PAPER R

RECONSTRUCTION USING WAVE ASYMPTOTICS

Guan Y. Wang

ABSTRACT

An important and useful point of view is to describe wave propagation in terms of rays, rather than wave motion itself. It is natural to associate wave propagation with a Hamiltonian system, which can then be thought of as forming the skeleton of the waves in the short-wavelength limit, as in the relationship between waves and rays. In this paper, we review asymptotic wave theory and construct the Green's function using the traveltime and ray spreading from two point raytracing. To take into account caustics, we use the Maslov canonical operator method to obtain uniform asymptotic wave field that is valid even in regions where the regular ray theory is not applicable. We apply pseudodifferential operator theory, together with distorted Born approximation to develop an asymptotic inversion theory for variable background medium.

WAVE ASYMPTOTICS

Connections between areas of physics often involve limits. Wave optics reduces to ray optics in the limit of small wavelength, and statistical mechanics reduces to thermodynamics in the limit of many particles. These limits or asymptotics are usually singular and generate divergent series. A classical description of asymptotic wave theory for the example of light waves in the geometric optics limit is given in (Born and Wolf, 1970). In this section we shall review the assumptions, methods and results of the conventional eikonal approach to the approximate, asymptotic solution of a wave equation. In the study of the reduced wave equation

$$\nabla^2\psi + k_0^2 n^2 \psi = 0, \tag{1.1}$$

the basic assumption underlying the eikonal approach to the solution of the wave equations is that the waves described by $\psi(\mathbf{x}, t)$ are characterized by a typical wavenumber k and frequency ω which are large compared with the spatial and temporal rates of variation of the medium described by a wave equation operator. If this is the case, it is reasonable to assume that at a point $(\mathbf{x}, t)$ the wave solution looks roughly like a plane wave, but over a larger scale the amplitude, wavenumber and frequency may vary, as do the properties of the medium. These concepts are embodied in the solution to wave equation [1]:

$$\psi(\mathbf{x}, t) \equiv A(\mathbf{x}, t) e^{i\phi(\mathbf{x},t)} \tag{1.2}$$

In analogy with the plane-wave solutions in a uniform medium where the phase $\phi(\mathbf{x}, t)$ is $\mathbf{k} \cdot \mathbf{x} - \omega t$, the local wavenumber and frequency are defined to be the measure of the local rate of variation of the phase

$$k(\mathbf{x}, t) \equiv \partial_x \phi(\mathbf{x}, t), \qquad \omega(\mathbf{x}, t) \equiv -\partial_t \phi(\mathbf{x}, t). \tag{1.3}$$

If the length and time scales of variation of the medium are L and T respectively, we define the small dimensionless parameter ϵ

$$k(\mathbf{x}, t)L \sim \omega(\mathbf{x}, t)T \equiv \epsilon^{-1} >> 1 \tag{1.4}$$

and impose

$$A^{-1}(\partial_x A) \sim k^{-1}(\partial_x k) \sim \omega^{-1}(\partial_x \omega) \sim L^{-1} \tag{1.5}$$

$$A^{-1}(\partial_t A) \sim k^{-1}(\partial_t k) \sim \omega^{-1}(\partial_t \omega) \sim T^{-1} \tag{1.6}$$

[1] These locally quasi-plane waves also called eikonal waves which are asymptotically associated with Lagrangian submanifolds that don't have a singular projection. When an eikonal wave evolve in time, the dynamics may bend the corresponding Langranian submanifold over. At such times, the originally wave has ceased to be eikonal. The image of the points with a singular projection forms the caustic of the wave.

by which the assumptions about the variation of the amplitude $A(\mathbf{x}, t)$, wavenumber and frequency are made explicit. All higher derivatives are assumed to be of corresponding higher order in ϵ. Inserting Equation (1.2) into (1.1) and equating the coefficient of the power of k_0^2 to zero one obtains the eikonal equation

$$|\nabla\phi(\mathbf{x})|^2 - n(\mathbf{x}) = 0. \tag{1.7}$$

Equating the coefficient of the power of (ik_0), one obtains the transport equations [2].

$$2\nabla\phi(\mathbf{x}) \cdot \nabla A_0 + A_0 \nabla^2 \phi = 0 \tag{1.8}$$

Solution of eikonal equation

To solve eikonal equation with the initial conditions, it is necessary to construct the system of rays. One way of doing this is to write out the Hamilton system that is a system of ordinary differential equations in the phase space $(\mathbf{x}, p)$. In the case of our interest, the Hamiltonian $H(\mathbf{x}, p) = |p|^2 - n(\mathbf{x})$ which represents the total energy of the system as a function of the generalized position and momentum coordinates. The phase curves of the Hamilton system in the phase space $\mathcal{R}^4_{\mathbf{x},p}$ with the initial conditions are called bicharacteristics. Hamilton is constant along each bicharacteristics. Consequently, $H(\mathbf{x}, \mathbf{p}) = 0$ provide it is satisfied by the initial data. Based on the Hamiltonian formulation of ray tracing, the rays $\mathbf{x}(\sigma)$ are defined by projections of the bicharacteristic $\{\mathbf{x}(\sigma), \mathbf{k}(\sigma)\}$ of the eikonal equation. The bicharacteristics are given by the equations

$$\frac{d\mathbf{x}}{d\sigma} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \frac{d\mathbf{p}}{d\sigma} = -\frac{\partial H}{\partial \mathbf{x}} \tag{1.9}$$

The rays emitted by a point source at $\mathbf{x} = \mathbf{x}_0$ can be parameterized by their take-off angle θ and the solution to the equation (1.7) can be written in the form

$$\mathbf{x} = \mathbf{x}(\sigma, \theta), \qquad \mathbf{p} = \mathbf{p}(\sigma, \theta) \tag{1.10}$$

[2]In general, the amplitude should also be expressed as a power series in ϵ: $A = \sum_n \epsilon^n A_n$. As we shall consider only the lowest two orders in the approximation treatment, this expansion is not necessary.

It can be shown [3] that along the bicharacteristics, $|p|^2 = n(\mathbf{x})$. The explicit solutions for the eikonal equation in terms of the initial values ϕ_0 are:

$$\phi = \phi_0 + \int_0^{\sigma} n(\mathbf{x}(\sigma'))d\sigma' \tag{1.11}$$

As for the rays of the problem (1.9), they may intersect. A typical

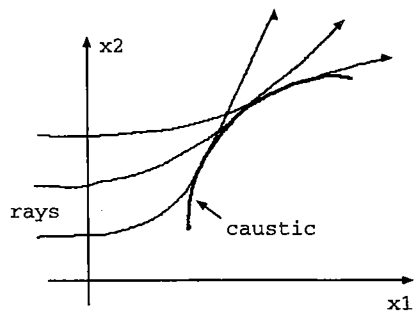

Figure 1.1: When rays intersect and their envelop is called caustic

picture of rays is shown in Figure 1.1 where starting from some instant the rays intersect. Their envelope is called a caustic. If we add the p-axis to the Figure and show the bicharacteristics, then these bicharacteristics will not intersect. We will have more descriptions in the later sections.

Solution of transport equation

The transport equation (1.8) can also be solved by integrating along the rays. To do so, we express $\nabla^2\phi$ in terms of its variation along a ray. Consider a region V bound laterally by a tube of rays and capped by two segment of wave fronts $\phi = constant$ which are denoted as s_0 and s_1 as shown in Figure 1.2. Let $\mathbf{N}$ be the exterior unit normal of V and apply the divergence theorem in V i.e.,

$$\begin{aligned} \int_{s_1} nda - \int_{s_0} nda &\approx n[J|_{\phi+d\phi} - J|_{\phi}]d\xi d\zeta, \\ \int_v \nabla^2\phi dv &\approx \nabla^2\phi J d\xi d\zeta d\phi. \end{aligned}$$

[3]Take the derivative of $H = |p|^2 - n(\mathbf{x})$ along the trajectories of the system (1.9): $\partial H/\partial\sigma = 2pdp/d\sigma - \nabla n(\mathbf{x}), dx/d\sigma = 0$; Since $H = 0$ for $\sigma = 0$, this conclude $|p|^2 = n(\mathbf{x})$.

In the limit of $d\phi \to 0$ one obtain

$$\nabla^2 S = \frac{n}{J} d(nJ)/d\phi = \frac{1}{J} d(nJ)/d\sigma \tag{1.12}$$

where J is the Jacobean transformation between the coordinates $\mathbf{x}$ and (σ, θ).

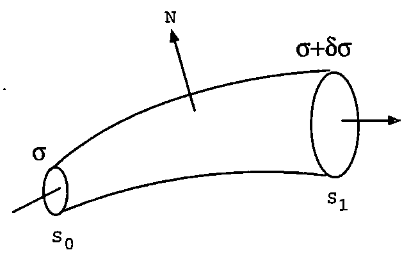

Figure 1.2: Propagation along a ray tube

This Jacobian characterizes the density of the rays in the "light tube" as depictured in Figure 1.2. Inserting Equation (1.12) into (1.8) yields

$$2n\frac{dA_0}{d\sigma} + \frac{1}{J}\frac{d(nJ)}{d\sigma}A_0 = 0 \tag{1.13}$$

This can be written as

$$\frac{2n}{nJ}\frac{d(A_0\sqrt{nJ})}{d\sigma} = 0 \tag{1.14}$$

Integrating along the ray $x(\sigma)$ from σ_0 to σ we have

$$A_0 = A_0(\sigma_0)\sqrt{\frac{n(\sigma_0)J(\sigma_0)}{n(\sigma)J(\sigma)}} \tag{1.15}$$

However, the ray field theory breaks down as soon as the ray intersect. That leads to $J = 0$. At those regions, ray field forms a caustics. This signals that a dramatic change of wave propagation occurs in the vicinity of a caustic. The solutions to the eikonal equation and transport equation break down at a caustic in two ways: continuation of the phase by ray tracing beyond the caustic and the determination of the amplitude by transport equation on a caustic. The former difficult arise because the solution of the eikonal equation is generally multivalued. The caustic coincides with the join of the branches of the phase function. The branch of the phase function changes as the phase is continued through a caustic, and the characteristics $\pi/2$ phase shifts resulted. The ray tracing fails to give a prescription for the choice of the branch on which the continuation should proceed, hence, of the phase shift. Amplitude transport fails at a caustic because the tube of rays in which the intensity is being conserved has zero cross section there; thus the ray theory incorrectly predicts an infinite amplitude, the ray spreading estimated from the Jacobian J at a caustic. As shown in the Figure 1.3.

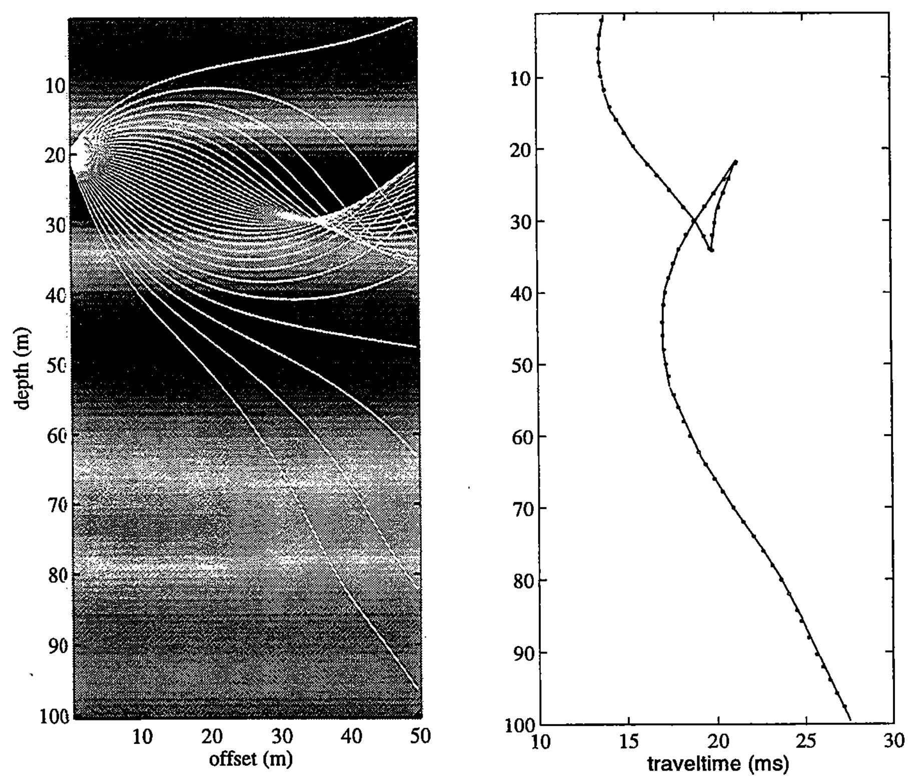

Figure 1.3: Raytracing through a velocity model. The left panel shows the Caustics occurred in the low velocity zone for the given source location; The right panel shows the triplication of the traveltime at the receiver positions.

Maslov uniform solution

Maslov introduced the concept of Lagrangian submanifolds (Maslov, 1988) while generalizing earlier one-dimensional work of Keller (Keller, 1958) [4] to overcome the difficulties of representing eikonal wave field at caustics. The main idea of Maslov's method is that the asymptotic wave field should be constructed not for the wave function itself but rather for its Fourier transform in the phase space where the trajectories present no caustics. The desired field in the caustic region is the Fourier transform of the asymptotic field of the hybrid space that is a projection of Lagrangian submanifold.

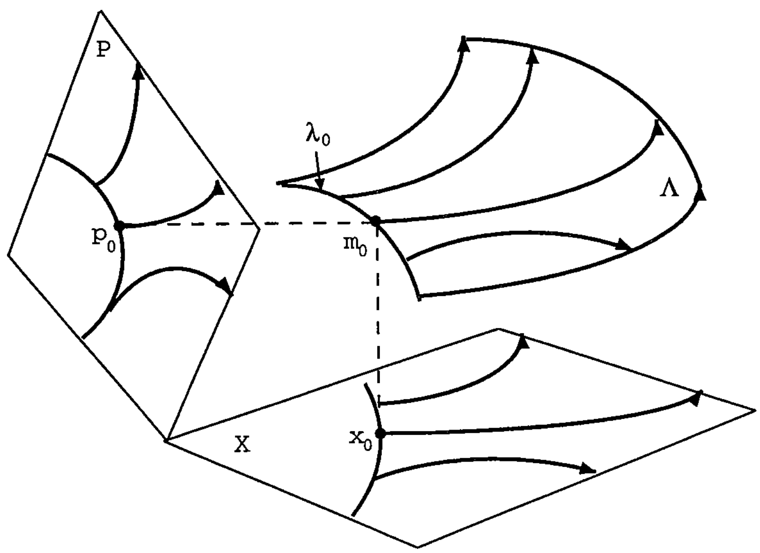

Figure 1.4: The rays and their velocity vector in the space X and P are projections of phase space trajectories and their velocity vectors. Λ is a Lagrangian submanifold in the phase space

Note that the phase space $\mathcal{M}$ consists of a position vector $\mathbf{x}$ and a wave slowness vector $\mathbf{p}$. The Lagrangian submanifold Λ of the phase space $\mathcal{M}$ can be viewed as a surface of the dispersion relation, on which the bicharacteristics of the Hamiltonian

[4] Ordinary asymptotic wave theory breaks down when diffraction occurs where the medium scale length is as small as the wavelength, or near turning points. Keller has developed a geometric diffraction theory which uses geometric optics away from the bad regions in the medium and glues in the extra rays due to diffraction emanating from these regions using matched asymptotics.

evolve. The importance of the Λ is to produce global solutions to the ray field problem as well as uniform representations in the caustic regions. Intuitively, a locally a plane wave with propagation vector $k = \phi_x$ is restricted to the particular values $k = \phi_x$ that is defined naturally on phase space $\mathcal{M}$ with equal footing in both $\mathcal{R}$ and $\mathcal{P}$ space (Ziolkwski, 1983), as depictured in Figure 1.4. The allowed values in $\mathcal{M}$ of $\mathbf{p}$ for any $\mathbf{x}$ are determined by the dispersion relation $H(\mathbf{x}, \mathbf{p}) = 0$. The local projection of $H(\mathbf{x}, \mathbf{p}) = 0$ onto $\mathcal{R}$ through the relation $k = \phi_x$ returns the eikonal equation.

As a example of choosing mixed coordinates, denote the position vector $\mathbf{x} = (x_1, x_2, x_3)$, momentum $\mathbf{p} = (p_1, p_2, p_3)$ and the coordinates in the phase space $\mathbf{m} = (p_1, x_2, x_3)$ (Kravtsov, 1993). Consider the phase space $\mathbf{m} = (p_1, x_2, x_3)$. In this space, the wave function $\tilde{\phi}(\mathbf{m})$ is related to $\phi(\mathbf{x})$ by the Fourier transform

$$\phi(\mathbf{x}) \sim \int \tilde{\phi}(\mathbf{m}) e^{ik_0 p_1 x_1} dp_1. \tag{1.16}$$

As in the real space, we are interested in the asymptotics of $\tilde{\phi}(\mathbf{m})$. Follow the same procedure of conventional asymptotic wave theory, one obtain the eikonal and transport equations of the phase space and their solutions:

$$\tilde{\phi} = \psi - x_1 p_1 = \psi_0 + \int_0^\sigma n(\mathbf{x}(\sigma'))d\sigma' - x_1 p_1 \tag{1.17}$$

$$\tilde{A}(\mathbf{m}) = \frac{\tilde{A}_0}{\sqrt{\tilde{J}}} \tag{1.18}$$

The leading wave asymptotics in real space is obtained via Fourier transform:

$$\phi(\mathbf{x}) = \sqrt{ik_0/2\pi} \int \frac{\tilde{A}_0}{\sqrt{\tilde{J}}} e^{ik_0 \int_0^\sigma n(\mathbf{x}(\sigma'))d\sigma' - x_1(\mathbf{m})p_1 + ik_0 p_1 x_1} dp_1 \tag{1.19}$$

Notice that a suitable choice of coordinates in $\mathbf{m}$ can always eliminate the singularity of $\tilde{A}$ at caustics, thus secure the $\phi(\mathbf{x})$ at caustic is finite. A choice of mixed coordinates in which $\tilde{J} \neq 0$, though $J = 0$, relies on the properties of the Lagrangian submanifold of the phase space (Ziolkwski, 1983).

Numerical implementation

Maslov's solution to the continuation of a field through a caustic region has the alternate compact representation (Kendall, et. al., 1985):

$$\phi(\mathbf{x}) = \begin{cases} \phi_0(\mathbf{x}) & \text{if } \mathbf{x} \text{ is away from any caustic} \\ \{\mathcal{F}^{-1} \circ \mathcal{F}_0\}[\phi_0[\mathbf{x})] & \text{if } \mathbf{x} \text{ is near a caustic} \end{cases} \tag{1.20}$$

The asymptotic Fourier operator $\mathcal{F}_0$ (see Appendix D) effectively cancels the singularities in the regular ray theory field $G_0(\mathbf{x})$. Clearly, if $\mathbf{x}$ is sufficiently far from a caustic, the operator $\mathcal{F}^{-1}$ can be replaced with $\mathcal{F}_0^{-1}$ and returns to regular ray theory solution immediately.

At the positions away from caustics, the wave fields are estimated using (1.11) and (1.15). We implement two point raytracing and, for each ray, additional two neighboring rays are traced. With the raypath information, one can calculate the Jacobean with a finite difference scheme shown in Figure 1.5. The wave field estimation is completed by the KMAH index that will determine the phase shift when a ray passes through a caustic. At caustics, the wave fields are calculated using

$$\phi(\mathbf{x}, \omega) \approx \frac{|\omega|^{1/2}}{(2\pi)^{3/2}} e^{-i\pi sgn(\omega)/4} \int A(\mathbf{x}_s) |\frac{\partial \mathbf{x}}{\partial \mathbf{p}}|_{x_s} e^{-iInd\gamma\pi/2} e^{i\omega[\Phi(\mathbf{x}_p) - \mathbf{p}\cdot\mathbf{x}_p]} d\mathbf{p} \tag{1.21}$$

Again, additional two rays neighboring each central ray are traced in phase space in order to calculate the corresponding Jacobean.

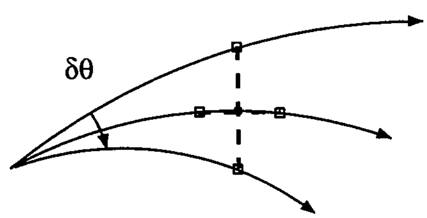

Figure 1.5: Calculation of Jacobian with neighboring rays

To check computer program, we compare the theoretical ray spreading of a homogeneous medium with that of calculated using the raypath information resulted from raytracing. As indicated in Figure 1.6, the calculated value agree with theoretic value.

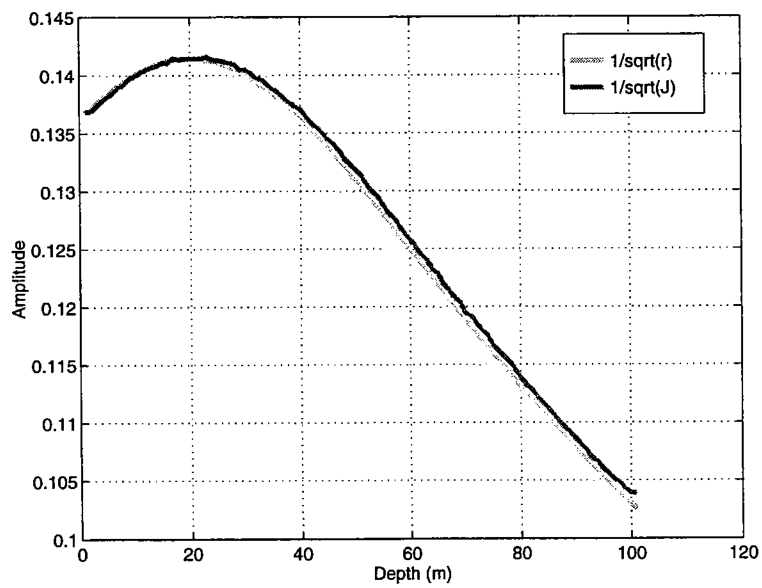

Figure 1.6: A comparison between the $1/\sqrt{r}$ and $1/\sqrt{J}$, where Jacobean J is calculated using raypath information

In Figure 1.7, we show that the phase and amplitude of the wave are correctly simulated in a three layer model.

In the following example, we show the wave amplitude of a array of sources and a array of receivers. The velocity model is a random, characterized by an ellipsoidal autocorrelation function (see paper P of this report). The correlation lengths are chosen such as the model approach looks like a 1-D. From this complicate model, we can see that there are some shadow zones where the raytracing is not go through. Therefore the amplitude is not calculated there.

In summary, the Maslov, or Lagrangian manifold, technique provides a means of constructing a uniform asymptotic solution to the wave equation. To find an asymptotic solution that is valid in the vicinity of a caustic, we express the wavefield as a summation over neighboring rays, rather than just considering the contribution from a single ray. Away from caustics, Maslov synthetics agree with synthetics based on classical ray theory.

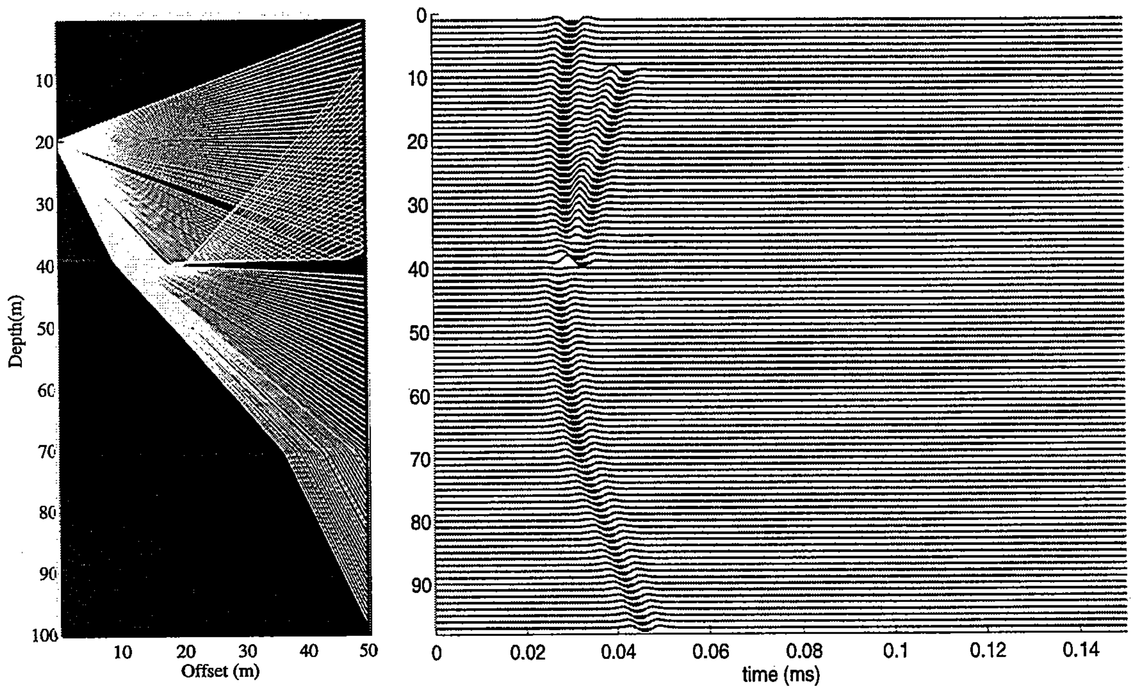

Figure 1.7: The left panel is raypath image; right panel is a time section. Both phase and amplitude are correct predicted.

ASYMPTOTIC INVERSION OF THE SCALAR WAVE

We assume that the background wave field propagating in a smoothly inhomogeneous medium. The condition of smooth inhomogeneity assume that the characteristic scale of the field variations is much smaller than the scale of medium property variations. We use the results from previous section to calculate wave asymptotics of the variable background and use distorted born approximation to estimate scattered field. The inversion theory is constructed using generalized Fourier transform or, precisely, pseudodifferential operator.

Inversion theory

The scattering integral for scalar wave can be written as

$$U(\mathbf{r}, \mathbf{s}) = \int o(\mathbf{x}) A(\mathbf{x}, \mathbf{r}, \mathbf{s}) e^{i\omega\phi(x,\mathbf{r},\mathbf{s})} d\mathbf{x} \tag{1.22}$$

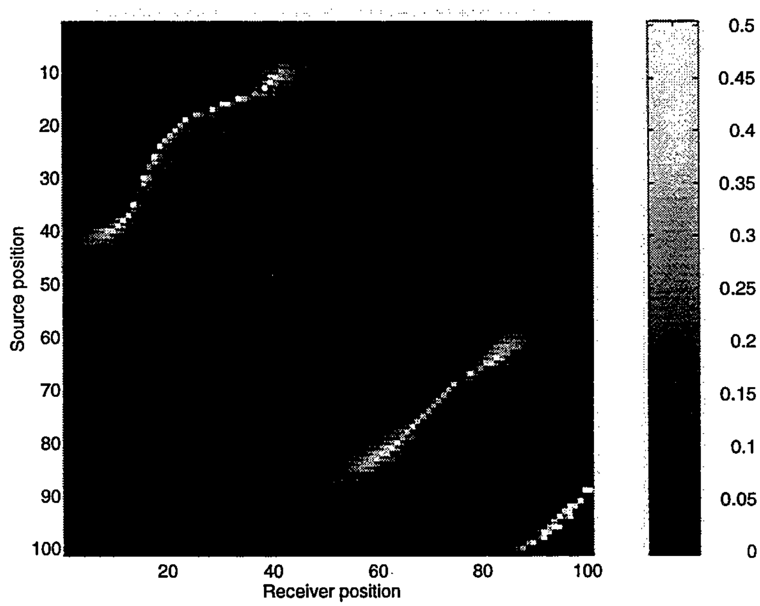

Figure 1.8: Wave amplitude of array sources and receivers

where $U(\mathbf{r}, \mathbf{s})$ is the scattered field at the receiver position for a given source at location $\mathbf{s}$, $A(\mathbf{x}, \mathbf{r}, \mathbf{s}) = \tilde{A}(\mathbf{x}, \mathbf{s})\hat{A}(\mathbf{r}, \mathbf{x})$ is amplitude, and $\phi(\mathbf{x}, \mathbf{r}, \mathbf{s}) = \tilde{\phi}(\mathbf{x}, \mathbf{s}) + \hat{\phi}(\mathbf{x}, \mathbf{r})$ is total traveltime of the background field which are calculated use the procedure described in previous section.

According to the pseudodifferential operator theory (see Appendix D), it is possible to invert the integral (1.22) with asymptotic Fourier transform, i.e.,

$$\hat{o}(\mathbf{x}) = \int U(\mathbf{r}, \mathbf{s}) H(\mathbf{r}, \mathbf{s}, \mathbf{x}) \frac{e^{-i\omega\phi(x,\mathbf{r},\mathbf{s})}}{A(\mathbf{x}, \mathbf{r}, \mathbf{s})} d\mathbf{r} d\mathbf{s} \tag{1.23}$$

where $H(\mathbf{r}, \mathbf{s}, \mathbf{x})$ is a weighting function to be determined. Substitute Equation (1.22) into (1.23), one obtain

$$\hat{o}(\mathbf{x}) = \int o(\mathbf{x}) R(\mathbf{x}, \mathbf{x}') d\mathbf{x}' \tag{1.24}$$

where

$$R(\mathbf{x}, \mathbf{x}') = \int \frac{A(\mathbf{x}', \mathbf{r}, \mathbf{s})}{A(\mathbf{x}, \mathbf{r}, \mathbf{s})} H(\mathbf{r}, \mathbf{s}, \mathbf{x}) e^{i\omega[\phi(\mathbf{x}', \mathbf{r}, \mathbf{s}) - \phi(\mathbf{x}, \mathbf{r}, \mathbf{s})]} d\mathbf{r} d\mathbf{s}. \tag{1.25}$$

The Fourier integral $R(\mathbf{x}, \mathbf{x}')$ is a pseudodifferential operator (Treves, 1980) and its principle symbol is

$$\frac{A(\mathbf{x}',\mathbf{r},\mathbf{s})}{A(\mathbf{x},\mathbf{r},\mathbf{s})}\frac{H(\mathbf{r},\mathbf{s},\mathbf{x})}{|J(k;\mathbf{r},\mathbf{s})|}, \tag{1.26}$$

where $J(k;\mathbf{r},\mathbf{s})$ is the coordinate transformation from $\mathbf{k} \to (\mathbf{r},\mathbf{s})$. Note that the pseudodifferential operator relates the object function and its estimate. Obviously, it can be understood as a model resolution operator.

If the operator $R(\mathbf{x},\mathbf{x}')$ is a δ-like function, then $\hat{o}(\mathbf{x})$ would be identical to $o(\mathbf{x})$. Apply Taylor expansion at the neighborhood of $\mathbf{x}$ to the amplitude and traveltime and take only leading term, i.e.,

$$\begin{aligned}
A(\mathbf{x}',\mathbf{r},\mathbf{s}) &\approx A(\mathbf{x},\mathbf{r},\mathbf{s}),\\
\tilde{\phi}(\mathbf{x}',\mathbf{s})-\tilde{\phi}(\mathbf{x},\mathbf{s}) &\approx \tilde{\phi}_{,j}(\mathbf{x},\mathbf{s})(x_j-x_j'),\\
\hat{\phi}(\mathbf{x}',\mathbf{r})-\hat{\phi}(\mathbf{x},\mathbf{r}) &\approx \hat{\phi}_{,j}(\mathbf{x},\mathbf{r})(x_j-x_j').
\end{aligned}$$

Let spatial variant wave vectors $\tilde{k}_j=\omega\tilde{\phi}_{,j}(\mathbf{x},\mathbf{s})$, $\hat{k}_j=\omega\hat{\phi}_{,j}(\mathbf{x},\mathbf{r})$, and $k_j(\mathbf{x})=\tilde{k}_j+\hat{k}_j$, $j=1,2,3$, and change variable such that $d\mathbf{k}=|J(\mathbf{k};\mathbf{r},\mathbf{s})|d\mathbf{r}d\mathbf{s}$, one obtains

$$R(\mathbf{x},\mathbf{x}')=\int\frac{H(\mathbf{r},\mathbf{s},\mathbf{x})}{|J(\mathbf{k};\mathbf{r},\mathbf{s})|}e^{ik_j(x_j'-x_j)}d\mathbf{k} \tag{1.27}$$

If we choose $H(\mathbf{r},\mathbf{s},\mathbf{x})=|J(\mathbf{k};\mathbf{r},\mathbf{s})|$, then $R(\mathbf{x},\mathbf{x}')\to\delta(\mathbf{x},\mathbf{x}')$. Consequently, the inversion Equation (1.23) takes the form of

$$\hat{o}(\mathbf{x})=\int U(\mathbf{r},\mathbf{s})|J(\mathbf{k};\mathbf{r},\mathbf{s})|\frac{e^{-i\omega\phi(x,\mathbf{r},\mathbf{s})}}{A(\mathbf{x},\mathbf{r},\mathbf{s})}d\mathbf{r}d\mathbf{s} \tag{1.28}$$

where the amplitude and the $A(\mathbf{x},\mathbf{r},\mathbf{s})$ and the phase $\phi(x,\mathbf{r},\mathbf{s})$ are calculated with the asymptotic analysis described in the previous section. By using the wave asymptotics and choosing weighting function $H(\mathbf{r},\mathbf{s},\mathbf{x})=|J(\mathbf{k};\mathbf{r},\mathbf{s})|$, we reconstructed the object function in the sense of consisting with the information containing in the wave asymptotics. The local wavenumber can be expressed as

$$k_x=k_0(\mathbf{x})(\cos\theta(\mathbf{x},\mathbf{s})+\cos\theta(\mathbf{r},\mathbf{x}))$$

$$k_x = k_0(\mathbf{x})(\sin\theta(\mathbf{x},\mathbf{s}) + \sin\theta(\mathbf{r},\mathbf{x})),$$

the Jacobian transformation J can be obtained

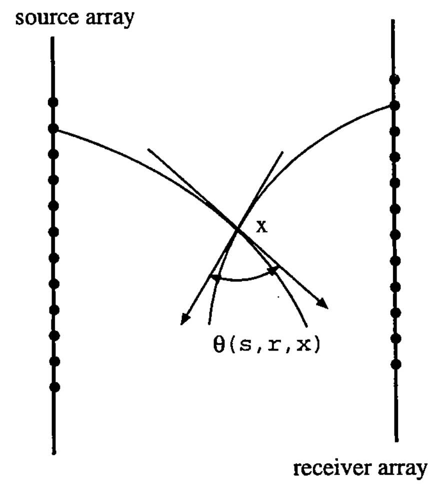

Figure 1.9: At scattering point $\mathbf{x}$ the coming and departure rays make a bisect angle

$$J(\mathbf{k};\mathbf{r},\mathbf{s}) = k_0^2(\mathbf{x}) sin\theta(\mathbf{g},\mathbf{s}) \frac{\partial\theta(\mathbf{x},\mathbf{s})}{\partial s} \frac{\partial\theta(\mathbf{g},\mathbf{x})}{\partial g} \tag{1.29}$$

where $\theta(\mathbf{g},\mathbf{s}) = \theta(\mathbf{g},\mathbf{x}) - \theta(\mathbf{x},\mathbf{s})$ is the bisect angle [5], as indicated in Figure 1.9. In the time domain (1.28) becomes

$$\hat{o}(\mathbf{x}) = \Re \int \frac{|J(\mathbf{k};\mathbf{r},\mathbf{s})|}{A(\mathbf{x},\mathbf{r},\mathbf{s})} U(\mathbf{r},\mathbf{s},t)|_{t=\phi(\mathbf{x},\mathbf{r},\mathbf{s})} d\mathbf{r} ds \tag{1.30}$$

where $\Re$ denotes the real part of the integral. In the expression (1.29), the image condition $t = \phi(\mathbf{x},\mathbf{r},\mathbf{s})$ is the same as in the case of migration.

[5]Notice that

$$\begin{aligned} J(\mathbf{k};\mathbf{r},\mathbf{s}) &= k_0^2(\mathbf{x})(\frac{\partial cos\theta(\mathbf{x},\mathbf{s})}{\partial s}\frac{\partial sin\theta(\mathbf{g},\mathbf{x})}{\partial g} - \frac{\partial cos\theta(\mathbf{g},\mathbf{x})}{\partial g}\frac{\partial sin\theta(\mathbf{x},\mathbf{s})}{\partial s}) \\ &= k_0^2(\mathbf{x})\frac{\partial\theta(\mathbf{x},\mathbf{s})}{\partial s}\frac{\partial\theta(\mathbf{g},\mathbf{x})}{\partial g}(sin\theta(\mathbf{g},\mathbf{x})cos\theta(\mathbf{x},\mathbf{s}) - sin\theta(\mathbf{x},\mathbf{s})cos\theta(\mathbf{g},\mathbf{x})) \end{aligned}$$

Asymptotic inversion of elastic wave

In the case of elastic wave inversion, the object function of the scattering operator has multiple terms and each term has different polarization factor. We have to modify the procedure applied to scale wave. In order to recover each term of the object function, we project the scattered to different polarization direction, then apply asymptotic Fourier transform to these projections. The resultant multiple "raw" images are then used to solve individual elastic parameter.

Asymptotic Green's function of elastic medium

For an isotropic elastic medium, suppose the density and elastic constant of the medium can be written as

$$c_{lmpq} = c^0_{lmpq} + c'_{lmpq}, \qquad \rho = \rho^0 + \rho',$$

where ρ^0, c^0_{lmpq} are background density and elastic constant, and ρ', c'_{lmpq} are their perturbations. The scattered wave field can be expressed as (Aki and Richards, 1980)

$$U_{jk}(\mathbf{s},\mathbf{r},t) = -\int [\rho' \partial_t^2 u_{jl} *_t \hat{G}_{kl} + c'_{lmpq} u_{jp,q} *_t \hat{G}_{kl,m}] d\mathbf{x}. \tag{1.31}$$

In the frequency domain, it is

$$U_{jk}(\mathbf{s},\mathbf{r},\omega) = -\int [\rho' \omega^2 u_{jl} \hat{G}_{kl} + c'_{lmpq} u_{jp,q} \hat{G}_{kl,m}] d\mathbf{x}. \tag{1.32}$$

For a smoothly varying background medium, the asymptotic Green's function for (1.31) can be written as

$$G_{ij} \approx A_{jl} e^{i\omega\Phi} \tag{1.33}$$

where the phase Φ is a slowly varying, real-valued function and the amplitude A is a slowly varying, complex-valued function. The leading term of the Green's function can be written in the form

$$\tilde{G}_{jl} = \tilde{G}^p_{jl} + \tilde{G}^s_{jl}, \qquad \hat{G}_{kl} = \hat{G}^p_{kl} + \hat{G}^s_{kl}, \tag{1.34}$$

where

$$\tilde{G}^p_{jl} = \tilde{A}^p_{jl} e^{i\omega\tilde{\phi}^p}, \qquad \tilde{G}^s_{jl} = \tilde{A}^s_{jl} e^{i\omega\tilde{\phi}^s},$$
$$\hat{G}^p_{kl} = \hat{A}^p_{kl} e^{i\omega\hat{\phi}^p}, \qquad \hat{G}^s_{kl} = \hat{A}^s_{kl} e^{i\omega\hat{\phi}^s}.$$

Following the procedure described in section 1, one obtains eikonal equations for the phase functions $\tilde{\phi}^p, \hat{\phi}^p, \tilde{\phi}^s$ and $\hat{\phi}^s$.

$$\sum_{j=1,2,3} [\tilde{\phi}^p_{,j}]^2 = \sum_{k=1,2,3} [\hat{\phi}^p_{,k}]^2 = \frac{1}{c_p^2}$$
$$\sum_{j=1,2,3} [\tilde{\phi}^s_{,j}]^2 = \sum_{k=1,2,3} [\hat{\phi}^s_{,k}]^2 = \frac{1}{c_s^2}$$

where c_p and c_s are the p-wave and the s-wave velocity respectively. The main term of the amplitude $\tilde{A}^p_{jl}, \hat{A}^p_{kl}, \tilde{A}^s_{jl}$ and $\hat{A}^s_{kl}$ satisfy transport equations

$$(\rho^0 c_p^2 \tilde{A}^p_{jp} \tilde{A}^p_{jp} \tilde{\phi}^p_{,m})_{,m} = 0$$
$$(\rho^0 c_p^2 \hat{A}^p_{kp} \hat{A}^p_{kp} \hat{\phi}^p_{,m})_{,m} = 0$$
$$(\rho^0 c_s^2 \tilde{A}^s_{jp} \tilde{A}^s_{jp} \tilde{\phi}^s_{,m})_{,m} = 0$$
$$(\rho^0 c_s^2 \hat{A}^s_{kp} \hat{A}^s_{kp} \hat{\phi}^s_{,m})_{,m} = 0$$

where no summation over j and k. Again the problem of finding the asymptotic solution resulting from given sources then decomposes into two parts: ray tracing which defines the continuation of the phase independently of the amplitude; and determination of the amplitude which is carried out by following intensity variations along the rays.

Inversion theory

We can rewrite (1.31) as

$$U(\mathbf{r}, \mathbf{s}, \omega) = \int \sum_{l=1,2,3} o_l(\mathbf{x}) w_l(\cos\theta(\mathbf{x}, \mathbf{r}, \mathbf{s}) A(\mathbf{x}, \mathbf{r}, \mathbf{s}) e^{i\omega\phi(x,\mathbf{r},\mathbf{s})} d\mathbf{x} \qquad (1.35)$$

where $U(\mathbf{r}, \mathbf{s}, \omega)$ represent one of components of a specific the elastic scattered wave mode such as U_{jk}, $o_l(\mathbf{x})$ is the object function, and $w_l(\cos\theta)$ is denoted as polarization factor. Again as in previous section amplitude $A(\mathbf{x}, \mathbf{r}, \mathbf{s}) = \tilde{A}(\mathbf{x}, \mathbf{s})\hat{A}(\mathbf{x}, \mathbf{r})$ and traveltime $\phi(\mathbf{x}, \mathbf{r}, \mathbf{s}) = \tilde{\phi}(\mathbf{x}, \mathbf{s}) + \hat{\phi}(\mathbf{x}, \mathbf{r})$. In the case of *p-p* scattering,

$$o_l(\mathbf{x}) \subset \{\frac{\lambda'}{\lambda^0 + 2\mu^0}, \frac{\rho'}{\rho^0}, \frac{2\mu'}{\lambda^0 + 2\mu^0}\}, \qquad w_l(\cos\theta) \subset \{1, \cos\theta, \cos^2\theta\}.$$

Utilizing the results of the scalar wave inversion, we define a pseudodifferential operator $\mathcal{F}_m$ such that

$$\mathcal{F}_m\{U(\mathbf{r}, \mathbf{s}, \omega)\}(\mathbf{x}) = \int \frac{U(\mathbf{r}, \mathbf{s}) w_m(\cos\theta)|J|}{A(x, \mathbf{r}, \mathbf{s})} e^{-i\omega\phi(\mathbf{x},\mathbf{r},\mathbf{s})} d\mathbf{s} d\mathbf{r} \tag{1.36}$$

where $|J|$ is Jacobian transformation as in the case of scalar wave. Again the pseudodifferential operator $\mathcal{F}_m$ generalized Fourier transform. Notice that the additional weighting function w_m is used to facilitate recovering individual component o_l of the object function. Substitute Equation (1.34) into (1.35) and apply Taylor expansion to the amplitude, the phase, as well as the weighting functions w_l at neighborhood of $\mathbf{x}$, one obtains

$$\mathcal{F}_m\{U(\mathbf{r}, \mathbf{s}, \omega)\}(\mathbf{x}) = \iint \sum_{l=1,2,3} o_l(\mathbf{x}') g_{lm}(\cos\theta) e^{ik_j(x'_j - x_j)} d\mathbf{x}' |J| d\mathbf{s} d\mathbf{r} \tag{1.37}$$

where

$$g_{lm}(\cos\theta) = w_l(\cos\theta) w_m(\cos\theta),$$

$$\tilde{k}_j = \omega\tilde{\phi}_{,j}(\mathbf{x}, \mathbf{s}), \qquad \hat{k}_j = \omega\hat{\phi}_{,j}(\mathbf{x}, \mathbf{r}), \qquad \mathbf{k}_j = \tilde{k}_j + \hat{k}_j$$

Explicitly, equation (1.36) can be rewritten as

$$\mathcal{F}_m\{U(\mathbf{r}, \mathbf{s}, \omega)\}(\mathbf{x}) = \sum_{l=1,2,3} \int g_{lm}(\cos\theta)|J| d\mathbf{s} d\mathbf{r} \int o_l(\mathbf{x}') e^{ik_j(x'_j - x_j)} d\mathbf{x}' \tag{1.38}$$

If we transform back to the time domain, then

$$\mathcal{F}_m\{U(\mathbf{r},\mathbf{s},t)\}(\mathbf{x}) = \sum_{l=1,2,3} o_l(\mathbf{x}) \int \frac{g_{lm}(\cos\theta)}{|\tilde{\phi}_{,j} + \hat{\phi}_{,j}|} |J| d\mathbf{s} d\mathbf{r} \tag{1.39}$$

from which o_l is solved. Notice that the inner integral in the (1.38) is evaluated as

$$\int o_l(\mathbf{x}') e^{-ik_j(x'_j - x_j)} d\mathbf{x}' = \frac{1}{|\tilde{\phi}_{,j} + \hat{\phi}_{,j}|} o_l(\mathbf{x}). \tag{1.40}$$

Similar to the case of scalar wave, we have

$$\mathcal{F}_m\{U(\mathbf{r},\mathbf{s},t)\}(\mathbf{x}) = \int \frac{w_m(\cos\theta)|J|}{A(x,\mathbf{r},\mathbf{s})} U(\mathbf{r},\mathbf{s},t)|_{t=\phi(\mathbf{x},\mathbf{r},\mathbf{s})} d\mathbf{s} d\mathbf{r} \tag{1.41}$$

which function as "raw images", from which individual elastic parameter is reconstructed.

CONCLUSIONS

We apply the Maslov theory which provides a means of constructing a uniform asymptotic solution to the wave equation. Using the the wave asymptotics we derived an general Fourier transform that has an approximate inverse in the sense that the inverse operator recover most singular part of the discontinuities of the medium. The algorithm is practical and very much like Kirchoff migration. The algorithm is currently being implemented and results are presented late.

ACKNOWLEDGMENTS

I would like to thank Jesse Costa for many enlighting discussions, particularly on Maslov ray theory.

REFERENCES

Aki, K., and Richards, P.G., 1980, Quantitative Seismology, vol. I: First edition, Freeman and Company, 273–286.

Beylkin, G., and Burridge, R., 1990, Linearized inverse scattering problems in acoustics and elasticity: Wave Motions, 12, p.15-52

Born, Max and Wolf, Emil, 1970, Principles of optics; electromagnetic theory of propagation, 4th ed. Oxford, New York, Pergamon Press

Keller, J. B., 1958, Corrected Bohr-Sommerfeld quantum conditions for non-separable systems: Ann. Physics 4, p.180-188

Keller, J. B., 1962, Geometrical theory of diffraction, J. Opt. Soc. Am. 52(2), p.116-130

Kendall, J. M. and Thomson, C. J., 1993, Maslov ray summation, pseudo-caustics, Lagrangian equivalence and transient seismic waveforms: Geophysical Journal International (April 1993) vol.113, no.1, p.186-214.

Kravtsov, Y. A. and Orlov, Yu. I., 1990, Geometrical optics of inhomogeneous media, Berlin ; Springer-Verlag

Kravtsov, Y. A. and Orlov, Yu. I., 1993, Caustics, catastrophes and wave field, Berlin ; Springer-Verlag

Maslov, V. P P., 1988, Asymptotics of operator and pseudo-differential equations: Consultants Bureau

Treves, F. 1980, Introduction to pseudodifferential and Fourier integral operators, New York: Plenum

Ziolkowski, R. W., 1984, Asymptotic evaluation of high-frequency fields near a caustic: An introduction to Maslov's method: Radio Science, vol. 19, 4 p.1001-1025

APPENDIX

PSEUDODIFFERENTIAL OPERATORS

Let $\mathcal{R}^n$ be the usual Euclidean space. On $\mathcal{R}^n$, the simplest differential operators are ∂_j or $D_j = -i\partial_j$. The most general linear partial differential operator of order m on $\mathcal{R}^n$ may be written as

$$\sum_{\alpha_1+\alpha_2+...+\alpha_n \leq m} a_{\alpha_1,\alpha_2,...,\alpha_n}(x) D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n} \qquad \text{(R.42)}$$

where $\alpha_1, \alpha_2, ..., \alpha_n$ are nonnegative integers and $a_{\alpha_1,\alpha_2,...,\alpha_n}(x)$ is an infinitely differentiable complex-valued function on $\mathcal{R}^n$. To simplify the expression (R.42), we let

$$\begin{aligned} \alpha &= (\alpha_1, \alpha_2, ..., \alpha_n), \\ |\alpha| &= \sum_{j=1}^{n} \alpha_j, \\ D^\alpha &= D_1^{\alpha_1} D_2^{\alpha_2} ... D_n^{\alpha_n}, \end{aligned}$$

the differential operator (R.42) can be rewritten as

$$\sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha. \qquad \text{(R.43)}$$

For each fixed x in $\mathcal{R}^n$, the operator (R.43) is a polynomial in $D_1, D_2, ..., D_n$. therefore it is natural to denote the operator (R.43) by $P(x, D)$. if D in (R.43) is replaced by a point $\xi = (\xi_1, \xi_2, ..., \xi_n)$ in $\mathcal{R}^n$, then a polynomial

$$\sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \qquad \text{(R.44)}$$

is obtained, where $\xi^\alpha = \xi_1^{\alpha_1} \xi_2^{\alpha_2} ... \xi_n^{\alpha_n}$. This polynomial is denoted by $P(x, \xi)$ which is called the *symbol* of the operator $P(x, D)$. The partial differential operator $P(x, D)$ can be represented in terms of its symbol by means of the Fourier transform. Let us define Fourier transform operation as "$\wedge$" and inverse Fourier transform operation as

"V".

$$\begin{aligned}
(P(x,D)\phi)(x) &= \sum_{|\alpha|\leq m} a_\alpha(x)(D^\alpha\phi)(x) \\
&= \sum_{|\alpha|\leq m} a_\alpha(x)(\widehat{D^\alpha\phi})^{\vee}(x) \\
&= \sum_{|\alpha|\leq m} a_\alpha(x)(\xi^\alpha\hat{\phi})^{\vee}(x) \\
&= \sum_{|\alpha|\leq m} a_\alpha(x)(2\pi)^{-n/2}\int \xi^\alpha\hat{\phi}(\xi)e^{ix\cdot\xi}d\xi \\
&= (2\pi)^{-n/2}\int P(x,\xi)\hat{\phi}(\xi)e^{ix\cdot\xi}d\xi
\end{aligned}$$

This representation suggests that one can get operators more general than partial differential operators if the symbol $P(x,\xi)$ is replaced by more general symbols $\sigma(x,\xi)$ which are no longer polynomials in ξ. The operators so obtained are called pseudo-differential operators. An important notion in the theory of pseudo-differential operators is the asymptotic expansion of a symbol.