

Beam Displacement at Total Reflection: The Goos-Hänchen Effect, I*

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Abstract

The Goos-Hänchen effect, arising in the total reflection of a parallel beam linearly polarized, is treated comprehensively. It was originally discovered in optics, but is also encountered in other branches of physics such as, for example, acoustics, quantum mechanics, plasma physics and nonlinear optics. This paper places emphasis on optics for historical reasons, but the results including the discussion on the flow of energy at total reflection, properly interpreted, apply to those other branches as well. The paper starts out with a mathematical description of the reflection and refraction of a beam of light at a plane interface, based upon an approximate solution of Maxwell's equations. This theory was originally proposed by *Schaefer* and *Pichl*, but their treatment had to be modified by *Lotsch* to comply with the principle of conservation of energy. The general description of the Goos-Hänchen effect, applying *v. Fraunhofer's* derivation, is consistent with *Renard's* viewpoint and reduces to the classical formulations of *Armann* and *Wolter* in the neighborhood of the critical angle for total reflection. Different approximations valid in this region are discussed and compared by numerical evaluation as well as with *Wolter's* measurements. It is concluded that the *Armann-Wolter* expressions are simple and accurate enough for most practical applications.

The Goos-Hänchen effect is shown to have physical ties to a number of other phenomena including the Schoch effect in acoustics. In the case of a slightly diverging beam the Goos-Hänchen effect is accompanied by *v. Schmidt's* lateral wave, originally discovered in seismology and acoustics. This wave is responsible for the illumination-trailing phenomenon observed by *Acloque* and *Gaullinet*. Several applications and/or utilizations, including total reflection holography and internal reflection spectroscopy, are briefly discussed to complete the treatment. It is shown that the well-known phenomenon of long-distance propagation along the ionosphere is, from a physical point of view, related to the Goos-Hänchen effect, modified for the case of a diverging beam.

The paper is divided into four parts. Part I comprises the Introduction and the two chapters entitled "Reflection and Refraction of a Beam of Light" and "Total Reflection of an E-Polarized Beam". Part II treats the Goos-Hänchen effect in classical optics. The different descriptions are discussed and their results are compared with *Wolter's* measurements. Part III deals with the Goos-Hänchen effect in other branches of physics such as acoustics, quantum mechanics, plasma physics and nonlinear optics. The Schoch effect is introduced; and further more the total reflection of diverging and converging waves is investigated. The final Part IV is devoted to several

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applications of the Goos-Hänchen effect, including the case of absorbing media. In addition, it contains the Summary and Conclusion, and the extensive list of references.

Inhalt

Strahlversetzung bei der Totalreflexion. Der Goos-Hänchen-Effekt, der bei der Totalreflexion eines parallelen, linear polarisierten Lichtstrahles auftritt, wird eingehend behandelt. Dieser Effekt wurde ursprünglich in der Optik entdeckt, tritt aber auch in anderen Zweigen der Physik wie z. B. Akustik, Quanten-Mechanik, Plasmaphysik und nichtlineare Optik auf. Die vorliegende Arbeit betont aus historischen Gründen die Optik; die Ergebnisse, einschließlich der Beschreibung des Energieflusses bei Totalreflexion, wenn sie entsprechend interpretiert werden, gelten auch für jene anderen Zweige. Die Arbeit beginnt mit einer mathematischen Beschreibung der Reflexion und Brechung eines Lichtstrahles an einer ebenen Trennfläche, die auf einer Näherungslösung der Maxwell'schen Gleichungen beruht. Diese Theorie wurde ursprünglich von *Schaefer* und *Pichl* vorgeschlagen, aber ihre Behandlung mußte von *Lotsch* modifiziert werden, um dem Prinzip der Energieerhaltung zu genügen. Die allgemeine Beschreibung des Goos-Hänchen-Effektes auf der Grundlage der *v. Fraunhofer'schen* Ableitung stimmt mit dem *Renard'schen* Gesichtspunkt überein und verknüpft sich zu den klassischen Ergebnissen von *Armann* und *Wolter* in der Nähe des Grenzwinkels der Totalreflexion. Verschiedene Näherungen, die in diesem Bereich gültig sind, werden besprochen und sowohl durch numerische Auswertungen als auch mit den *Wolter'schen* Messungen verglichen. Dabei zeigt sich, daß die relativ einfachen *Armann-Wolter'schen* Ausdrücke für die meisten Anwendungen in der Praxis ausreißend genau sind.

Der Goos-Hänchen-Effekt wird mit einer Reihe anderer physikalischer Erscheinungen, einschließlich dem Schoch-Effekt in der Akustik, in Verbindung gebracht. Im Falle eines schwach divergierenden Strahles wird der Goos-Hänchen-Effekt von der *v. Schmidt'schen* Flankenwelle begleitet, die ursprünglich in der Seismologie und Akustik gefunden worden ist. Diese Welle zeichnet für die von *Acloque* und *Gaullinet* beobachtete Lichtwanderung bei der Totalreflexion verantwortlich. Einige Anwendungen, einschließlich der Totalreflexion-Holographie und der Totalreflexion-Spektroskopie, werden kurz besprochen, um die vorliegende Behandlung abzurunden. Es wird unter anderem gezeigt, daß die bekannte Erscheinung der anomalen Rundfunkwellen-Fortpflanzung entlang der Ionosphäre physikalisch gesehen auf den für einen divergierenden Strahl modifizierten Goos-Hänchen-Effekt zurückzuführen sein kann.

Die vorliegende Arbeit ist in vier Teile unterteilt. Teil I umfaßt die Einleitung und die beiden Kapitel „Reflexion und Brechung eines Lichtstrahles“ und „Totalreflexion eines E-polarisierten Lichtstrahles“. Teil II behandelt den Goos-Hänchen-Effekt in der klassischen Optik. Die verschiedenen Beschreibungen werden besprochen, und ihre Ergebnisse mit den Messungen von *Wolter* verglichen. Teil III befaßt sich mit dem Goos-Hänchen-Effekt in anderen Zweigen der Physik, nämlich der Akustik, der Quanten-Mechanik, der Plasma-Physik und der nichtlinearen Optik. Der Schoch-Effekt wird eingeführt, und außerdem wird die Totalreflexion divergierender und konvergierender Wellen untersucht. Der abschließende Teil IV ist einigen Anwendungen des Goos-Hänchen-Effektes, einschließlich dem Fall absorbierender Medien, gewidmet. Außerdem enthält er die Zusammenfassung und Schlussbetrachtung und das ausführliche Literaturverzeichnis.

Introduction

Goos and (*Linberg*)-*Hänchen* [1] demonstrated experimentally in 1943 that, in the plane of incidence, the totally reflected, linearly polarized beam of light is displaced parallel to a ray which would be reflected geometrically at the interface between the two optically transparent, homogeneous and isotropic media. They concluded, as was already suspected by *Newton* [2] about three centuries ago, that, even at total reflection, the incident beam

penetrates into the optically less-dense medium but then re-emerges into the optically denser medium; in other words, that the beam is reflected at some virtual surface located a small distance within the less-dense medium. This phenomenon, illustrated in Fig. 1, was named "the *Goos-Hänchen effect*" by Professor *H. Wöller* [16]. It can be expressed equivalently either as a beam displacement, or as a depth of penetration, or as a shift of the reflection center.

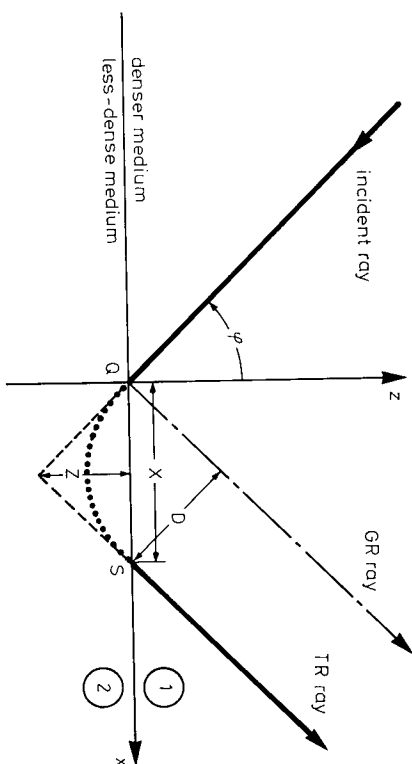


Fig. 1. Schematic representation of the path assumed by a ray of light at total reflection. The incident ray is not reflected geometrically (GR) but rather totally (TR) with the displacement D . The path in the less-dense medium from Q to S may, according to *Newton*, be visualized by a parabola, as indicated by the string of dots. The shift of the reflection center is denoted by X and the depth of penetration into the less-dense medium by Z .

Newton [2] anticipated that total reflection does not take place at the physical interface between the two media. He suspected that the path of a ray of light is a parabola, the vortex being within the less-dense medium. This problem has been investigated experimentally for many decades, but with little success. The approach had always been to measure somehow the flow of light energy inside the less-dense medium. However, energy cannot be detected unless it is extracted from the physical process. Therefore, it seems impossible to measure the flow of energy in the less-dense medium without disturbing the mechanism of total reflection. Several of the ingenious but unsuccessful experiments that were carried out are briefly outlined in Reference 1a. Finally in 1943 *Goos* and *Hänchen* [1a] found the key to the solution of the intriguing problem. They devised a clever experiment designed to show in the denser medium what happens to the totally reflected beam in the less-dense medium without affecting it in this medium. As illustrated in Fig. 2, *Goos* and *Hänchen* deposited a strip of silver on the back surface of a totally reflecting prism. The part of the incident beam falling upon the silvered strip is metallically reflected and serves as a reference to measure the displacement of the totally reflected portions of the beam. In order to magnify the

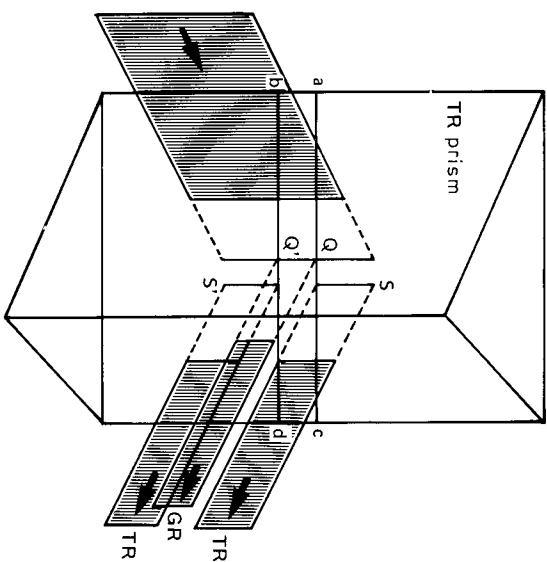


Fig. 2. Schematic illustration of the historical experiment devised by *Goos* and *Hänchen* [1a]. The strip abcd on the back side of the totally reflecting (TR) prism is silvered. The beam incident from the left-hand side, therefore, produces totally reflected (TR) beams as well as a geometrically reflected (GR) beam due to the metallic reflection.

minute displacement, *Goos* and *Hänchen* utilized a scheme involving multiple total reflections (up to 133) between two plane and parallel interfaces.

The beam displacement is generally small in the entire angle range of total reflection, except in the immediate neighborhood of the critical angle for total reflection. It increases rather rapidly as the angle of incidence approaches this critical angle. Consequently, the depth of the virtual plane of reflection, and thus the depth of the light penetration into the less-dense medium, is significant only in the immediate neighborhood of the critical angle for total reflection. *Lotsch* [3] has, therefore, recognized that, if the less-dense medium is nonlinear, the production of harmonics, measured in the totally reflected beam, should strongly increase as the angle of incidence approaches the critical angle. This phenomenon was meanwhile demonstrated quite convincingly by *Bloembergen*, *Lee* and *Simon* at the second harmonics [4] and by *Bay*, *Griškani* and *Rabin* at the third harmonics [5]. Hence, nonlinear optics has provided a new indirect method for demonstrating the *Goos-Hänchen* effect.

The paper at hand presents a comprehensive treatment of the *Goos-Hänchen* effect, placing emphasis on the viewpoint of light penetration into the less-dense medium. It starts out in Chapter I with a brief outline of the theory of reflection and refraction for a beam of light. The beam is assumed to be linearly polarized either in the plane of incidence or at right angles to it. The mathematical description is based upon an approximate, plane-wave

type solution of Maxwell's equations. The groundwork for this theory was originally laid by *Schaefer* and *Picht* [6], but they adopted simplifying assumptions which led to a violation of the principle of conservation of energy. This shortcoming was discovered and corrected by *Lotsch* [3].

Chapter 2 treats the special case of total reflection in greater detail, assuming an E-polarized beam. It is shown that, due to the limited extent of the beam in its transverse direction, light penetrates into the less-dense medium and then, after having propagated along the interface, re-emerges into the denser medium. This interpretation, being consistent with the classical investigations of *Picht* [7] and of *Noether* [8], substantiates *Voigt's* contention [9] dating back to the turn of this century.

Chapter 3 is devoted to the *Goos-Hänchen* effect in classical optics. The light penetration into the less-dense medium is interpreted as a forward shift of the line of "gravity" of the reflected beam, as originally suggested by *v. Progsstein* [12]. This interpretation, if derived from the properly modified *Schaefer* and *Picht* theory [3], leads to the general description of the *Goos-Hänchen* effect, being in full agreement with *Renaud's* contention [13]. *Renaud* had questioned the earlier formulations in the German literature since they predict non-zero beam displacements in the limit of grazing incidence. Basing his derivation upon *Arzelles'* theory of reflection and refraction [14], he proposed a new description which yields a vanishing beam displacement in the limit of grazing incidence and reduces to those formulations for angles of incidence close to the critical angle for total reflection. Other investigators including, for example, *Arntmann* [15], *Wolter* [16], *Maecker* [17], *Schilling* [18] and [19] and *Paasgeau* [20] formulated the problem on grounds of physical optics, either approximating it by only two plane-wave components like *Noether* [8]. In any case the plane-wave components undergo phase shifts at total reflection depending upon their respective angles of incidence. Therefore, the plane-wave components of the incident beam are superimposed in the reflected beam with changed mutual phase relations. *Arntmann* [15] had originally adopted this approach and evaluated the resulting integral with the method of stationary phase; and later *Wolter* [16] considered it on the basis of the minimum-ray definition for a beam of light, introduced by himself [21]. The latter theory not only provides a method to measure the *Goos-Hänchen* effect with high accuracy, but also has the additional advantage of interpreting the effect of absorption in the two adjacent media. — All these theoretical treatments are discussed individually and in

¹ *Voigt* contended that light penetrates into the less-dense medium at the leading edge of the incident beam in order to build up an evanescent wave. This light is somehow scattered into space at the trailing edge since the evanescent wave is no longer required. *Dryde* [10] voiced his opinion in the first edition of his *Lehrbuch der Optik* (about 1900) that the light returns to the denser medium at that time and removed by *Gehrke* in a revision of *Dryde's* book. *Wolter* [11] has recognized that both viewpoints are indeed correct, but that their validity depends upon the experimental circumstances. With regard to the *Goos-Hänchen* effect *Dryde's* viewpoint is applicable.

relation to others. Their results are compared by numerical evaluation and are related to *Wolter's* very precise measurements of the beam displacement for both polarizations. It is concluded that the formulas for the *Goos-Hänchen* effect, derived by *Arntmann* and *Wolter*, are not only relatively simple but also accurate enough for most practical applications. Their inconsistency at grazing incidence is immaterial since the *Goos-Hänchen* effect is significant only in the neighborhood of the critical angle for total reflection.

Chapter 4 deals with the *Goos-Hänchen* effect in a broader sense. It is shown that this phenomenon is also encountered in other branches of physics such as, for example, acoustics [22], plasma physics [23], quantum mechanics [24] and [13] and nonlinear optics [4] and [5]. In contrast to optics an acoustical problem may, in addition, require a longitudinal wave for its complete description. The occurrence of a longitudinal wave obviously increases the degree of complexity, and the *Goos-Hänchen* effect may arise at more than one angle of incidence. Furthermore, a large beam shift localized at the angle for excitation of the Rayleigh wave is encountered in the case of an air or liquid/solid interface. This phenomenon is termed the Schoch effect. — Nonlinear optics became attractive only recently through the advent of the laser producing a high-intensity beam. The connection between nonlinear optics and the *Goos-Hänchen* effect, in particular if the less-dense medium is nonlinear, was first recognized by *Lotsch* [3].

Chapter 5 treats phenomena which are physically related to the *Goos-Hänchen* effect, placing emphasis on the total reflection of diverging (cylindrical or spherical) waves. The model of the "traveling-reflection", introduced by *v. Schmidt* [25] to explain observations in seismic shock experiments, is expressed by a conical wavefront in the denser medium. In conformity with the electromagnetic nomenclature [26] this wave, identified with a boundary-layer wave in optics [17], is generally termed *v. Schmidt's* lateral wave. *Maecker* [17] recognized that it arises only in the total reflection of a diverging wave, but not in the case of a converging wave. Therefore, the following notation is tacitly adopted throughout the present paper: The word "beam" is always used in the sense of a parallel beam; if the beam is converging or diverging, it will be stated so explicitly. Section 5.3 is devoted to the general case of a parallel beam with superimposed diffraction (cylindrical or spherical) waves.

The present paper assumes that the beam of light is linearly polarized either in the plane of incidence or at right angles to it. The problem of an arbitrary polarization either linear, elliptical, or circular would introduce an additional degree of complexity, as briefly indicated in Section 5.1. This is because the light energy would not necessarily be confined to the plane of incidence. The Poynting vector would have an orthogonal component, as pointed out by *Boguslawski* [27] and *Wiegrefe* [28]. The flow of energy leads to a beam displacement perpendicular to the plane of incidence, as suggested by *Fedorov* [29]. This beam displacement is at least one order of magnitude smaller than that of the *Goos-Hänchen* effect and, to the author's knowledge, has not been measured to date.

Chapter 6 is devoted to several important cases in which the *Goos-Hänchen* effect is either applied or utilized. After a brief discussion of the physics behind Brewster's law the *Goos-Hänchen* effect in absorbing media is treated. The phenomenon of negative penetration into a strongly absorbing less-dense medium, discovered by *Wöller* [16], is readily understood on the basis of Brewster's law. It plays a role in the total reflection holography. The case of weak absorption is discussed in connection with the internal reflection spectroscopy. Several more examples are briefly indicated, including reflection phenomena at radio frequencies.

The literature on the *Goos-Hänchen* effect and related topics will be reconsidered and reviewed in the main text where the need arises. A rather comprehensive treatment may be found in *Brekho'skii's* book [30]. The review papers [31] touching on the subject under consideration will, however, not be mentioned again.

1. Reflection and Refraction of a Beam of Light

A beam of light is, for the sake of nomenclature, defined as an inhomogeneous "plane" electromagnetic wave linearly polarized². Its amplitude is appreciably different from zero only in a limited range perpendicular to the direction of propagation. The reflection and refraction, when such a beam encounters a plane interface, is here treated on the basis of an approximate solution of Maxwell's equations. This solution is approximate due to the limited extent of the electromagnetic beam-field and good only if the beam-field changes slowly over the distance of one wavelength. It was originally derived by *Lorentz* [32] from the rigorous "plane-wave" solution by assuming an appropriate amplitude function in place of the constant amplitude factor. The amplitude function is, for simplicity, taken to be real, thus discounting the inherent diffraction of a beam of light. *Artmann* [33] noted correctly that this assumption is unrealistic from a physical point of view since, as he could prove, the amplitude function must be complex in general. *Von Fraunhofer* and *Schaefer* [34], however, defended the assumption on the basis of the constraints (3), mentioned above, since diffraction is of no concern to the subject under consideration. They argued that the imaginary part can be made so small that the amplitude function appears to be essentially real. Diffraction effects will be discussed in Section 5.3.

Nevertheless, we have adopted the theory of approximation because of its great clarity in describing the physical processes we intend to investigate. The fact that this theory leads straightforwardly to the familiar formulas in the limit as the beam becomes a plane wave produces a degree of confidence. The mathematical description of the *Goos-Hänchen* effect, thus derived, is not only consistent with other approaches, but even more general than the classi-

² The reader should bear in mind that the word "wave" appearing alone is used throughout this paper to denote the propagation of a transversely unlimited disturbance, in most cases (with the exception of Sections 5.2 and 5.3) in the sense of a plane wave.

cal results. The theoretical predictions are fully substantiated by measurements; and this, the close agreement between theory and experiment, is the ultimate test of any theory.

The problem of reflection and refraction can be formulated on the basis of the plane-wave expansion. According to *Debye* [35] the field of a transversely limited wave can be represented by an integral over plane waves with different directions of propagation. This representation can be modified to account for arbitrary distortions in two or three dimensions [36]. Applying Fresnel's formulas to each wave component individually, *Pitch* [7] rigorously solved the problem for the case of total reflection. However, his analysis is so complex and involved that the physical mechanism remains obscure. Physical clarity can be obtained by considering only two or three plane waves propagating under small angles with respect to one another [8], [15] and [16]), but then the formulation is again approximate. A similar difficulty was recently encountered by *Pawqean* [20]: he could express the problem with a convolution integral in a rigorous manner, but was seemingly unable to solve for it. — The approach adopted in this paper is an approximate theory from the beginning, offering the advantage of great clarity in interpreting the physical processes under investigation. This approach is likely to be superior to others, in particular to those utilizing plane-wave expansions, if nonlinearities are involved. As an example, *Lotsch* [3] recognized that the *Goos-Hänchen* effect can be interpreted in terms of nonlinear optics. *Bloembergen* et al. [4b], in turn, suggested that the *Goos-Hänchen* effect should also be observable at light harmonics. This suggestion will be theoretically substantiated in Section 4.4 for the special case where the less-dense medium is nonlinear only.

The approximate solution of Maxwell's equations, mentioned above, is utilized in setting up the problem of reflection and refraction. The beam of light is assumed to be linearly polarized either in the plane of incidence or at right angles to it. The continuity of the electromagnetic field across the interface between the optically transparent media, combined with the principle of conservation of energy [3], leads to unique relations for the amplitudes of the reflected and refracted beams in terms of the amplitude of the incident beam. These relations represent the well-known Fresnel formulas plus additional terms due to the limited extent of the beam-fields.

This chapter is divided into two sections. The first section deals with the explicit representation of the electromagnetic beam-fields for both polarizations. The second section presents the formulas for the amplitudes of the reflected and refracted beams, including the laws of reflection and refraction.

Needless to say, the following derivation applies to acoustics as well and can be readily interpreted in the notation of a beam of sound. This is because both longitudinal and transverse acoustical waves obey the wave equation (2b). However, it does not account for coupling between longitudinal and transverse waves since the optical field only supports transverse waves. For a comprehensive treatment of the acoustical problem the reader is referred elsewhere (e.g., [22a] and [30]).

1.1 Representation of Electromagnetic Beam-Fields

Consider Maxwell's equations for a homogeneous and isotropic medium, free of charges, with the (relative) real dielectric constant (permittivity) ϵ and the (relative) magnetic permeability of unity. When the problem is completely independent of one cartesian coordinate, say y , Maxwell's equations may be split up into two independent sets:

$$\left. \begin{aligned} \frac{\epsilon}{c} \frac{\partial E_y}{\partial t} &= \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, & \frac{\partial H_x}{\partial x} + \frac{\partial H_z}{\partial z} &= 0, \\ \frac{1}{c} \frac{\partial H_x}{\partial t} &= \frac{\partial E_y}{\partial z}, & -\frac{1}{c} \frac{\partial H_z}{\partial t} &= \frac{\partial E_y}{\partial x}, \end{aligned} \right\} \quad (1a)$$

and

$$\left. \begin{aligned} \frac{\epsilon}{c} \frac{\partial E_x}{\partial t} &= -\frac{\partial H_y}{\partial z}, & \frac{\epsilon}{c} \frac{\partial H_z}{\partial t} &= \frac{\partial H_y}{\partial x}, \\ \frac{1}{c} \frac{\partial H_y}{\partial t} &= \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}, & \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} &= 0, \end{aligned} \right\} \quad (1b)$$

where the Gaussian system of units is used and

- $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$: vector of electric field strength
 $\mathbf{H} = \mathbf{H}(\mathbf{r}, t)$: vector of magnetic field strength
 $\mathbf{r} = \mathbf{r}(x, y, z)$: vector of position in cartesian coordinate system
 c : speed of light in vacuum
 t : variable of time.

The first group involves only E_y , H_x , H_z and the second only E_x , E_z , H_y . Simplification can, therefore, be obtained by separating any solution into a linear combination of the two solutions for which every member of one of the above sets is zero. For the sake of nomenclature we characterize the two types of fields as follows:

a) E at right angles to the plane of incidence, E polarization

$$E_y, H_x, H_z \quad \text{and} \quad E_x = E_z = H_y = 0. \quad (2a)$$

As is evident on substituting for H_x and H_z into the first equation of (1a), the complete field is here specified in terms of E_y . This component, of course,

satisfies the two-dimensional form of the scalar wave equation, i.e., using the shorthand ψ for E_y

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} \quad (2b)$$

where $v = c/\sqrt{\epsilon}$ is the speed of light in the dielectric medium.

The plane electromagnetic wave having a uniform and spatially unimitted wavefront represents a rigorous solution of Maxwell's equations. Replacing its constant amplitude factor by the real amplitude function $A(\beta)$, where β , to be defined below, depends upon the transverse coordinate ζ , leads to an approximate solution of Maxwell's equations if the constraints

$$\left| \frac{d^2 A}{d\beta^2} \right| \lesssim \left| \frac{dA}{d\beta} \right| \lesssim |A| = |A(\beta)| \quad \text{for} \quad (-\infty \leq \zeta \leq +\infty) \quad (3)$$

are satisfied, where \lesssim means "at most of the orders of magnitude of", [6]. In the limit, as $|d^2 A/d\beta^2|$ approaches zero, the approximate solution thus characterized becomes the rigorous plane-wave solution again, [33a]. The amplitude function $A(\beta)$ is, for simplicity, assumed in the following manner: Let $\zeta_a > \zeta_1 > 0$ and $\zeta_1 \gg (\zeta_a - \zeta_1) \gg \lambda_0$, where λ_0 is the spatial period (wavelength) in vacuum. Then: for $|\zeta| < \zeta_1$, $A(\beta)$ is constant, for $|\zeta| > \zeta_a$ $A(\beta)$ is zero, and for $\zeta_a > |\zeta| > \zeta_1$ $A(\beta)$ decreases gradually to zero. This behavior must be symmetric with respect to the plane $\zeta = 0$; and the amplitude must decay to zero in accordance with (3), i.e., the amplitude and its derivative can only change slowly over the distance of one wavelength.

Let us assume that the beam of light propagates in the positive ξ direction. The ξ axis intersects the positive x axis at the angle $(\pi/2 - q)$ where q , measured counterclockwise from the z axis as explained in Fig. 3, is restricted to the range $(0, \pi/2)$. Then, the E-polarized beam field is described by the approximate solution of Maxwell's equations, using the real part of the complex notation, [3]

$$\left. \begin{aligned} E_y &= A_{\perp} A(\beta) e^{i\beta} - i\alpha_{\perp} (dA/d\beta) e^{i\beta} \\ H_x &= \sqrt{\epsilon} E_y \cos q - i\sqrt{\epsilon} \alpha_{\perp} A_{\perp} (dA/d\beta) e^{i\beta} \sin q \\ H_z &= \sqrt{\epsilon} E_y \sin q + i\sqrt{\epsilon} \alpha_{\perp} A_{\perp} (dA/d\beta) e^{i\beta} \cos q \\ E_x &= E_z = H_y = 0 \quad \text{and} \quad |\alpha| \ll 1 \end{aligned} \right\} \quad (4)^3$$

³ This ansatz is written such, although $\epsilon = 1$ for λ_0 , that it is readily applicable in Section 1.2. The argument β can equivalently be expressed in terms of the propagation constant k for the angular frequency ω , i.e., $\beta \equiv (\omega t - k\zeta)$.

with

$$\beta = (2\pi/\lambda_0) \zeta \alpha, \quad \zeta = x \cos \varphi + z \sin \varphi,$$

$$\vartheta = (2\pi/\lambda_0) (ct/\sqrt{\epsilon} - \xi); \quad \xi = x \sin \varphi - z \cos \varphi,$$

where A_{\perp} , a_{\perp} and α are constants. The largest value which a_{\perp} may assume depends upon the magnitudes of A_{\perp} , $dA/d\beta$ and α , as pointed out in Reference 3.

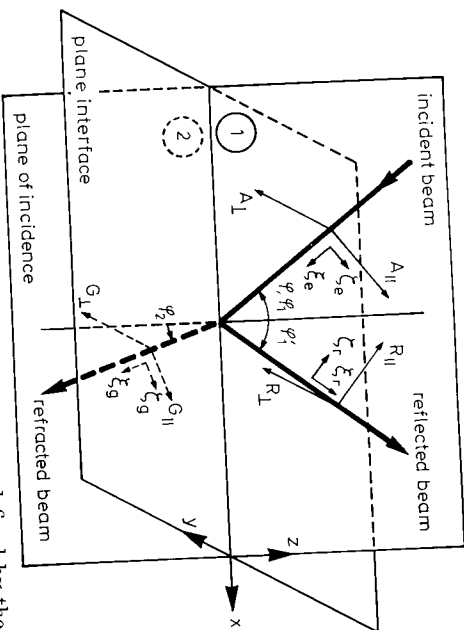


Fig. 3. The incident, reflected and refracted beams are defined by the respective amplitudes A_{\perp} , R_{\perp} and G_{\perp} in the cartesian coordinate system adopted. The symbol \perp is used to designate E polarization since the E vectors are at right angles to the plane of incidence, and the symbol \parallel to designate H polarization since the E vectors are in the plane of incidence.

b) *E in plane of incidence, H polarization*

$$E_x, E_z, H_y \quad \text{and} \quad E_y = H_x = H_z = 0. \quad (5)$$

This case can be treated in a manner similar to the E polarization. With the corresponding notation the H-polarized beam-field is described by the approximate solution of Maxwell's equations, using the real part of the complex notation, [3]

$$\left. \begin{aligned} H_y &= -\sqrt{\epsilon} A_{\parallel} A(\beta) e^{i\vartheta} + i\sqrt{\epsilon} \alpha a_{\parallel} (dA/d\beta) e^{i\vartheta} \\ E_x &= -(1/\sqrt{\epsilon}) H_y \cos \varphi - i\alpha A_{\parallel} (dA/d\beta) e^{i\vartheta} \sin \varphi \\ E_z &= -(1/\sqrt{\epsilon}) H_y \sin \varphi + i\alpha A_{\parallel} (dA/d\beta) e^{i\vartheta} \cos \varphi \\ E_y &= H_x = H_z = 0 \quad \text{and} \quad |\alpha| \ll 1, \end{aligned} \right\} \quad (6)$$

where A_{\parallel} , a_{\parallel} and α are constants. The largest value which a_{\parallel} may assume depends upon the magnitudes of A_{\parallel} , $dA/d\beta$ and α , as pointed out in Reference 3.

1.2 *Laws of Reflection and Refraction, Fresnel's Formulas*

Suppose that the relative dielectric constant is ϵ_1 in the half space $z > 0$ and $\epsilon_2 \approx \epsilon_1$ in the half space $z < 0$. On substituting ϵ_1 for ϵ and φ_1 for φ in (4), the real part of this equation describes an E-polarized beam, designated the incident beam. As depicted in Fig. 3, the incident beam encounters the interface between the two media at the angle of incidence φ_1 , measured counterclockwise from the positive z axis. In the same manner we deduce the representations for the incident (index "e"), the reflected (index "r") and the refracted (index "g") E-polarized beams⁴ from (4), [3] and [5])

$$\left. \begin{aligned} E_y^e &= A_{\perp} A(\beta_e) e^{i\vartheta_e} - i\alpha_1 a_{\perp} (dA/d\beta_e) e^{i\vartheta_e} \\ H_x^e &= \sqrt{\epsilon_1} E_y^e \cos \varphi_1 - i\sqrt{\epsilon_1} \alpha_1 A_{\perp} (dA/d\beta_e) e^{i\vartheta_e} \sin \varphi_1 \\ H_z^e &= \sqrt{\epsilon_1} E_y^e \sin \varphi_1 + i\sqrt{\epsilon_1} \alpha_1 A_{\perp} (dA/d\beta_e) e^{i\vartheta_e} \cos \varphi_1 \end{aligned} \right\} \quad (7)$$

with

$$\beta_e = (2\pi/\lambda_1) \zeta_e \alpha_1, \quad \zeta_e = x \cos \varphi_1 + z \sin \varphi_1,$$

$$\vartheta_e = (2\pi/\lambda_1) (ct/\sqrt{\epsilon_1} - \xi_e), \quad \xi_e = x \sin \varphi_1 - z \cos \varphi_1;$$

$$\left. \begin{aligned} E_y^r &= R_{\perp} A(\beta_r) e^{i\vartheta_r} - i\alpha_1 r a_{\perp} (dA/d\beta_r) e^{i\vartheta_r} \\ H_x^r &= \sqrt{\epsilon_1} R_{\perp} \cos \varphi_1 - i\sqrt{\epsilon_1} \alpha_1 R_{\perp} (dA/d\beta_r) e^{i\vartheta_r} \sin \varphi_1 \\ H_z^r &= \sqrt{\epsilon_1} R_{\perp} \sin \varphi_1 + i\sqrt{\epsilon_1} \alpha_1 R_{\perp} (dA/d\beta_r) e^{i\vartheta_r} \cos \varphi_1 \end{aligned} \right\} \quad (8)$$

with

$$\beta_r = (2\pi/\lambda_1) \zeta_r \alpha_1, \quad \zeta_r = x \cos \varphi_1 + z \sin \varphi_1,$$

$$\vartheta_r = (2\pi/\lambda_1) (ct/\sqrt{\epsilon_1} - \xi_r), \quad \xi_r = x \sin \varphi_1 - z \cos \varphi_1;$$

and

$$\left. \begin{aligned} E_y^g &= G_{\perp} A(\beta_g) e^{i\vartheta_g} - i\alpha_2 g a_{\perp} (dA/d\beta_g) e^{i\vartheta_g} \\ H_x^g &= \sqrt{\epsilon_2} G_{\perp} \cos \varphi_2 - i\sqrt{\epsilon_2} \alpha_2 G_{\perp} (dA/d\beta_g) e^{i\vartheta_g} \sin \varphi_2 \\ H_z^g &= \sqrt{\epsilon_2} G_{\perp} \sin \varphi_2 + i\sqrt{\epsilon_2} \alpha_2 G_{\perp} (dA/d\beta_g) e^{i\vartheta_g} \cos \varphi_2 \end{aligned} \right\} \quad (9)$$

with

$$\beta_g = (2\pi/\lambda_2) \zeta_g \alpha_2, \quad \zeta_g = x \cos \varphi_2 + z \sin \varphi_2,$$

$$\vartheta_g = (2\pi/\lambda_2) (ct/\sqrt{\epsilon_2} - \xi_g), \quad \xi_g = x \sin \varphi_2 - z \cos \varphi_2;$$

⁴The indices e, r and g refer to the German words "einfallend", "reflektiert", and "gebrochen" meaning in translation incident, reflected and refracted, respectively.

where φ_1 is the angle of incidence, φ_1' is the angle of reflection, and φ_2 is the angle of refraction. The magnitudes of the constants α_1 , α_1' and α_2 are assumed to be much smaller than one.

All the electric-field vectors of the E-polarized beam are at right angles to the plane of incidence, i.e., the xz plane. For the electromagnetic field to be continuous across the interface at $z = 0$, the following boundary conditions must be satisfied

$$E_y e + E_y' e = E_y e' \quad \text{and} \quad H_x e + H_x' e = H_x e'. \quad (10)$$

In the case of a plane-wave "ansatz" these conditions are sufficient to determine the laws of reflection and refraction, as well as the amplitudes of the reflected and refracted waves in terms of that of the incident wave. For a beam of light described by an approximate solution of Maxwell's equations they are insufficient and must be supplemented by the principle of conservation of energy. [3]. Complying with all conditions yields the *law of reflection*

$$\varphi_1' = \varphi_1 \quad \text{and} \quad \lambda_1' = \lambda_1 \quad (\text{since } \sin \varphi_1' = \sin \varphi_1), \quad (11)$$

the *law of refraction* (Snell's law)

$$\sqrt{\epsilon_2} \sin \varphi_2 = \sqrt{\epsilon_1} \sin \varphi_1 \quad \text{and} \quad \sqrt{\epsilon_2} \lambda_2 = \sqrt{\epsilon_1} \lambda_1 \quad (12a)$$

or defining the *index of refraction* $n = \sqrt{\epsilon_2/\epsilon_1}$

$$n \sin \varphi_2 = \sin \varphi_1 \quad \text{and} \quad n \lambda_2 = \lambda_1, \quad (12b)$$

the *Fresnel formulas*

$$R_{\perp} = \left(\frac{\cos \varphi_1 - n \cos \varphi_2}{\cos \varphi_1 + n \cos \varphi_2} \right) A_{\perp} \equiv - \frac{\sin(\varphi_1 - \varphi_2)}{\sin(\varphi_1 + \varphi_2)} A_{\perp} \quad (13a)$$

$$G_{\perp} = \left(\frac{2 \cos \varphi_1}{\cos \varphi_1 + n \cos \varphi_2} \right) A_{\perp} \equiv \frac{2 \sin \varphi_2 \cos \varphi_1}{\sin(\varphi_1 + \varphi_2)} A_{\perp}, \quad (13b)$$

the extra quantities a_{\perp} , r_{\perp} and g_{\perp} , which, in a sense, represent the corrections to Fresnel's formulas due to the amplitude variation of the incident beam, [3]

$$a_{\perp} = - \tan \varphi_2 \left(\frac{\cos \varphi_1 - n \cos \varphi_2}{\cos \varphi_1 + n \cos \varphi_2} \right) A_{\perp} \quad (14a)$$

$$r_{\perp} = \tan \varphi_2 \left(\frac{3 \cos \varphi_1 + n \cos \varphi_2}{\cos \varphi_1 + n \cos \varphi_2} \right) R_{\perp} \quad (14b)$$

$$g_{\perp} = - \frac{\tan \varphi_1}{\cos \varphi_1} \left(\frac{2 \cos \varphi_1 + n \cos \varphi_2}{\cos \varphi_1 + n \cos \varphi_2} \right) (\cos \varphi_1 - n \cos \varphi_2) G_{\perp}, \quad (14c)$$

and the relations between the constants α_1 , α_1' and α_2 , noting that $\cos \varphi_1' = -\cos \varphi_1$, [3]

$$\alpha_1' = -\alpha_1 \quad \text{and} \quad \alpha_2 = \alpha_1 \cos \varphi_1 \sqrt{n^2 - \sin^2 \varphi_1}. \quad (15)$$

Obviously, if $\sin \varphi_1$ can get equal to n , α_2 becomes singular and the condition $|\alpha_1| \ll 1$ stated in (4) cannot be met. To comply with this condition we must, therefore, exclude a narrow angle range about the angle $\Phi = \arcsin(n)$, representing the *critical angle for total reflection*. This restriction is not serious since the range in question is very narrow. It can be removed in the limit of a plane-wave ansatz since $(dA/d\beta)$ becomes identically zero.

The case of H polarization can be treated in a similar manner. The important results are the Fresnel formulas

$$R_{\parallel} = \left(\frac{n \cos \varphi_1 - \cos \varphi_2}{n \cos \varphi_1 + \cos \varphi_2} \right) A_{\parallel} \equiv \frac{\tan(\varphi_1 - \varphi_2)}{\tan(\varphi_1 + \varphi_2)} A_{\parallel} \quad (16a)$$

$$G_{\parallel} = \left(\frac{2 \cos \varphi_1}{n \cos \varphi_1 + \cos \varphi_2} \right) A_{\parallel} \equiv \frac{2 \cos \varphi_1 \sin \varphi_2}{\sin(\varphi_1 + \varphi_2) \cos(\varphi_1 - \varphi_2)} A_{\parallel} \quad (16b)$$

and the extra quantities due to the amplitude variation of the incident beam, [3]

$$a_{\parallel} = - \left[\frac{n^2 \cos \varphi_1 - n \cos \varphi_2}{\cos \varphi_2 (n \cos \varphi_1 + \cos \varphi_2)} \right] A_{\parallel} \quad (17a)$$

$$r_{\parallel} = \frac{2}{n \cos \varphi_1 + \cos \varphi_2} \left(\frac{\cos^2 \varphi_1 \tan \varphi_2 - n \sin \varphi_1 \cos \varphi_2}{n \cos \varphi_1 - \cos \varphi_2} \right) + \frac{n^2 \cos \varphi_1 - n \cos \varphi_2}{2 \cos \varphi_2} R_{\parallel} \quad (17b)$$

$$g_{\parallel} = - \frac{n \cos \varphi_1 - \cos \varphi_2}{\cos \varphi_1 (n \cos \varphi_1 + \cos \varphi_2)} \times \left(\frac{\cos \varphi_1 \sin \varphi_2 - n \tan \varphi_1 \cos^2 \varphi_2}{n \cos \varphi_1 - \cos \varphi_2} + n^2 \right) G_{\parallel}. \quad (17c)$$

The above results are essentially familiar to us from text books on classical optics; new are only the formulas (14) and (17) resulting from the amplitude variation of the incident beam. These extra quantities lose their significance the farther the beam extends in its transverse direction, as may be inferred from (4) and (6), leaving behind the well-known Fresnel formulas. These classical formulas are utilized in many present-day applications [31] and are gaining further importance in modern research topics such as, for example, internal reflection spectroscopy [37], planar dielectric waveguides [38], and enhancement of detector sensitivity [39]. They have recently been reconsidered in the literature; for instance, *Shieh* [40] suggested a simplified derivation; and *Komrska* [41], and *Ghezzi* and *Busen* [42] proposed graphical methods of interpretation.

The extra quantities (14) and (17) are only stated here without explanation. Their significance will become apparent in the following chapter dealing with the special case of total reflection.

2. Total Reflection of an E-Polarized Beam

The theory of total reflection is outlined in this chapter. We assume that the incident beam with E polarization propagates in an optically denser medium toward an optically less-dense medium. If the angle of incidence exceeds the critical angle for total reflection Φ , defined by

$$\sin \Phi = n \quad (18)$$

where $n = \sqrt{\epsilon_2/\epsilon_1} < 1$ is the index of refraction, the refracted beam disappears. The entire light energy impinging upon the interface is reflected back into the denser medium. However, continuity of the electromagnetic field across the interface requires an evanescent wave in the less-dense medium. This inhomogeneous wave has a (time-average) Poynting vector parallel to the interface, if the incident beam is approximated by a plane wave. It decays very rapidly in the negative z direction pointing into the less-dense medium. Since there is no net flow of energy into the less-dense medium physicists have been puzzled by the flow of energy in the less-dense medium for several decades, as outlined in Reference 3. *Voigt* [9]⁵ was the first at the turn of this century to recognize that the limited extent of the electromagnetic beam-field, which has been avoided in theoretical investigations by using the plane-wave ansatz, plays a significant role in this respect. Finally, about three decades later, *Picht*, when investigating the total reflection of a beam of light, reached the remarkable conclusion stated on p. 496 of his paper [7]: „Das Auftreten einer Energieströmung im zweiten Medium (optically less-dense medium) ist dadurch hervorgerufen, daß es an der Trennungsebene beider Medien Stellen gibt, an denen unter sehr geringer Neigung gegen jene Ebene *im zeitlichen Mittel* dauernd Energie vom ersten ins zweite Medium übertritt, die dann an anderen Stellen der Trennungsebene restlos wieder ins

erste Medium zurückfließt. Die über die Trennungsebene hin und her pendelnde Energie ist von der Größenordnung der einfallenden Energie.“ *Noether* [8] demonstrated *Picht's* conclusion in a simple manner, thus shedding much light on the physics involved which remained somewhat obscure in *Picht's* complex mathematical treatment. This famous conclusion is here illustrated by means of our approximate solution of Maxwell's equations in a manner similar to *Schaefer* and *Pich* [5]. It is shown that, due to the limited extent of the beam-field, energy is extracted from one side of the incident beam and added to the other side of the reflected beam, in the plane of incidence, after having propagated parallel to the interface in the less-dense medium.

This chapter⁶ is divided into two sections. The first section presents the mathematical description of the electromagnetic beam-fields in both media. The time-averaged Poynting vector is considered in the second section, placing emphasis on the interpretation of the light penetration into the less-dense medium.

2.1 Description of Electromagnetic Beam-Fields at Total Reflection

At total reflection $\cos \varphi_2$ is purely imaginary since $1 > \sin^2 \varphi > n^2$ and can thus be written in the form, [3]

$$\cos \varphi_2 = -iw/n \quad \text{with} \quad w = +\sqrt{\sin^2 \varphi - n^2}. \quad (19)$$

The negative sign of the square root is rejected on physical grounds. With the time convention $\exp(i\omega t)$, adopted in this paper, a negative w would lead to an exponentially increasing field in the less-dense medium, as may be inferred from (27) below.

The Fresnel formulas (13) can be transformed into⁷

$$R_{\perp} = A_{\perp} e^{i\delta}, \quad \text{and} \quad G_{\perp} = \frac{2 \cos \varphi}{\sqrt{1-n^2}} A_{\perp} e^{i\delta} \epsilon_2 \quad (20a)$$

where the phase shift δ at total reflection is given by

$$\delta_{\perp} = 2\delta_g = 2 \arctan \left(\frac{\sqrt{\sin^2 \varphi - n^2}}{\cos \varphi} \right). \quad (20b)$$

⁶ The reader should bear in mind that, in order to simplify the notation, the electromagnetic field and the conclusions drawn from it are expressed in terms of quantities defined in the region I of the optically denser medium. The subscript “1” is henceforth dropped in the interest of brevity, i.e., $\alpha_1, \epsilon_1, \varphi_1$ and A_1 will tacitly be replaced by $\alpha, \epsilon, \varphi$ and A , respectively. If exceptions should be required for clarity, the subscripts will conform with Fig. 3.

⁷ It should be noted that all quantities δ are defined for E polarization, but the indicator \perp is omitted for simplicity of the notation.

⁵ See footnote 1.

The extra quantities (14) can be written in the form, [3]

$$a_{\perp} = -i a A_{\perp} e^{i\delta_r} \quad \text{with} \quad a = \sin q/w \quad (21a)$$

$$r_{\perp} = i r A_{\perp} e^{i(\theta_g/2 + \delta')} \quad \text{with} \quad r = \frac{\sin q \sqrt{1 + 8 \cos^2 q - n^2}}{w \sqrt{1 - n^2}} \quad (21b)$$

$$g_{\perp} = -2 g A_{\perp} e^{i(\theta_g + \delta'')} \quad \text{with} \quad g = \tan q \frac{\sqrt{1 + 3 \cos^2 q - n^2}}{\sqrt{1 - n^2}}, \quad (21c)$$

where $3 \tan \delta' = 2 \tan \delta'' = -w/\cos q$, defining δ' about the point π .

The argument θ_g defined in (9) becomes complex owing to (19). With the shorthand

$$\theta_g' = (2\pi/\lambda) (ct/\sqrt{\epsilon} - x \sin q)$$

the exponential function $\exp(i\theta_g)$ can be written in product form

$$\exp(i\theta_g) = \exp(i\theta_g') \exp(2\pi z w/\lambda). \quad (22)$$

Hence, the electromagnetic field (9) decays rapidly as z decreases from zero at the interface to negative values in the less-dense medium. Eq. (22) describes a wave propagating parallel to the interface with the phase velocity $c/(\sqrt{\epsilon} \sin q)$ determined only by q and the properties of the denser medium, [38].

The argument β_g defined in (9) becomes complex, too. Using (15) and (19) it can be expressed in the form

$$\beta_g = (2\pi/\lambda) [x \cos q + i z \sin q (\cos q/w)] \alpha \equiv (\chi + i\Omega). \quad (23)$$

The function $A(\beta_g)$ is, therefore, complex and should be known explicitly since only the real parts of (7) through (9) are physically meaningful. *Schaefer* and *Pich* [6] argued, however, that due to the rapid decay of the electromagnetic field in the less-dense medium the explicit form of $A(\beta_g)$ does not need to be specified as long as z is restricted to such values that $|z/\lambda| \lesssim 1$. Hence, if $A(\beta_g)$ is a regular, analytic function of $\beta_g = (\chi + i\Omega)$ we can, in a first approximation, write for the region $|z/\lambda| \lesssim 1$, since $|\Omega| \ll 1$ due to the constraint on α

$$A(\beta_g) \approx A(\chi) + i\Omega \frac{\partial A}{\partial \chi} = A(\chi) + i2\pi \frac{z}{\lambda} \alpha \sin q (\cos q/w) \frac{\partial A}{\partial \chi} \quad (24a)$$

and

$$\frac{dA}{d\beta_g} \approx \frac{\partial A}{\partial \chi} + i\Omega \frac{\partial^2 A}{\partial \chi^2} \approx \frac{\partial A}{\partial \chi}, \quad (24b)$$

where the imaginary part of (24b) is neglected on the basis of (3).

Finally, we substitute the various results derived above into (7) through (9) and consider only the real parts. Then, we obtain, at total reflection, the approximate solutions of Maxwell's equations for the incident beam with E polarization, [3]

$$\left. \begin{aligned} E_y^e &= A_{\perp} [A(\beta_e) \cos \delta_e - a \alpha (dA/d\beta_e) \cos(\delta_e + \delta_r)] \\ H_x^e &= \sqrt{\epsilon} E_y^e \cos q + \sqrt{\epsilon} \alpha A_{\perp} (dA/d\beta_e) \sin \delta_e \sin q \\ H_z^e &= \sqrt{\epsilon} E_y^e \sin q - \sqrt{\epsilon} \alpha A_{\perp} (dA/d\beta_e) \sin \delta_e \cos q \end{aligned} \right\} \quad (25)$$

with

$$\begin{aligned} \beta_e &= (2\pi/\lambda) \alpha \zeta_e, & \zeta_e &= x \cos q + z \sin q, \\ \delta_e &= (2\pi/\lambda) (ct/\sqrt{\epsilon} - \xi_e), & \xi_e &= x \sin q - z \cos q; \end{aligned}$$

for the reflected beam

$$\left. \begin{aligned} E_y^r &= A_{\perp} [A(\beta_r) \cos(\delta_r + \delta_r) - r \alpha (dA/d\beta_r) \cos(\delta_r + 3\delta_r/2 + \delta'')] \\ H_x^r &= -\sqrt{\epsilon} E_y^r \cos q - \sqrt{\epsilon} \alpha A_{\perp} (dA/d\beta_r) \sin \delta_r + \delta_r \sin q \\ H_z^r &= \sqrt{\epsilon} E_y^r \sin q - \sqrt{\epsilon} \alpha A_{\perp} (dA/d\beta_r) \sin \delta_r + \delta_r \cos q \end{aligned} \right\} \quad (26)$$

with

$$\begin{aligned} \beta_r &= (2\pi/\lambda) \alpha \zeta_r, & \zeta_r &= -x \cos q + z \sin q, \\ \delta_r &= (2\pi/\lambda) (ct/\sqrt{\epsilon} - \xi_r), & \xi_r &= x \sin q + z \cos q; \end{aligned}$$

and for the electromagnetic field in the narrow region $|z/\lambda| \lesssim 1$ of the less-dense medium ($z < 0$)

$$\begin{aligned} E_y^g &= \frac{2 \cos q}{\sqrt{1 - n^2}} e^{-2\pi w z/\lambda} A_{\perp} \left\{ A(\chi) \cos(\theta_g' + \delta_g) \right. \\ &\quad - g \frac{\cos q}{w} \alpha \frac{\partial A}{\partial \chi} \cos(\theta_g' + 2\delta_g + \delta'') + g \alpha \frac{\partial A}{\partial \chi} \sin(\theta_g' + 2\delta_g + \delta'') \\ &\quad \left. - 2\pi \sin q \frac{\cos q}{w} \alpha \frac{\partial A}{\partial \chi} \sin(\theta_g' + \delta_g) \right\} \quad (27a) \end{aligned}$$

$$\begin{aligned} \mathbf{H}_z^E = & \frac{2w}{\sqrt{1-n^2}} \sqrt{\epsilon} \cos q e^{2\pi w z/\lambda} A_{\perp} \left\{ A(\chi) \sin(\theta_g' + \delta_g) \right. \\ & - g \frac{\cos q}{w} \alpha \frac{\partial A}{\partial \chi} \sin(\theta_g' + 2\delta_g + \delta'') - g \alpha \frac{\partial A}{\partial \chi} \cos(\theta_g' + 2\delta_g + \delta'') \\ & \left. + \alpha \sin q \frac{\cos q}{w^2} \frac{\partial A}{\partial \chi} \left[2\pi \frac{z}{w} + 1 \right] \cos(\theta_g' + \delta_g) \right\} \end{aligned} \quad (27b)$$

$$\begin{aligned} \mathbf{H}_z^E = & \frac{2\sqrt{\epsilon} \sin q \cos q}{\sqrt{1-n^2}} e^{2\pi w z/\lambda} A_{\perp} \left\{ A(\chi) \cos(\theta_g' + \delta_g) \right. \\ & - g \frac{\cos q}{w} \alpha \frac{\partial A}{\partial \chi} \cos(\theta_g' + 2\delta_g + \delta'') + g \alpha \frac{\partial A}{\partial \chi} \sin(\theta_g' + 2\delta_g + \delta'') \\ & \left. - \alpha \sin q \frac{\partial A}{\partial \chi} \left[2\pi \frac{z}{w} + \frac{\cos q}{\lambda} + \frac{\cos q}{\sin^2 q} \right] \sin(\theta_g' + 2\delta_g + \delta'') \right\} \end{aligned} \quad (27c)$$

with

$$\theta_g' = (2\pi/\lambda) (ct/\sqrt{\epsilon} - x \sin q), \quad \chi = (2\pi/\lambda) \alpha x \cos q,$$

where $E_x = E_y = H_z = 0$, $w = +\sqrt{\sin^2 q - n^2}$, $1 > \sin^2 q > n^2$, $n = \sqrt{\epsilon_2/\epsilon_1} < 1$, $\tan(\delta_r/2) = \tan \delta_g = -3 \tan \delta' = -2 \tan \delta'' = w/\cos q$, and the quantities a , g and r are defined in (21).

The special case of an incident plane wave is contained in the above description for $dA/d\beta = \partial A/\partial \chi = 0$. To obtain it, we may simply set $\alpha = 0$. The electromagnetic field based on an incident beam with H polarization can be represented in a similar manner.

2.2 Light Energy at Total Reflection

The flow of light energy is meaningfully described by the Poynting vector $\mathbf{S} = (c/4\pi) \mathbf{E} \times \mathbf{H}$ time-averaged over the period $T = \lambda/\bar{v}c$. As a consequence of the assumed polarization the flow of energy is confined to the plane of incidence; the general case is briefly indicated in Section 5.1. Ignoring terms which involve squares of α , the time-averaged Poynting vectors are obtained for the incident beam with E polarization, [3]

$$\bar{\mathbf{S}}_e = \frac{c\sqrt{\epsilon}}{8\pi} A_{\perp}^2 (\hat{i} \sin q - \hat{k} \cos q) [A^2(\beta_0) - 2a \alpha A(\beta_0) (dA/d\beta_0) \cos \delta_r] \quad (28)$$

with

$$\beta_e = (2\pi/\lambda) \alpha \zeta_e, \quad \zeta_e = x \cos q + z \sin q;$$

for the reflected beam

$$\bar{\mathbf{S}}_r = \frac{c\sqrt{\epsilon}}{8\pi} A_{\perp}^2 (\hat{i} \sin q + \hat{k} \cos q) [A^2(\beta_1) - 2r \alpha A(\beta_1) (dA/d\beta_1) \cos(\delta_r/2 + \delta')] \quad (29)$$

with

$$\beta_r = (2\pi/\lambda) \alpha \zeta_r, \quad \zeta_r = -x \cos q + z \sin q;$$

and for the electromagnetic field in the narrow region $|z/\lambda| \approx 1$ of the less-dense medium ($z < 0$)

$$\begin{aligned} \bar{\mathbf{S}}_x^E = & \frac{c}{2\pi} \frac{\sqrt{\epsilon} \sin q \cos^2 q}{1-n^2} e^{4\pi w z/\lambda} A_{\perp}^2 \left\{ A^2(\chi) \right. \\ & - 2g \frac{\cos q}{w} \alpha A(\chi) \frac{\partial A}{\partial \chi} \cos(\delta_g + \delta'') \\ & \left. + \alpha A(\chi) \frac{\partial A}{\partial \chi} \left[2g - 2\pi \frac{z \sin q \cos q}{\lambda} - ct \sin q \right] \sin(\delta_g + \delta'') \right\} \end{aligned} \quad (30a)$$

and

$$\bar{\mathbf{S}}_z^E = -\frac{c}{2\pi} \frac{\sqrt{\epsilon} \cos^3 q \sin q}{1-n^2} \frac{\sin q}{w} e^{4\pi w z/\lambda} A_{\perp}^2 \alpha A(\chi) \frac{\partial A}{\partial \chi} \quad (30b)$$

with

$$\chi = (2\pi/\lambda) \alpha x \cos q,$$

where \hat{i} and \hat{k} are the unit vectors in the x and the z direction, respectively. All other quantities have been summarized in connection with (25) through (27), i.e., $w = +\sqrt{\sin^2 q - n^2}$, $1 > \sin^2 q > n^2$, $n = \sqrt{\epsilon_2/\epsilon_1} < 1$; $\tan(\delta_r/2) = \tan \delta_g = -3 \tan \delta' = -2 \tan \delta'' = w/\cos q$; a , g and r are defined in (21).

In the denser medium the beams are well defined and the flow of energy is determined by the similarly looking Poynting vectors (28) and (29). As may be inferred from (27), the electromagnetic field in the less-dense medium is rather involved. Its propagation of energy is, therefore, more complicated and given by the components (30) of the Poynting vector. In the limit of grazing incidence, i.e., as $q \rightarrow \pi/2$, the above expressions, however, reduce to the simple relation $\bar{\mathbf{S}}_r \equiv \bar{\mathbf{S}}_e$, revealing the conservation of energy, [3]. The special case of an incident plane wave is also contained in the above expressions. The familiar results are simply obtained by setting $\alpha = 0$.

Schaefer and Pich [6] developed a pictorial interpretation of the complex mathematical treatment presented in order to provide an insight into the

physics involved. The model of light penetration into the less-dense medium is outlined here, but for further details the reader is referred to Reference 3. Let us consider the narrow region $|z/\lambda| \ll 1$ within the less-dense medium. The Poynting vector intersects the interface at the angle $\gamma = \arctan(\sqrt{S_z^{\#}}/S_x^{\#})$, i.e., [3]

$$\tan \gamma = -\frac{\cos \varphi}{w} \alpha \frac{\partial A}{\partial x} \left\{ A(\gamma) - 2g \frac{\cos \varphi}{w} \alpha \frac{\partial A}{\partial x} \cos(\theta_g + \delta'') \right. \\ \left. + \left[2g - 2\pi \frac{z \sin \varphi \cos \varphi}{\lambda} - c \tan \varphi \right] \sin(\theta_g + \delta'') \right\}^{-1}. \quad (31)$$

This equation is illustrated in Fig. 4, assuming $\alpha > 0$. In the central portion M, where the amplitude function is constant and $\partial A/\partial x = 0$, we have $\tan \gamma = 0$. Consequently, the energy flows parallel to the interface in the less-dense medium, and in the denser medium $|\mathbf{S}^{\#}| = |\mathbf{S}^e|$ according to (28) and (29).

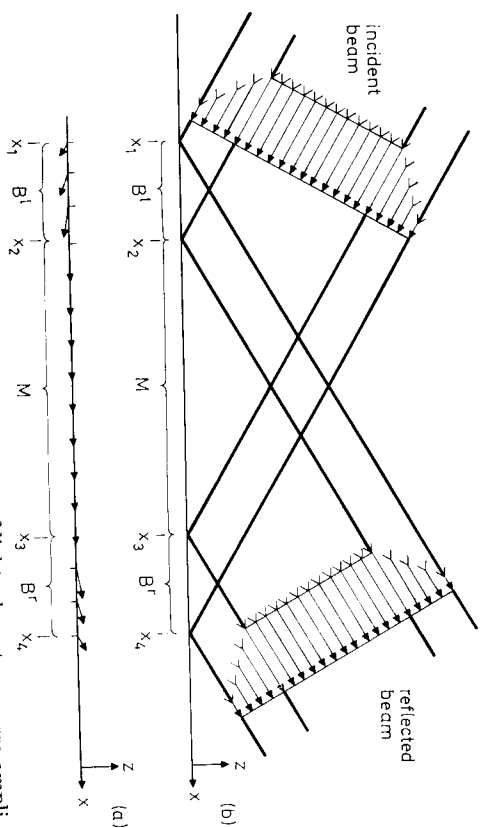


Fig. 4. Illustrating the total reflection of a beam of light whose transverse amplitude distribution is indicated according to the prescription adopted in Subsection 1.1a. Fig. (a) explains the pictorial interpretation discussed in Section 2.2, and Fig. (b) shows the flow of energy in the less-dense medium, as expressed by (31).

In the left border zone B^l, see Fig. 4a, where $\partial A/\partial x > 0$, the Poynting vector points into the less-dense medium at the small angle γ . This angle decreases along the interface toward the center of the beam. In the right border zone B^r, see Fig. 4a, where $\partial A/\partial x < 0$, the situation is reversed. This interpretation lends itself to the conclusion, illustrated in Fig. 4b, that in B^l energy enters the less-dense medium in the time average and the reflection is slightly less than total. The energy flows along the portion M of

the interface in the less-dense medium. In B^r it re-emerges into the denser medium where the reflection is thus slightly more than total. As indicated, the deviation from ideal total reflection is very minute since $|\alpha \partial A/\partial x| \ll |A(\gamma)|$ according to basic constraints.

Wolter [11] developed an interpretation of the light penetration into the less-dense medium which leads to the same overall effect, but differs from ours in details. In Wolter's picture the fraction of incident energy, which crosses the interface in B^l and is thus extracted from the reflected beam, is necessary to establish the evanescent wave in the less-dense medium. This amount of energy is then returned to the reflected beam in the other border zone B^r below which the evanescent wave dies away. Wolter demonstrated the existence of circulatory waves⁸ carrying energy back and forth across the interface. These waves provide a clear understanding of the corresponding phenomenon discovered by Picht [7] and later utilized by Armann [33a]. Wolter [11] has pointed out that its mathematical description requires the superposition of six or four waves (two plane waves of incidence plus two plane waves of reflection with or without two evanescent waves). Therefore, the phenomenon does not arise in the total reflection of a single plane wave. This fact resolves the puzzling result obtained when investigating the flow of energy on the basis of Fresnel's formulas, as indicated in Section 2.0. Fresnel's formulas are conventionally derived on the assumption of one single plane wave impinging upon an interface. Only in the case of a transversely limited wave which, as is well known [35], can be expanded accurately in a sum of plane waves, is the light penetration into the less-dense medium easily understood.

Needless to say, the foregoing discussion is also applicable, if properly interpreted, to the total reflection of a beam of sound. This acoustical problem was extensively treated by Schoch [22a] and Brekhovskikh [30a].

(To be continued)

⁸ Such waves were recently utilized by Lange [43] in investigating the electromagnetic field at complicated boundaries.