SOLUTION SELECTION IN GEOTHERMAL HEAT PIPES

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SUMMARY – Steady heat pipe solutions with capillarity and conduction effects included are considered. Phase plane trajectories of temperature vs. saturation typically approach zero capillarity solutions, one liquid-dominated and one vapor-dominated. The way in which choice of boundary conditions selects from these solutions is studied. Comparisons with previous work on the effect of upstreaming on solution selection are made. Implications for actual geothermal reservoirs are that liquid-dominated counterflow is typically selected when saturation and temperature are fixed at the top of the counterflow region, and vapor-dominated counterflow is typically selected when temperature and saturation are fixed at the bottom.

1 Introduction

In a geothermal heatpipe, it is possible to transfer heat with very little net mass flow, by having steam flowing upwards and liquid flowing downwards, in contact with each other. Such a model has been put forward by White et al (1971) for a part of the Geyser steam field, and also to Larderello in Italy, Matsukawa in Japan: and Kawah Kamojang in Indonesia.

In a number of studies of the heatpipe mechanism, it has become clear that there are sometimes two solutions possible for a given steady heat flow, one vapor-dominated and one liquid-dominated. However, often only one of these solutions is actually obtained, in an experiment or in a computer study. Computer studies have solved both the steady-state problem, and the full time-dependent problem, which is run for long enough to achieve steady-state.

A fundamental question that arises is whether both solutions are realistic. Recent work by Satik et al (1991) suggests that only the vapor-dominated solution is typically obtained. Also, Schubert and Strauss (1979) obtain only the vapor-dominated solution when two-phase counterflowing fluid overlies stationary liquid.

However, in laboratory experiments, Udel (1985), Bau and Torrance (1982) and Cornwell et al (1978) obtain only liquid-dominated solutions.

Further, Shen et al (1979) find when numerically integrating the steady-state conditions down from the surface: no counterflow solutions exist when net mass flow is small.

This is apparently directly contradictory to the results of Schubert and Strauss (1979). Satik et al (1991) have a similar model (three layers, vapor, two-phase and liquid), and obtain vapor-dominated solutions for zero net mass flow. Satik et al integrate upwards from a liquid layer, in the geothermal case.

Some understanding of which solution is selected by geothermal simulators that solve the full time-dependent problem, depending on the boundary conditions applied: has already been reached (McGuinness et al, 1991; McGuinness, 1990; McGuinness, 1988). In these studies, the role played by the numerical technique called upstream differencing was explored. It was shown that upstreaming leads to a map for steady-state conditions. The solution which is obtained depends on which vertical direction the map is iterated in. In particular, if saturation is fixed at the top of the counterflow region, the map is iterated downwards, and the liquid-dominated solution is typically obtained. If saturation is fixed at the bottom of the two-phase region, the map is iterated upwards, and the vapor-dominated solution is typically obtained.

That is, a selection mechanism is operating, whereby if saturation is fixed at the top the liquid-dominated solution is selected, and if saturation is fixed at the bottom the vapor-dominated solution is selected.

Further, in McGuinness et al (1991) and in McGuinness (1990) this was shown to be consistent with wavelike behaviour in saturation for the full time-dependent problem. In other Words, this selection mechanism is a property of the conservation equations, and not merely a numerical artifact.
In the studies mentioned in the previous two paragraphs, capillarity and conductive effects were ignored. Capillarity is not generally important over the length scales of geothermal reservoirs. However, any consideration of boundary layer effects should include capillarity, as boundary layers are thin, and capillarity is important over short distances.

In this paper, the work of Satik et al. (1991) is re-examined and modified. Capillarity and conduction are included in their treatment. However, their development is limited by reliance on numerical integration from one end only, and by the three-layer nature of their model.

It is shown that the diffusive effects of capillarity lead to the same phase space behaviour as the effects of up-streaming. In retrospect this is not too surprising, as it is well-known that up-streaming introduces numerical diffusion. What is interesting is that this numerical diffusion mimics closely the diffusive effects of capillarity on solution behaviour.

2 Steady Counterflow

No assumption is made here about where liquid or vapor layers are. We consider only the two-phase region. Otherwise the development here is similar to that of Satik et al. (1991), and the nomenclature and sign conventions are the same. A sign error on the heat flux terms in Satik et al. (1991) is corrected here. Also, in this paper the full dependence of thermodynamic quantities on temperature is included, whereas in Satik el al. (1991) this is ignored for simplicity.

A steady heat flux is imposed on a two-phase region, with zero net mass flux. Steam and liquid are present, and flow in opposite directions, so as to give zero net mass flow. The difference in their specific enthalpies allows a nonzero imposed net heat flow.

Capillary pressure and/or gravity can drive this counterflow, so that this formulation encompasses both engineering and geothermal heat pipes.

The flow is one-dimensional, and inclined at an angle $\theta$ to the horizontal, as illustrated in Fig. 1.

Darcy's law gives the momentum balance for the two phases,

$$u_l = -\frac{k_r}{\nu_l} \left( \frac{\partial P_l}{\partial x} + \rho_g g \sin \theta \right)$$

$$u_v = -\frac{k_r}{\nu_v} \left( \frac{\partial P_v}{\partial x} + \rho_v g \sin \theta \right)$$

where $u$ is the mass flux density ($kg s^{-1} m^{-2}$); $k$ is permeability; $k_r, k_v$ are relative permeabilities for liquid and vapor phases respectively, which depend on liquid saturation $S$; $\nu$ is kinematic viscosity; $P$ is pressure, different in liquid and vapor phases; $\rho$ is specific density; $g$ is the acceleration due to gravity; subscripts $l$ and $v$ refer to liquid and vapor phases respectively.

**Capillary pressure is taken to be**

$$P_c(S) = P_v - P_l = \frac{\sigma}{\sqrt{k}} J(S),$$

where the Leverett $J$ function is shown in Fig. 2. $\sigma$ is surface tension.

![Figure 1. The 1D heat pipe configuration](image)

Vapor-pressure lowering (due to the Kelvin effect) is incorporated in the form

$$P_v = P_{v0} \exp(-bJ)$$

where

$$b = \frac{\sigma \nu_l}{RT \sqrt{k}}$$

and $\nu_l$ is liquid molar volume, $R$ is the gas constant (8.3 J/°C mole), and $T$ is the temperature in °K.

When steam and liquid are in equilibrium with each other, pressure and temperature are related by the Clausius-Clapeyron relationship

$$P_{v0} = P_0 \exp \left(\frac{h_{v0} \rho_l \nu_l}{R} \left( \frac{1}{T_0} - \frac{1}{T} \right) \right),$$

where $P_0$ depends on the arbitrary reference temperature.
Then steady solutions to the heat pipe are naturally presented as trajectories in the \((S, \tau)\) or \((S, T)\) phase plane. The paths taken by these trajectories are strongly influenced by the curves \(G = 0\). Off these curves, trajectories are almost constant in \(\tau\) for typical geothermal parameter values. It is near the \(G = 0\) curves that temperature can change appreciably for solution trajectories. This is because

\[
\frac{\partial \tau}{\partial S} = \frac{H J'}{G}
\]

and \(J'\) is typically much smaller than \(G\) in the geothermal context, so that \(\tau\) varies only slowly compared to \(S\) except when \(G\) is small.

Curves resulting from the relationship \(G = 0\) are plotted for different values of heat flux in Fig. 3. These have been computed using full thermodynamic relationships (International Formulation Committee, 1987), and linear relative permeabilities have been used for illustration purposes.

For simplicity, in the geothermal case, the temperature is taken to increase in the downwards direction.

These considerations lead to trajectories that typically (depending on the magnitude of the heat flow) look like those in Fig. 4. The steady geothermal heat pipe will have values of temperature and saturation that lie on one of these trajectories, depending on the two remaining boundary conditions and on the length of the heat pipe.

\[ T_{li} \text{ and } h_{ui} \text{ is the latent heat of vaporisation.} \]

Figure 2. The Leverett J function, used to represent the saturation dependence of capillary pressure

In the steady-state, mass and energy are conserved. Two boundary conditions are used to integrate the usual conservation equations, these conditions being zero mass flow and a fixed energy flow. Two boundary conditions remain, to specify the solution to the following mass and energy conservation equations:

\[
\begin{align*}
&\frac{\partial u}{\partial t} + \nabla \cdot (u \rho) = 0, \\
&\frac{\partial h}{\partial t} + \nabla \cdot (u h) = \dot{Q} + \lambda \frac{\partial^2 T}{\partial x^2},
\end{align*}
\]

\(\lambda\) is thermal conductivity. \(u\) is mass flux density, \(\text{kg} \cdot \text{s}^{-1} \cdot \text{m}^{-2}\).

These two equations may be rewritten in the form

\[
A \begin{pmatrix} \frac{\partial T}{\partial x} \\ \frac{\partial S}{\partial x} \end{pmatrix} = \begin{pmatrix} g \sin \theta \left( \frac{\rho c_p \chi}{\nu_s} + \frac{\rho k x}{\nu_t} \right) \\ g \sin \theta \frac{k x \chi h}{\nu_s} + \dot{Q} \end{pmatrix}
\]

where \(A\) is given in the appendix.

This may be solved to give two coupled first-order differential equations for temperature and saturation:

\[
\begin{align*}
\frac{\partial \tau}{\partial \zeta} &= \frac{H}{F}, \\
\frac{\partial S}{\partial \zeta} &= \frac{G}{F} J',
\end{align*}
\]

where the right-hand sides are nonlinear functions of temperature and saturation, given in the appendix. Temperature has been nondimensionalised to \(\tau\), and distance to \(\zeta\).

Figure 3. The curves \(G = 0\) for various values of heat flux, \(Q\), shown for each curve in kW/m². Other parameter values here and in the following plots are \(\lambda = 3 \text{W/°C} / \text{m}\), \(\chi = 1\text{d}, \sin \theta = 1\).
3 Zero Capillarity Comparison

The $G = 0$ curves are precisely the solvability conditions for the zero capillarity case. This is clear both from direct calculation of the expression $G = 0$, and from a consideration of equation 8. The matrix $d$ is singular when capillarity is zero. The singularity arises from the second column being zero.

Then simple algebra provides an explanation, if the equation 8 is written in the form

$$
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2
\end{pmatrix} =
\begin{pmatrix} b_1 \\ b_2
\end{pmatrix},
$$

then the solution for $x_2 (dS/d\zeta)$ by Cramer's rule or by direct manipulation has on its top line

$$
G = \det
\begin{pmatrix} a_{11} & b_1 \\ a_{21} & b_2
\end{pmatrix}.
$$

If $a_{12}$ and $a_{22}$ are zero, as happens with zero capillarity, the solvability condition is obtained by eliminating $x_1$, and it is precisely $G = 0$.

4 Conclusions

A variety of conclusions may be drawn directly from the nature of the trajectories in Fig. 4.

With capillarity included, boundary layers are possible (the constant temperature sections), and solutions quickly approach the usual solutions $G = 0$ without capillarity. One branch is liquid-dominated (large $S$), the other is vapor-dominated (small $S$).

When integrating upwards, for example from a fixed saturation and temperature boundary, the vapor-dominated solution is typically obtained (see Fig. 5). Exceptions are when $S$ is started at a value close to or greater than the right-hand branch of $G = 0$. Then the heat pipe solution is either a shock with the liquid-dominated branch underlying the vapor-dominated branch, or the trajectory moves rapidly to the right to $S = 1$, and the counterflow region is small in extent, or if the starting value is close enough to that branch, the solution is the liquid-dominated one.

3.1 Upstreaming

The trajectories in the phase plane are almost identical in appearance to the 1D manifolds on which iterates under the 2D map induced by upstream differencing move (McGuiness et al., 1991). That is, the effect of numerical diffusion induced by upstreaming is qualitatively the same as the diffusive effect of capillarity. The main difference will be the thickness of the boundary layers.

Figure 4. Typical phase plane trajectories, sketched approximately for heat flux 2 kW/m²

Figure 5. Trajectories for the steady heat pipe, integrating upwards.

This suggests that integrating upwards from a quiescent liquid layer (where $S = 1$) will not give a steady heat pipe solution. However, the heat flow up through a quiescent liquid layer is limited to flow by conduction, and
is restricted by the requirement that pressure and temperature in the single phase liquid stay off the Clapeyron curve. In this case, heat flow is small enough that there is only one branch of the $G = 0$ curve, near $S = 0$. This is illustrated in Fig. 6. Hence for a quiescent liquid phase at the bottom, there is only one steady heat pipe solution, and it is vapor-dominated.

For larger heat flow values, integrating upwards, there is no smooth path of decreasing temperature taking saturations from values near one at depth to values near any branches of $G = 0$. It is only near these branches that the heat pipe extends over a significant region beyond a capillary boundary layer.

When integrating downwards, a liquid-dominated solution is typically obtained. The capillary boundary layer acts like the $D$ map associated with upstream differencing (McGuinness et al., 1991), quickly forgetting the starting values of saturation and temperature, and taking the trajectory close to the right-hand branch of the $G = 0$ curve, the zero capillarity solution.

The experiments mentioned in the Introduction are examples of this case, as they have a liquid layer at the surface of the porous medium, maintaining a fixed saturation and pressure there. This explains why these experiments always yield the liquid-dominated solution.

Dry-out at the bottom of these experimental set-ups corresponds to raising the heat flow to values such that the trajectory "falls off" the top of the $G = 0$ curve. The results of Sheu et al., in which they integrate from the top, are consistent with these trajectories also. They have a stationary or quiescent liquid layer at depth, which they shoot for from the top. They obtain no two-phase counterflow with quiescent liquid below. This corresponds to there being no way to go from a saturation near one and have increasing temperature, to reach the single branch of $G = 0$ in Fig. 6.

Consequences for two-phase counterflow regions in actual geothermal reservoirs are that provided the heat flow is large enough that $G = 0$ has two branches, there is a selection mechanism due to capillarity that typically chooses the liquid-dominated steady counterflow solution if saturation and temperature are held constant at the top, and that typically chooses the vapor-dominated solution if saturation and temperature are fixed at depth. For special values of the boundary conditions, composite or shock solutions with the liquid-dominated part at the hotter (typically deeper) end, and a vapor-dominated part at the other end are possible. The shock is actually a capillary boundary layer, which approaches a shock in the limit that capillarity goes to zero.

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6 References


### A Mathematical Details

Conservation of mass and energy for the steady heat pipe is expressed as

\[
\begin{align*}
    u_t + u_s &= 0, \\
    u_s h_v + u_t h_i &= Q + \lambda \frac{\partial T}{\partial z}. \\
\end{align*}
\]

\[ (10) \]

\[ (11) \]

\( \lambda \) is thermal conductivity.

These two equations may be rewritten in the form

\[ A \left( \frac{\partial T}{\partial z} \right) = \left( g \sin \theta \left( \frac{a_s k_v + a_t k_i}{u_v} \right) \right) \]

\[ \frac{\partial S}{\partial z} = (g \sin \theta \left( \frac{b_s k_v + b_t k_i}{u_v} \right) + Q), \]

\[ (12) \]

where

\[ A = \left| \begin{array}{cc} -P_v D (\lambda_v + \lambda_i) + \frac{\lambda_v J'}{\sqrt{\alpha}} & [P_v b (\lambda_v + \lambda_i) + \frac{\lambda_v J'}{\sqrt{\alpha}}] J' \\
- P_t D k \lambda_v h_{vi} - \lambda_i & P_t b k \lambda_v h_{vi} J' \end{array} \right|, \]

and

\[ D \equiv b' J + C' T - C' T^2, \] \[ C \equiv h_v \mu_p v_i / R; \] \[ \lambda_i \equiv k_i / \mu_p, \] \[ i \equiv v, i; \] \[ J' \equiv \sqrt{\alpha}, \] \[ P \] \[ \text{Primes indicate differentiation with respect to the appropriate variable (T for b, C for \sigma; S for J).} \]

All fluid properties are dependent on \( \mathcal{P} \) and \( \mathcal{T} \). This dependence has been fully accounted for in this development.

This may be solved to give two coupled first-order differential equations for temperature and saturation:

\[ \frac{\partial \tau}{\partial \zeta} = \frac{H}{F}, \]

\[ \frac{\partial \sigma}{\partial \zeta} = \frac{G}{F J} \]

\[ (13) \]

\[ (14) \]

where

\[ H = k_r k_v \sin \theta (b R_m A - R_v R_c) \]

\[ -k_r (R_h - b R_p R_h A) - k_r b \beta R_p R_h A, \]

\[ F = k_r k_v R_m A \left( \frac{K}{r^2} + \frac{b J}{r} - \frac{K}{r} \frac{d}{d \tau} (\ln h_v) \right) \]

\[ + k_p (1 + b R_p A) + k_r b \beta R_p A, \]

\[ G = k_r k_v \sin \theta \left( R_m A M + R_v R_c \right) \]

\[ + k_p (1 + b R_p A) + k_r b \beta R_p A, \]

\[ M \equiv \frac{K}{r^2} + \frac{b J}{r} - \frac{K}{r} \frac{d}{d \tau} (\ln h_v) - b J \frac{d}{d \tau} (\ln \sigma). \]

The temperature has been nondimensionalized to \( \tau : T / T_0 \), and distance to \( \zeta = \tau \cos \beta \mathcal{A} / \mathcal{J} \). Parameters have been grouped into nondimensional expressions.

\[ A = \exp \left( K \left( 1 - \frac{1}{\tau} \right) - b J \right), \]

\[ K = \frac{h_v \mu_p v_i}{T_0 K}, \]

\[ R_v \equiv \frac{k h_v \sigma}{\nu_v \sqrt{k T_0}}, \]

\[ R_h = \frac{\lambda T_0 \sqrt{k} \lambda T_0}{\mathcal{J}}, \]

\[ R_i = \frac{\rho \Delta \rho}{\Delta \rho}, \]

\[ R_m = \frac{k h_v \mu_p}{\nu_v T_0}, \]

\[ R_p = \frac{P_0 \sqrt{k} \sigma}{\Delta \rho}, \]

\[ R_v = \frac{\mu_p}{\Delta \rho}. \]

\[ \beta \equiv \nu_i / \nu_v, \]

\[ \Delta \rho \equiv \rho_i - \rho_v. \]