D-5)$$

Comparing Eq. (D-5) with Eq. (B-31):

$$\bar{C}_{wD}(s)_{pulse} = s\bar{C}_{wD}(s) \quad (D-6)$$

The analytical inversion of Eq. (D-6):

$$C_{wD}(t_D)_{pulse} = \frac{dC_{wD}(t_D)}{dt_D} \quad (D-7)$$

From this expression we observe the known fact that the derivative of the continuous tracer injection solution yields the pulse injection solution (Chen, 1987).

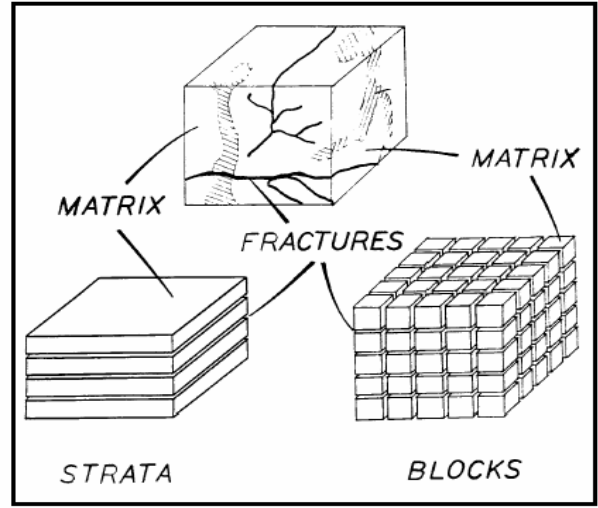


Fig. 1. Representation of a naturally fractured reservoir.

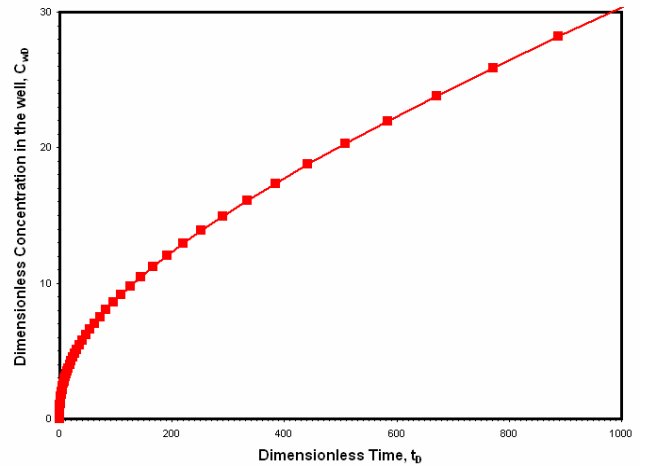


Fig. 2. Solutions for homogeneous continuous constant mass flux tracer injection, for radial flow in a slabs naturally fractured reservoir.

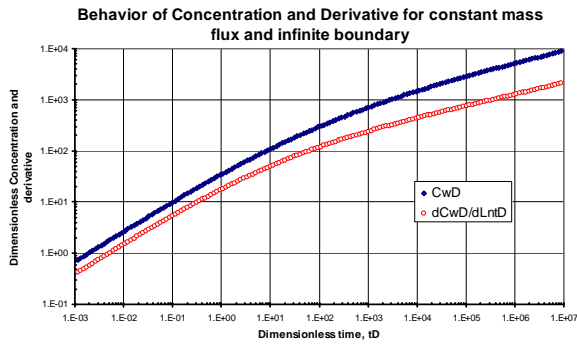


Fig. 3. Solutions for homogeneous continuous constant mass flux tracer injection, for radial flow in a slabs naturally reservoir with transient fracture-matrix transfer.

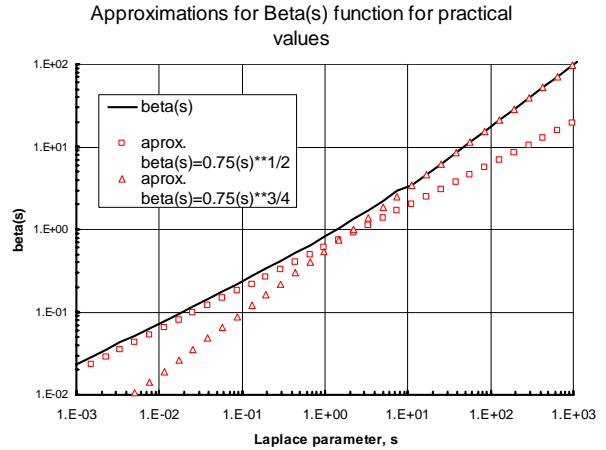


Fig. 6. Approximations for the Beta(s) function using the practical values of Table 1.

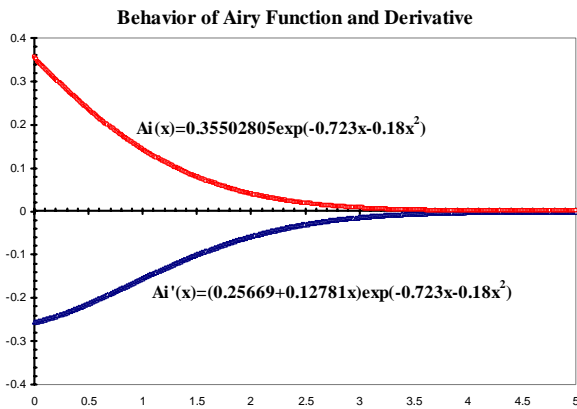


Fig. 4. Airy function and its derivative compared with data of Abramowitz and Stegun

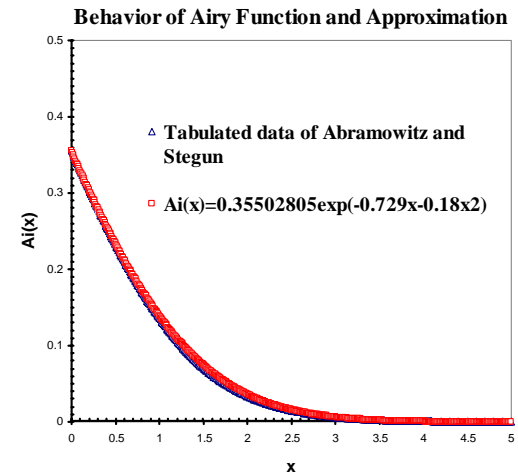


Fig. 7. New approximation for the Airy function.

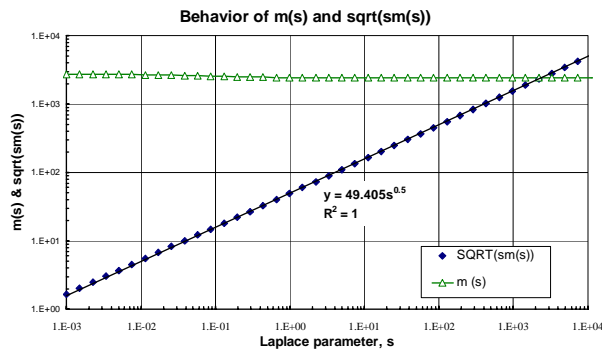


Fig. 5. The  $m(s)$  function is constant for several set of practical values; it only changes when the inaccessible pore volume is higher than 50%.

Table 1. Practical values used in the evaluation of  $m(s)$  function.

Q	h	a	$\alpha$	$M_m$	$D_m$	$\lambda$	$\sigma$
$m3/D$	m	$M2/H$	m	1/s	$m2/s$	1/s	adim
2	200	0.04	0.5	0.500	$1.E-04$	$1.E-06$	0.032

$\varphi_t$	$\varphi_{fb}$	$\varphi_{mb}$	H	$h_f$	$v_b$	$\varphi_{sm}$	$f = (\varphi_{mb} - \varphi_{sm}) / \varphi_{mb}$
fraction	fraction	fraction	m	m	M3	adim	adim
0.130	0.0169	0.1131	1.5	0.0085	3.43	0.01	0.89