A NEW MODEL FOR WELL TEST ANALYSIS IN A PURELY FRACTURED MEDIUM

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Abstract

In a porous medium the flow conduits are small and a large number of conduits are connected to the well. For this reason the medium appears to behave like a continuum on the scale of the well test, and volumetric averaging and continuum approximations are justified. On the contrary, in a fractured medium, only a small number of fractures may intersect the pumping well. These particular fractures will be stressed by a large gradient under well test conditions. Consequently, the early time behavior will be dominated by these fractures. The volumetrically averaged permeability does not control flow in the vicinity of an active well. The individual fractures close to the well must be characterized in order to understand the well test behavior especially if the hydraulic parameters of these fractures are significantly different from the average values for the entire system.

In the present study, a new analytical model is proposed for well test problems in fracture networks where the matrix is impermeable. The model accounts for the difference in the flow regime around the active well from that of the system as a whole. The analytical solutions are presented in a series of type curves for ranges of dimensionless parameters. The flow properties of the fracture system can be determined by curve matching.

Introduction

In the region near the pumping well, a single continuum approximation is not appropriate for modelling well tests. This is because the inner boundary condition of a well test causes flow to converge on the few fractures which intersect the well. These fractures are macroscopic features compared to the wellbore radius. However large the scale of observation may be, the characteristic length of the system still has to be the wellbore radius of the pumping well. In a porous medium the size of each flow conduit is microscopic and there are a large number of conduits connected to the well. In this case volumetric averaging and continuum approximations are justified. On the contrary, in a purely fractured medium, the small number of fractures that intersect the pumping well will be stressed by a large gradient. The volumetrically averaged permeability does not control flow in the vicinity of an active well. The individual fractures close to the well must be characterized in order to understand the well test behavior especially if the hydraulic parameters of these fractures are significantly different from the average values for the entire system.

In this study a new analytical solution is proposed for well test problems in purely fractured media based on a composite model with two concentric regions. The inner region contains a finite number of discrete fractures. The outer region is a classical porous medium. Similar composite models have been examined in the petroleum literature but none include the effects of fracture flow. The analysis of a well intersecting a single vertical fracture has also been developed. However, in this case flow enters the fracture from the faces. In our model the fluid enters and exits the fractures only from their intersections with other fractures. A solution is obtained for a finite radius well. The solution is presented in a series of type curves, so that the flow properties of the fracture system can be determined by curve matching.

Model Description

An isothermal well test problem in a homogenously fractured formation of uniform thickness \( H \) is considered. The fractures are assumed to be vertical and extend from the top to the bottom of the formation. It is further assumed that the rock matrix is impermeable and all the hydraulic parameters are independent of pressure. No wellbore storage or damage is considered. The conceptual model of the well test in this system consists of two zones. In the outer region the usual equivalent porous medium approximation is assumed to hold, i.e., the flow properties of the fractures are volumetrically averaged and a single continuum replaces the fractures. The hydraulic conductivity and the storage coefficient for the region are \( k_e \) and \( S_e \), respectively. The well is located in the center of the inner region and communicates with the outer region through a finite number \( n \) of fractures in the inner region. The radius of the well is \( r_w \) and the radius of the boundary between the inner and outer regions is \( r_f \). All the fractures in the inner region have the same hydraulic aperture \( a \) and the hydraulic parameters \( k_i \) and \( S_i \). It is assumed that there is an infinitesimally thin ring of infinite conductivity between the two regions so that the otherwise incompatible boundaries can be matched. Figure 2 illustrates the model. The details of the solution are given below and the final result can be found in Equation (28). We then examine the asymptotic behavior in small and large time in order to provide simpler solutions and check the results.

The governing equation for the inner region is that for one dimensional unsteady flow,
where the hydraulic diffusivity, $\alpha = k/S$. For the outer region the usual radial flow equation describes the flow.

\[
\frac{\partial^2 h_2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial h_2}{\partial r} = \frac{1}{\alpha_2} \cdot \frac{\partial h_2}{\partial t}.
\]  

The initial conditions and the boundary conditions for constant rate injection test are

\[
\begin{align*}
\text{(2)} & \quad h_1(r, 0) = 0 \quad (r_e \leq r \leq r_f) \\
\text{(3)} & \quad h_2(r, 0) = 0 \quad (r_f \leq r \leq \infty) \\
\text{(4)} & \quad -nk_1k_2 \frac{\partial h_1}{\partial r} = Q \quad (r = r_e).
\end{align*}
\]

For the continuity at the boundary of the inner and outer region, we have

\[
\begin{align*}
\text{(5)} & \quad h_1 = h_2 \quad (r = r_f) \\
\text{(6)} & \quad nk_1 \frac{\partial h_1}{\partial r} = 2r k_2 \frac{\partial h_2}{\partial r} \quad (r = r_f).
\end{align*}
\]

The following dimensionless parameters are defined:

\[
\begin{align*}
h_D &= \frac{2r k_2 J_R}{Q}, \quad t_D = \frac{\alpha_2 r^2}{k_2} \cdot \frac{r^2_f}{r_f^2} = \frac{\alpha_2 t_f}{r_f^2}, \quad r_D = \frac{r}{r_f}, \\
r_1 &= \frac{r_f}{r_1}, \quad \alpha_1 = \frac{\alpha_2}{2 \pi r_1}, \quad B = \frac{nk_1}{2 \pi r_1}.
\end{align*}
\]

Substituting into Eqs. (1a) and (1b) we obtain

\[
\begin{align*}
\text{(8)} & \quad \frac{\partial^2 h_D}{\partial t_D^2} = \frac{1}{\alpha_2} \cdot \frac{\partial h_D}{\partial t_D} \\
\text{(9)} & \quad \frac{\partial^2 h_D}{\partial r_D^2} + \frac{1}{r_D} \cdot \frac{\partial h_D}{\partial r_D} = \frac{\partial h_D}{\partial t_D}. \\
\text{In terms of the dimensionless parameters the boundary conditions become}
\end{align*}
\]

\[
\begin{align*}
\text{(10)} & \quad h_D(r_0, 0) = 0 \quad (r_e \leq r_0 \leq 1) \\
\text{(11)} & \quad h_D(1, 0) = 0 \quad (1 \leq r_0 \leq \infty) \\
\text{(12)} & \quad \frac{\partial h_D}{\partial t_D} = -\frac{1}{\beta} \quad (r_0 = 1) \\
\text{(13)} & \quad h_D(1, 1) = h_D(1, 0) \quad (r_0 = 1).
\end{align*}
\]

Solution Scheme

Laplace transforms can be used successfully to solve the equations simultaneously. The subsidiary equations are:

\[
\begin{align*}
\text{(15)} & \quad \frac{d^2 h_D}{dt_D^2} = \frac{p}{\alpha_1} \cdot h_D \\
\text{(16)} & \quad \frac{d^2 h_D}{dt_D^2} + \frac{1}{r_D} \cdot \frac{dh_D}{dr_D} = p h_D.
\end{align*}
\]

The transformed boundary conditions are

\[
\begin{align*}
\text{(17)} & \quad \frac{\partial h_D}{\partial t_D} = -\frac{1}{\beta p} \quad (r_0 = r_e) \\
\text{(18)} & \quad h_D = h_D(1, 1) \quad (r_0 = 1).
\end{align*}
\]

The general solutions for Eqs. (15) and (16) are of the form:

\[
\begin{align*}
\text{(20)} & \quad \hat{h}_D = A \cosh(\sqrt{p}/\alpha_1 \cdot r_0) + B \sinh(\sqrt{p}/\alpha_1 \cdot r_0) \\
\text{(21)} & \quad \hat{h}_D = C I_D(\sqrt{p}/\alpha_1 \cdot r_0) + D K_D(\sqrt{p}/\alpha_1 \cdot r_0)
\end{align*}
\]

where $I_D$ and $K_D$ are modified Bessel functions of zeroth order of the first and second kind, respectively. The coefficient $C$ in Eq. (21) is found to be nil since we expect the solution to be bounded for $r_0 \to \infty$. Eqs. (20) and (21) are substituted into Eqs. (17), (18), and (19), and the following set of equations are obtained.

\[
\begin{align*}
\text{(22)} & \quad A \sqrt{p}/\alpha_1 \cosh(\sqrt{p}/\alpha_1 \cdot r_0) + B \sqrt{p}/\alpha_1 \sinh(\sqrt{p}/\alpha_1 \cdot r_0) = \frac{1}{\beta p} \\
\text{(23)} & \quad A \sqrt{p}/\alpha_1 \sinh(\sqrt{p}/\alpha_1 \cdot r_0) + B \sqrt{p}/\alpha_1 \cosh(\sqrt{p}/\alpha_1 \cdot r_0) = \frac{D}{\beta} K_D(\sqrt{p}/\alpha_1)
\end{align*}
\]

Eqs. (22) are solved for $A$, $B$, and $D$.

\[
\begin{align*}
\Delta &= \sqrt{\alpha_1 K_D(\sqrt{p})} \cos \left[ \frac{\sqrt{p} / \alpha_1 \cdot (1 - r_0)}{\beta p} \right] + \\
&\quad + B K_D(\sqrt{p}) \sinh \left[ \frac{\sqrt{p} / \alpha_1 \cdot (1 - r_0)}{\beta p} \right]
\end{align*}
\]

Substituting $A$, $B$, and $D$ back into Eqs. (20) and (21), the solutions in the Laplace domain are obtained:

\[
\begin{align*}
\hat{h}_D &= \frac{\sqrt{\alpha_1}}{\beta p} \frac{K_D(\sqrt{p} \cdot r_0)}{K_D(\sqrt{p})} \cos \left[ \frac{\sqrt{p} / \alpha_1 \cdot (1 - r_0)}{\beta p} \right] + \\
&\quad + \frac{B}{\beta p} K_D(\sqrt{p}) \sinh \left[ \frac{\sqrt{p} / \alpha_1 \cdot (1 - r_0)}{\beta p} \right]
\end{align*}
\]

Inversion of $h_D$

The inversion theorem is applied to $\hat{h}_D$.

\[
\begin{align*}
\hat{h}_D &= \frac{\sqrt{\alpha_1}}{\beta p} \frac{K_D(\sqrt{p} \cdot r_0)}{K_D(\sqrt{p})} \cos \left[ \frac{\sqrt{p} / \alpha_1 \cdot (1 - r_0)}{\beta p} \right] + \\
&\quad + \frac{B}{\beta p} K_D(\sqrt{p}) \sinh \left[ \frac{\sqrt{p} / \alpha_1 \cdot (1 - r_0)}{\beta p} \right]
\end{align*}
\]
The integrand has a branch point at $p = 0$. We consider the contour $\Gamma$ in Figure-2. Since there are no singularities within $\Gamma$,

$$
\mathcal{J} = \oint_{\Gamma} \left( \int_{A}^{B} + \int_{D}^{E} + \int_{E}^{C} + \int_{C}^{D} + \int_{D}^{F} + \int_{F}^{A} \right),
$$

so that

$$
\int_{\Gamma} = - \left( \int_{B}^{C} + \int_{D}^{E} + \int_{E}^{F} + \int_{F}^{A} \right) \mathcal{J}.
$$

Therefore,

$$
L^{-1}(p) = \lim_{t \to 0} \frac{1}{2\pi i} \int_{\Gamma} \mathcal{J}.
$$

where $R$ is the radius of the outer circle and $\varepsilon$ the radius of the inner. It can be shown that

$$
\frac{C}{\varepsilon}, \quad \text{as} \quad \varepsilon \to 0
$$

On $CD$, let $p = \mu e^{i\theta}$ and on $EF$ let $p = \mu^2 e^{-i\theta}$ and using the identity:

$$
K_d(z) = \frac{1}{2\pi i} \int_{\Gamma} \mathcal{J}.
$$

we obtain

$$
L^{-1}(p) = \lim_{t \to 0} \int_{\Gamma} \mathcal{J}.
$$

Finally, by the convolution theorem we obtain

$$
\mathcal{J}(t) = \frac{1}{\pi} \int_{0}^{\infty} \mathcal{J}(\mu) \cos \left( \frac{t \mu}{\sqrt{\alpha}} \right) d\mu.
$$

Simplifying and using the recurrence formula:

$$
J_d(z) = \int_{0}^{\infty} \mathcal{J}(\mu) \cos \left( \frac{t \mu}{\sqrt{\alpha}} \right) d\mu.
$$

(3)

$$(\mathcal{J} - Y_d(z) J_d) = \frac{1}{\pi \varepsilon^2} \cos \left( \frac{t \varepsilon}{\sqrt{\alpha}} \right) \mu d\mu.
$$

(28)

In Figure-4, $h_d$ is evaluated using Eq.(28) for various values of $r_0$ with $a_1 = 0.2$ and $\beta = 0.7$. Note that when $r_0 = 1.0$, i.e., zero inner zone thickness, the curve is identical to that of van Everdingen and Hurst. As can be seen in Figure-4, $h_d$ is not very sensitive to $r_0$ for $r_0 < 0.1$. Therefore, it may be very difficult to determine $r_0$ directly from the match for $r_0 = r_0 = 0.01$. This would leave only four equations to work with. In such a case, either one of $S_{1}, \beta, \alpha$ or $r_1$ must be estimated a priori in order to determine the rest. An observation well is needed to determine $S_{1}$, a priori; $k_1$ and $r_1$ will probably never be known a priori. One might assume $S_{1}$ to be equal to incompressible fluid compressibility as a possible first trial.

Large Time Solution for Region 1

For small values of $z$,

$$
K_d(z) = \int_{0}^{\infty} \mathcal{J}(\mu) \cos \left( \frac{t \mu}{\sqrt{\alpha}} \right) d\mu.
$$

(29a)

$$
K_r(z) = \int_{0}^{\infty} \mathcal{J}(\mu) \cos \left( \frac{t \mu}{\sqrt{\alpha}} \right) d\mu.
$$

(29b)
\[
\begin{align*}
\cosh z &\approx 1 + \frac{z^2}{2} + \cdots \\
\sinh z &\approx z + \frac{z^4}{6} + \cdots
\end{align*}
\]
where \(\ln e = \gamma = 0.577215665\) (Euler's constant). Substituting Eq. (20) into (24) and rearranging,
\[
k_{D_1} \approx \frac{1}{p} \left\{ \frac{1-r_p}{\beta} - \ln \frac{E}{2} + \right.
\]
\[
\left. + \left[ \frac{1-r_0}{\alpha_s} - \frac{(1-r_0)^2}{2} \frac{\Xi}{\beta} \right] \cdot \ln \frac{E}{2} + \cdots \right\}
\]
\[
\approx \frac{1}{p} \left\{ \frac{1-r_p}{\beta} - \ln \frac{E}{2} + \right.
\]
\[
\left. + p \cdot \left[ \frac{1}{2} - \frac{\beta}{\alpha_s} \left( 1-r_0 \right) \right] \cdot \left( \ln \frac{E}{2} \right)^2 + \cdots \right\}
\]
so that
\[
h_{D_1} = L^{-1} \{ k_{D_1} \} \approx \frac{1-r_p}{\beta} + \frac{1}{2} \cdot \left( 1 + \frac{1}{\sigma_t} \cdot \frac{\beta(1-r_0)}{\alpha_s} \right) \times
\]
\[
\left( -\gamma + \ln 4\beta \right) + \cdots
\]
Eq. (30) shows that for large \(t_p\), \(h_{D_1}\) can be approximated by a logarithmic function of \(t_p\). The lower limit of \(t_p\) for which this approximation is valid is a function of \(\beta, \alpha_s, \) and \(r_0, \alpha_s\) can be seen in the derivation of Eq. (30). On a semi-log graph, Eq. (30) is would be a straight line identical to the solution for a homogeneous porous medium except for the constant \((1-r_0)/\beta\). This, in effect, can be thought of as a specific skin parameter. Therefore, \(k_s\) can be obtained in the usual manner but the calculation of the storage coefficient in the conventional manner may result in incorrect value of that parameter by a factor given by
\[
\frac{S_{11}}{S_{13}} = \frac{r_p}{r_f} \exp \left[ \frac{2(1-r_p/r_f)}{\beta} \right].
\]

**Small Time Solution for Region 1**

Remembering for small \(z\),
\[
K(z) = \left[ \frac{\sigma_t}{2\sigma_z} \right]^{1/2} \cdot e^{-\sigma_t} \cdot \left\{ 1 + \frac{4\sigma_t^2 - 1}{2} \frac{1}{\sigma_z} + O \left( \frac{1}{\sigma_z} \right) \right\}
\]
Since \(r_0 < 1\),
\[
k_{D_1} \approx \frac{\sqrt{\sigma_z}}{\beta \sigma_t} \cdot e^{-\sigma_t} \cdot \left( 1 + \frac{2}{\sigma_z} \right)
\]
\[
\times \left[ 1 - \frac{\sqrt{\sigma_z}}{\beta \sigma_t} \cdot e^{-\sigma_t} \cdot \left( 1 + \frac{2}{\sigma_z} \right) + \cdots \right]
\]
Then at the pumping well, \(r_f = r_0\),
\[
h_{D_1} = L^{-1} \{ k_{D_1} \} \approx \frac{2\sqrt{\sigma_z}}{\beta \sigma_t} \cdot \sqrt{\sigma_z} \cdot \left[ 1 - 2 \cdot \frac{\sqrt{\sigma_z}}{\beta \sigma_t} \cdot e^{-\sigma_t} \cdot \left( 1 - r_0 \right) \right]
\]
\[
+ 4 \frac{\sqrt{\sigma_z}}{\beta \sigma_t} \cdot \left( 1 - r_0 \right) \cdot \frac{1}{\sqrt{\sigma_z} \cdot \sigma_t} + \cdots
\]
From Eq. (32), it is assured that for small \(t_p\), \(h_{D_1}\) exhibits a straight line. Using the identities in Eqs. (7),
\[
h_{D_1} \approx \frac{4Q}{\beta \sigma_t} \cdot \left( 1 - r_0 \right) \cdot \frac{1}{\sqrt{\sigma_z} \cdot \sigma_t} e^{-\sigma_t} + \cdots
\]

**Inversion of \(h_{D_3}\)**

It now remains to invert \(h_{D_3}\) to real space:
\[
L^{-1} \{ h_{D_3} \} = -\frac{2\sqrt{\sigma_z}}{\pi \beta} \cdot \int_0^\infty e^{-\sigma_t^2} \cdot \phi \cdot \theta \cdot \gamma \cdot d\mu
\]
\[
(\sigma_t \geq 1),
\]
So that
\[
h_{D_3} \approx \frac{2\sqrt{\sigma_z}}{\pi \beta} \cdot \int_0^\infty e^{-\sigma_t^2} \cdot \phi \cdot \theta \cdot \gamma \cdot d\mu
\]
\[
(\sigma_t \geq 1),
\]

**Large Time Solution for Region 2**

Substituting Eqs. (29) into Eq. (25),
\[
k_{D_2} \approx \frac{1}{p} \left[ -\ln \frac{E}{2} \right] \cdot \ln \frac{E}{2} + \cdots
\]
Then,
\[
h_{D_2} \approx \frac{1}{2} \left\{ 1 + \left[ \frac{1}{2} - \frac{\beta(1-r_0)}{\alpha_s} \right] \cdot \frac{r_0^2}{\beta} \right\} \cdot (-\gamma + \ln 4\beta)
\]
\[
= \frac{1}{2} \left\{ -\gamma + \ln \frac{4\beta}{r_0^2} \right\}
\]

**Small Time Solution for Region 2**

For large \(p\), noting that \(r_0 < 1\), Eq. (32) becomes
\[
k_{D_2} \approx \frac{2\sqrt{\sigma_z}}{\beta \sigma_t} \cdot e^{-\sigma_t^2} + \cdots
\]
and now inverting \(h_{D_2}\) we have:
\[
h_{D_2} \approx \frac{2\sqrt{\sigma_z}}{\beta \sigma_t} \cdot e^{-\sigma_t^2} + \cdots
\]
This completes the large and small time solutions.
Example Application and Conclusion

Figure 4 illustrates an example of the procedure applied to find $k_f$, $k_s$, $S_{sv}$, and $r_f$ from a numerical well test simulation in the fracture network also shown in the figure. The numerical procedure employed here is discussed in Long' and Kanehiro. A fit was found assuming $S_{sv}$ was known. The flow parameters for both regions calculated from the match point and from the equations for $a_i$ and $b$ were found to be very close to the input values. The calculated $r_f$ was 4.9 m in this case. In the actual network, the distance to the nearest intersection was 7.3 m in one direction and 3.8 m in the other, averaging 5.5 m.

If a fit can be found this analytical model can successfully be used to accurately determine the average storage coefficient of a fracture system from one well test as well as the distance to the nearest fracture intersection, which in turn give us a clue to fracture spacing. Because there are a finite number of curves for various combinations of $a_i$, $b$, and $r_f$, an automatic curve fitting procedure by a computer is being investigated.

REFERENCES


Figure 3c  Dimensionless pressure for $r < 0.05$ and $\alpha = 0.1$.

Figure 3d  Dimensionless pressure for $r < 0.05$ and $\beta = 10$.

Figure 3e  Dimensionless pressure for $r < 0.05$ and $\beta = 1.0$.

Figure 3f  Dimensionless pressure for $r < 0.05$ and $\beta = 0.1$.

Figure 4  Dimensionless pressure for $\alpha = 2.0$ and $\beta = 0.7$.

Figure 5  An example of curve matching technique applied to numerical well test simulation results.