PAPER Q

RECONSTRUCTION USING RENORMALIZATION

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ABSTRACT

The renormalization theory is a broad concept whose content can be understood from various perspectives. It can be viewed as a technical device that get rid of infinite results in quantum electrodynamics; or as a up scaling theory in statistical mechanics; or as a regulative principle in the inverse problems. In this paper we use the renormalization to regulate Born expansion and extend Born approximation to the inverse scattering problem with strong fluctuations.

INTRODUCTION

Extensive numerical computations, e.g., finite difference methods (Alford, 1974), are feasible for elastic 2D and 3D models. If it were only necessary to study a few standard structures, such computations might be adequate, but given the great variety of scales and combinations of structure encounter in geophysics, it is necessary to use more efficient approximation methods. Even using finite difference computation, it is unlikely that all features of seismic data can be modeled, and again approximate methods are more appropriate. Finite difference calculations have to be interpreted using the same approximation methods used to interpret real data.

A number of previously intractable problems in several very different areas of physics have been successful solved using renormalization techniques. The goal of this paper is to present the renormalization techniques in the inverse scattering problem with strong fluctuations. To extend Born approximation we recast the Born approximation into an exponential form, i.e., to utilize the Rayleigh' renormalization technique to effectively sum the secular Born expansion. While the total field is approximated by incident field in the first Born approximation, with this method, the total field is replaced by the renormalized incident field. This provide a tool to simulate certain forward problems with reasonable accuracy. For the inverse problem, we first invert an auxiliary function which is the product of the renormalization factor and the object function. Since the renormalization factor is known, we can extract the object function from the auxiliary function by solve another easier nonlinear or linear problem (Torres-Verdin, 1994).

NONLINEARITY VS. RENORMALIZATION

The integral representation of scalar wave equation is expressed as

$$\mathcal{U} = \mathcal{U}_0 + \mathcal{G}\mathcal{V}\mathcal{U}, \tag{1.1}$$

where the symbols stand for operators (e.g., matrices), and their ordering is important and cannot be altered at will. In the operator notation, Equation (1.1) is valid regardless of whether $\mathcal{U}, \mathcal{G}$ and $\mathcal{V}$ are in the $\mathbf{r}$ domain representation or in the $\mathbf{k}$ domain representation. Multiple scattering correspond to nonlinear fields is intractable with hardly any rigorous mathematical foundations at all. The only hope for most practical purposes is to treat the multiple scattering, i.e., the nonlinear terms, as a perturbation and look for a power series expansion in terms of ϵ. In our case the parameter ϵ is the strength of velocity perturbation. Therefore, an alternative way to express Equation (1.1) is obtained by iterating on $\mathcal{U}$:

$$\mathcal{U} = \mathcal{U}_0 + \mathcal{G}\mathcal{V}\mathcal{U}_0 + \mathcal{G}\mathcal{V}\mathcal{G}\mathcal{V}\mathcal{U}_0 + ... \tag{1.2}$$

The smallness of the perturbation means that for low frequency and low contract one can have convergence in low orders, without having to struggle with the question of whether the power series itself converges. In strong fluctuations, the expansion parameter is of order large than one, the perturbation expansion does not converge at all; What is the physical meaning of those secular terms of the expansion? We may revisit "small deformation" assumption of the wave equation and conclude that the linear assumption is not intend to account for nonlinear terms. In this paper,

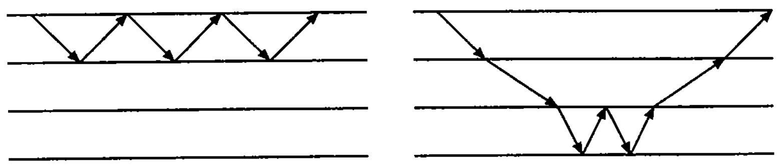

Figure 1.1: When perturbation parameters is not small, the recursive relation is not valid because of unobservable reflections.

we are not trying to pursing those nonlinear theory, rather to apply a practical fix, i.e., we consider some of interactions or reflections, as depicture in Figure 1.1, are not observable. They are "virtual" reflections. What one can do then is to apply the concept of renormalization to get rid of "infinite", i.e. to apply the regulations to the asymptotic expansion.

Lord Rayleigh (Optical Society of America, 1994) developed a renormalization technique to generalize his first scattering from a thin slab to scattering from many slabs. He obtained an expansion of the form

$$u = e^{ik_0 x}(1 + ik_0 nx), \tag{1.3}$$

for first scattering form one slab. Where $k_0 nx << 1$ n is refraction index. To obtain a solution valid for many slabs, he recasted this expansion into an exponential, i.e.,

$$u = e^{i(k_0 + k_0 n)x}. \tag{1.4}$$

In this manner he effectively summed the sequence

$$\sum_{m=1}^{\infty} (ik_0 nx)^m / m!, \tag{1.5}$$

of secular terms. Before we renormalized Equation (1.1), recall that its explicit form for the first Born approximation is

$$u(\mathbf{x}) \approx u_0(\mathbf{x}) + \int v(\mathbf{x}')G(\mathbf{x}')u_0(\mathbf{x}', \mathbf{x})d\mathbf{x}', \tag{1.6}$$

which can be rewritten in terms of Rayleigh's elementary scattering of Equation (1.3):

$$u(\mathbf{x}) \approx u_0(\mathbf{x})(1 + \frac{1}{u_0(\mathbf{x})} \int v(\mathbf{x}')G(\mathbf{x}, \mathbf{x}")G(\mathbf{x}', \mathbf{x})d\mathbf{x}'). \quad (1.7)$$

Recast Expansion (1.7) into an exponential form:

$$\begin{aligned} u &\approx \mathbf{u}_0 exp(\frac{1}{u_0(\mathbf{x})} \int o(\mathbf{x}')G(\mathbf{x}, \mathbf{x}")G(\mathbf{x}', \mathbf{x})d\mathbf{x}') \quad (1.8) \\ &= u_0 \sum_{m=0}^{\infty} (\frac{1}{u_0} \mathcal{G}\mathcal{V}\mathcal{G})^m / m!. \end{aligned}$$

Comparing to asymptotic expansion without renormalization

$$\mathcal{U} = \mathcal{U}_0(\mathcal{I} + \mathcal{V}\mathcal{G} + \mathcal{V}\mathcal{G}\mathcal{V}\mathcal{G} + ...), \quad (1.9)$$

we can see that Equation (1.7) is uniformly valid.

Let $R(\mathbf{x}, \mathbf{x}") = exp(\frac{1}{u_0(\mathbf{x})} \int o(\mathbf{x}')G(\mathbf{x}', \mathbf{x}")G(\mathbf{x}', \mathbf{x})d\mathbf{x}')$ be renormalization factor. With the renormalization relation, i.e., a localized nonlinear approximation to the total internal field, the scattering problem can be modified as

$$U(\mathbf{s}, \mathbf{r}) \approx \int v(\mathbf{x})R(\mathbf{x}, \mathbf{s})G(\mathbf{x}, \mathbf{s})G(\mathbf{r}, \mathbf{x})d\mathbf{x}. \quad (1.10)$$

FORWARD MODELING

In Figure 1.2 we compare the true solution to various approximations as a function of the velocity ratio between the sphere and the background. At the receiver location, scattered field, at frequency of 350 Hz is essentially in-phase with the primary field. The background velocity is kept constant at 3500 m/s, while the velocity of the sphere is varied. The renormalization method provides very good estimates of the true solution over 150 percent of contrast. At this frequency, the amplitudes of the scattered field and the amplitudes of the scattered field approximations are small compared to the background. The Born approximation, on the other hand, are inaccurate except when velocity perturbation is small.

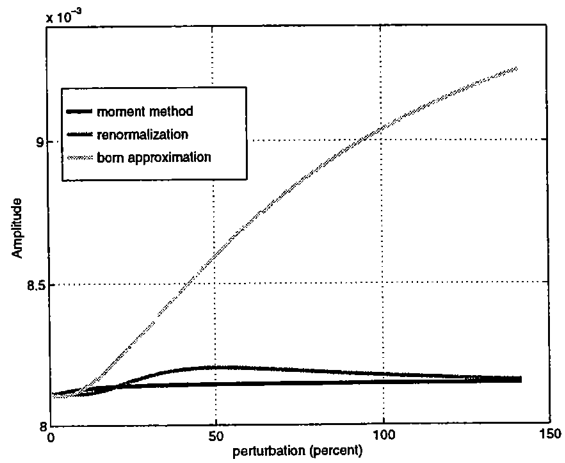

Figure 1.2: Comparison the field vs. the velocity contrast of a sphere model. The fields are calculated with the moment method, born approximation and renormalization method.

In Figure 1.3 we examine the accuracy of the approximations as a function of frequency. The velocity perturbation is fixed at 50 percent. The amplitude is given very accurately by both approximations for frequency up to 300 Hz. Therefore, the calculation with renormalization provides a better estimate of amplitude. Until at 1 KHz it is in error by about 50

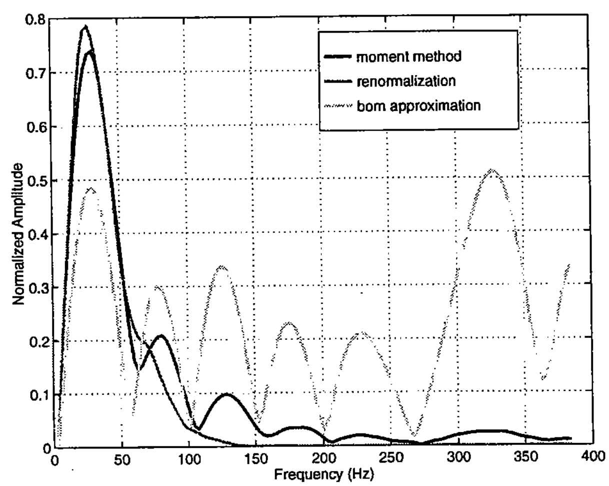

Figure 1.3: The comparison the moment method, born approximation and nonlinear born approximation over frequency. The model is a sphere with fixed contrast of 50 percent

In Figure 1.4, we show time sections of a two layer model. Again the calculations

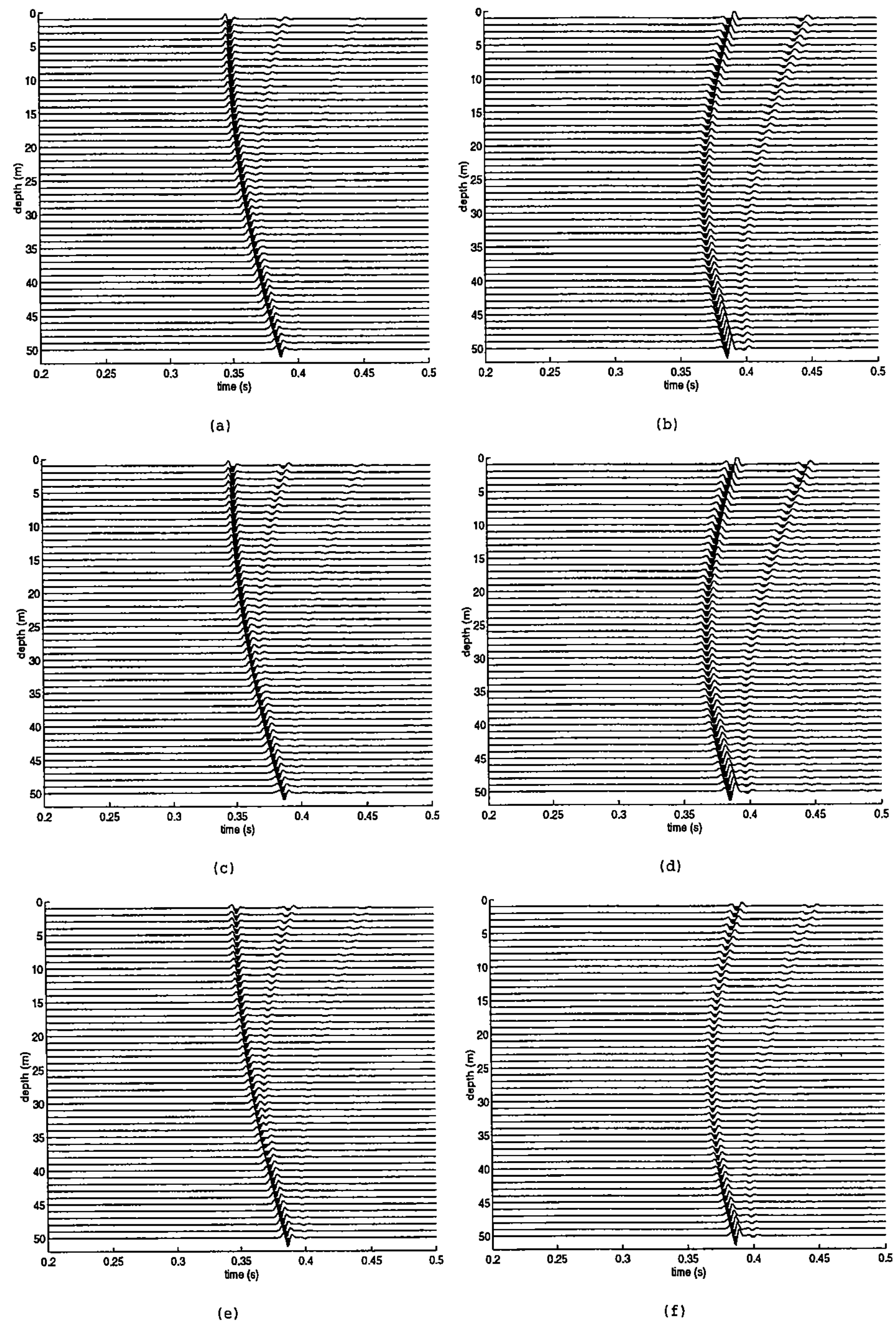

Figure 1.4: (a), (c), and (e) are total fields calculated using moment method, renormalization, and Born approximation respectively; (b), (d), and (f) are corresponding scattered fields.

using the moment method, born approximation and nonlinear born approximation are compared. We can see that the results by the renormalization method are agree to that of by the moment method.

INVERSE SCATTERING

When the observation points inside the scatterer, one must be careful in defining the volume integral because of the singularity of the field u and Green's function. For the interior points, the above equation can be rewrite (Habashy, 1992) as

$$u(\mathbf{x}) = u_0(\mathbf{x}) + u_0(\mathbf{x}) \int v(\mathbf{x}')G(\mathbf{x}', \mathbf{x})d\mathbf{x}' \tag{1.11}$$

$$+ \int v(\mathbf{x}')(u_0(\mathbf{x}') - u_0(\mathbf{x}))G(\mathbf{x}', \mathbf{x})d\mathbf{x}'\} \tag{1.12}$$

Since the field G is a localized function, i.e., $G(\mathbf{x}', \mathbf{x})$ is significant only when $\mathbf{x}' \to \mathbf{x}$. Therefore, the second term can be ignored, one obtain Rayleigh's single scattering form (1.4)

$$u(\mathbf{x}) = u_0(\mathbf{x})(1 + \int v(\mathbf{x}')G(\mathbf{x}', \mathbf{x})d\mathbf{x}') \tag{1.13}$$

An argument in favor of this approximation is as follows: From the singular nature of $G(\mathbf{x}', \mathbf{x})$ at $\mathbf{x}' = \mathbf{x}$, one may expect that the dominant contribution to the integral in Equation 1.12 comes from points in the vicinity of $\mathbf{x}' = \mathbf{x}$. if the internal field is approximated by its value at $\mathbf{x}' = \mathbf{x}$, the error is given by the second term on the right hand side of Equation ??. This approximation is particularly appropriate since the internal field $u_0(\mathbf{x}')$ is a smoothly varying function of the position. The error term can be expected to be small because $u_0(\mathbf{x}') - u_0(\mathbf{x})$ is zero where $G(\mathbf{x}', \mathbf{x})$ is singular. Hence, the accuracy of the approximation depends on $G(\mathbf{x}', \mathbf{x})$ falling off rate as $\mathbf{x}'$ moves away from $\mathbf{x}$ and the slow variations of the internal field. The validity of the local nonlinear approximation is shown in Figure 1.6.

We recast Equation (1.13) into exponential as in Equation (1.5), i.e.

$$U(\mathbf{s}, \mathbf{r}) \approx \int v(\mathbf{x})R(\mathbf{x}, \mathbf{s})G(\mathbf{x}, \mathbf{s})G(\mathbf{r}, \mathbf{x})d\mathbf{x} \tag{1.14}$$

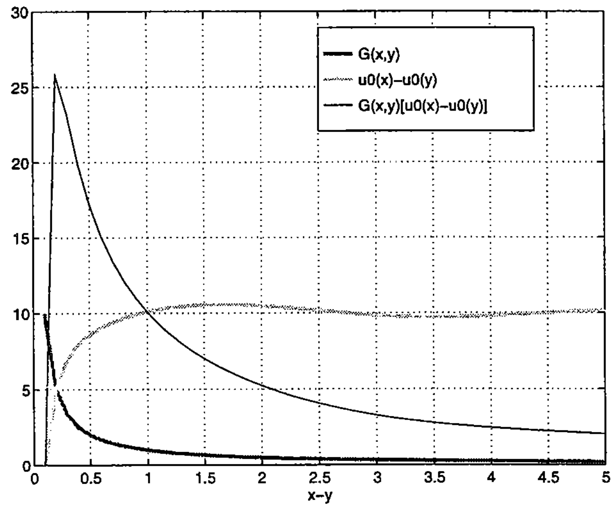

Figure 1.5: The validity of local nonlinear approximation.

where the renormalization factor $R(\mathbf{x}) = exp(\int v(\mathbf{x}')G(\mathbf{x}',\mathbf{x})d\mathbf{x}')$. If we treat the combination $v(\mathbf{x})R(\mathbf{x})$ as a new variable $D(\mathbf{x})$, then the conventional diffraction tomographic procedure can be applied to invert the variable $D(\mathbf{x})$ as described in (Harris, 1987) and (Wu, et al. 1987). The object function itself can be evaluated through the equation

$$D(\mathbf{x}) = v(\mathbf{x})e^{\int v(\mathbf{x}')G(\mathbf{x},\mathbf{x}')d\mathbf{x}'} \tag{1.15}$$

Notice that if $D(\mathbf{x}) = 0$, then $v(\mathbf{x}) = 0$. Therefore Equation 1.15 is degenerated, we only consider non-degenerated case. The integral in the Equation (1.15) may be discreatized as $o_j G_{lj}$, where $G_{lj} = \int_{s_j} G(\mathbf{x}', \mathbf{x}_j)d\mathbf{x}'$. Equation (1.15) can be rewritten as

$$-k_0^2 o_l e^{-k_0^2 o_j G_{lj}} = d_l \tag{1.16}$$

NONLINEAR INVERSION

We consider nonlinear least square inversion for Equation 1.16 through successive linearization. The inversion implement is based on the Levenberg-Marquardt algorithm (Numerical Recipes, 1989), which combines the best features of the gradient search with the method of linearizing the fitting function. This is achieved by increasing the

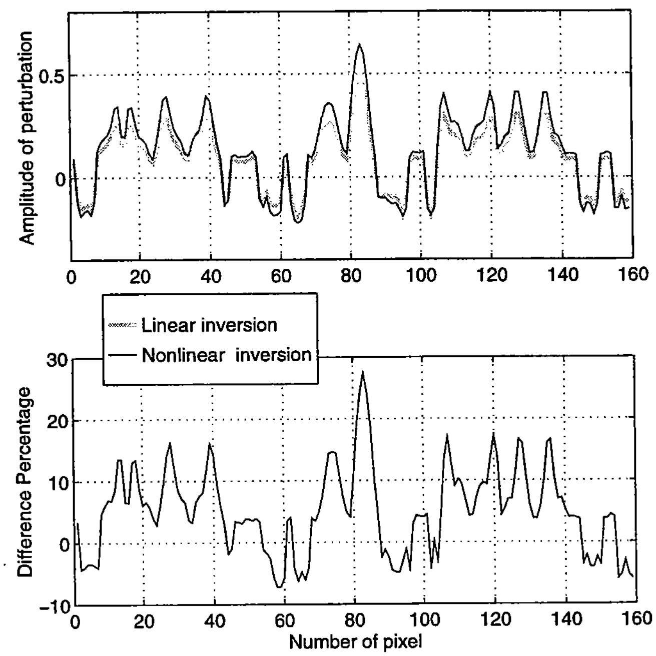

Figure 1.6: Comparison linear and nonlinear inversion on five diffractors. The top panel is the reconstructions with the Born and renormalization methods. The bottom panel is their difference.

diagonal terms of the curvature matrix a by a factor $1 + \lambda$ that controls the interpolation of the algorithm between the gradient search and linearizing the fitting function through Taylor expansion.

Initial parameter values are first guessed for a particular model, and a linear deviation is allowed from the starting model. The residual function r, i.e., the square sum of the difference between the modeled and the observed values, is computed during each iteration step. Parameter values corresponding to the smallest residual are accepted. If the iteration is convergent towards a reasonable parameter set, then the smallest residual is usually reached in the last iteration step. The initial value of the constant factor λ should be chose small enough to take advantage of the analytical solution, but large enough so that r function decreases.

The first estimates are very important because this method is very sensitive to those estimates. In our case, we start from the model inverted with the first Born approximation. To check the algorithm, we reconstructed a idea model of five diffractors. Results are shown in Figure 1.6 and 1.7. Since the scatterers are isolated we expect Born approximations works well.

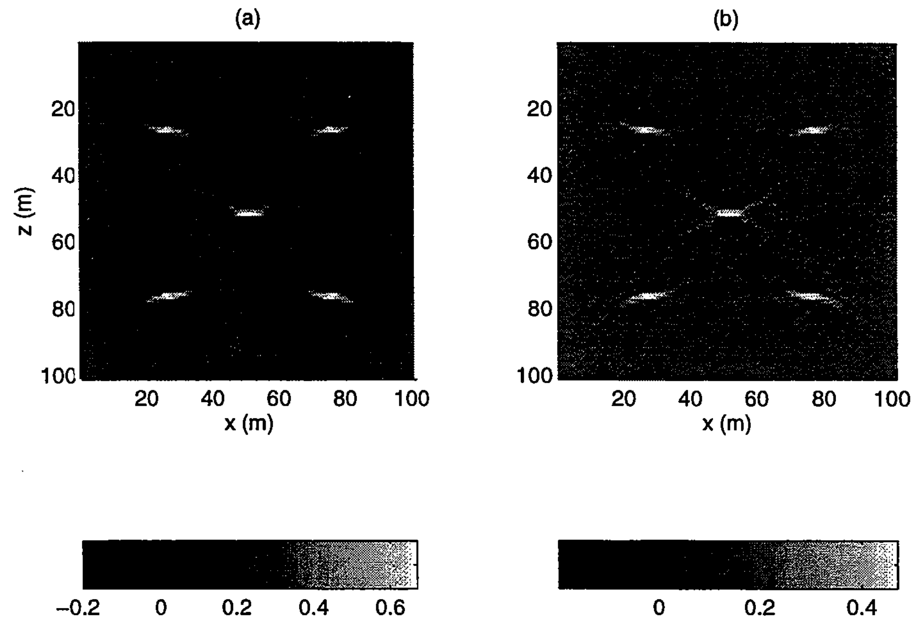

Figure 1.7: Comparison linear and nonlinear inversion on diffractor model. (a) is the reconstructed image with the renormalization method and (b) is the reconstructed image with the Born approximation.

In Figure 1.8 and 1.9, we shown reconstruction of a complicated model of three fractures. We can see that the nonlinear inversion has less distortion compared to linear Born approximation.

To invert Mcroly near offset data, we linearize Equation (1.16) as

$$-k_0^2(o_l - d_l \sum_j o_j G_{lj}) = d_l. \tag{1.17}$$

where d_l is first step inversion using linear Born approximation. Then we solve the linear system using conjugate gradient iterations. The results are shown in Figure 1.10. The residual error of the iterations is shown in Figure 1.11. Notice that we didn't perform any processing to the data set. The results are not intend to be the final.

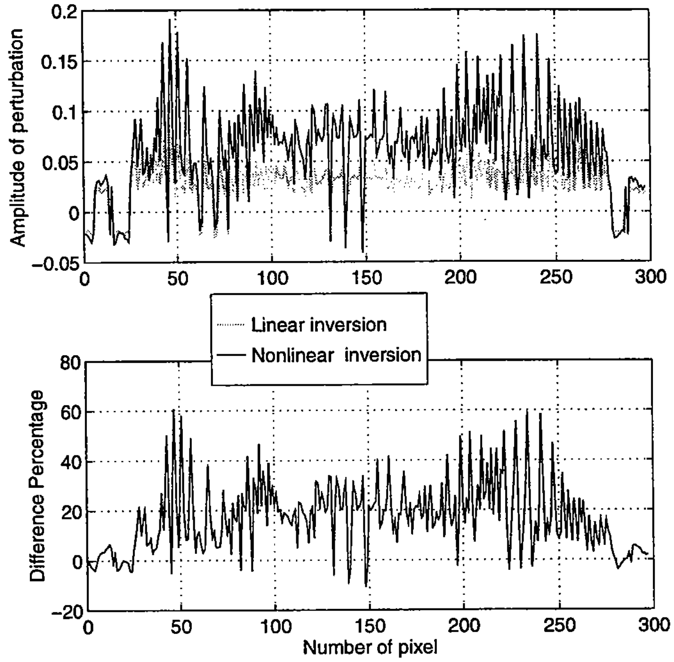

Figure 1.8: Comparison linear and nonlinear inversion on fracture model. The top panel is the reconstructions with the Born and renormalization methods respectively. The bottom panel is their difference

CONCLUSIONS

The renormalization is use to regulate Born expansion and extend Born approximation to the inverse scattering problem with strong fluctuations. This provide a tool to simulate certain forward problem with reasonable accuracy. This procedure also result in a inversion algorithm that can be applied to the case in which media have strong fluctuations.

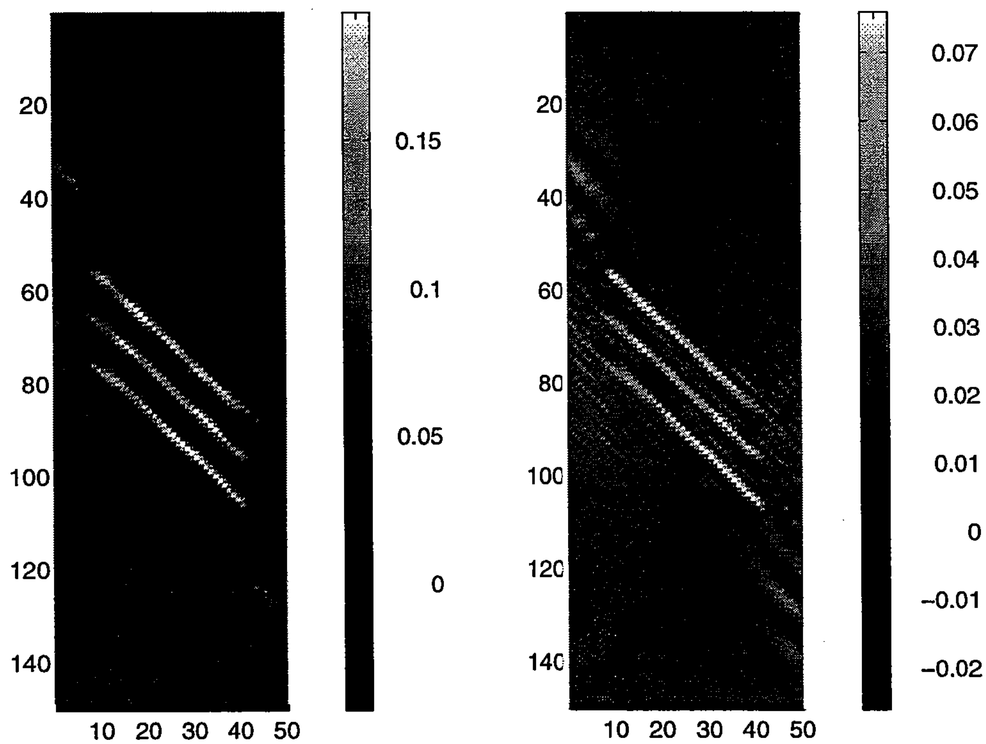

Figure 1.9: Comparison linear and nonlinear inversion on 3 fracture model. The left panel is from nonlinear inversion and the right panel is from linear Born inversion.

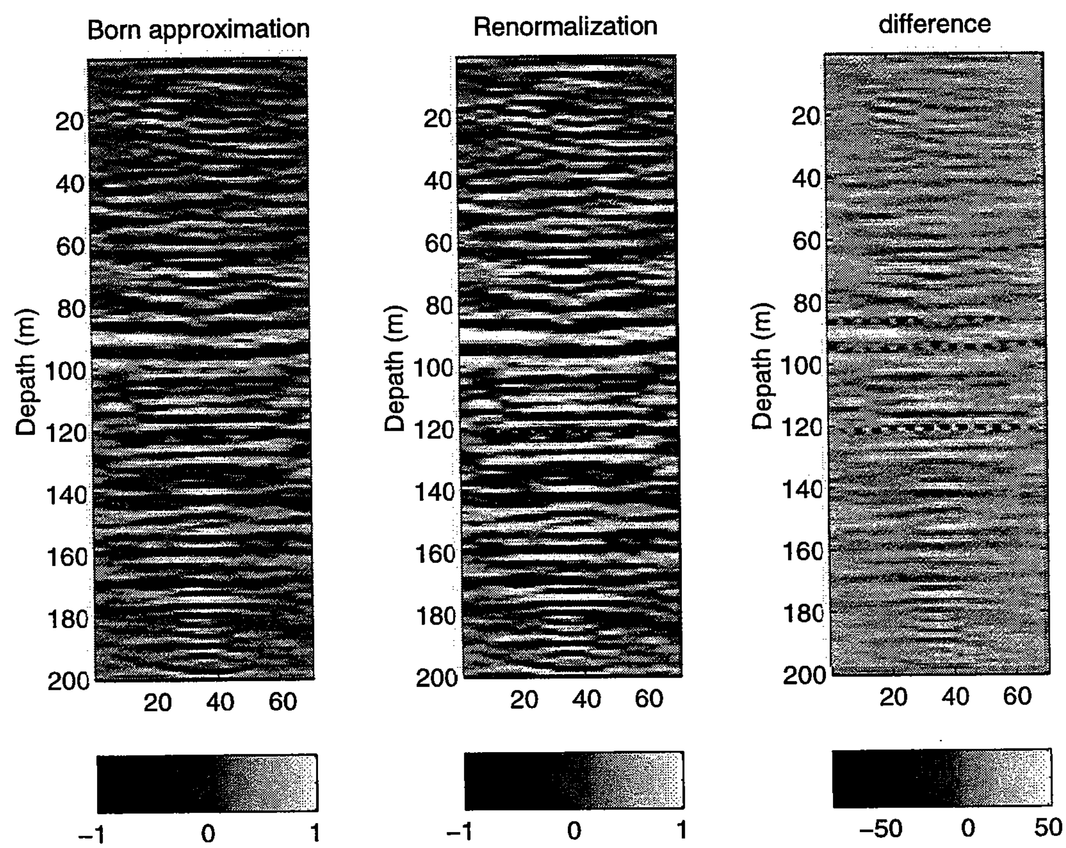

Figure 1.10: The reconstruction of McElroy near offset data. The left panel is reconstructed with the Born approximation, The middle panel is reconstructed with the renormalization, and the right panel is their relative difference.

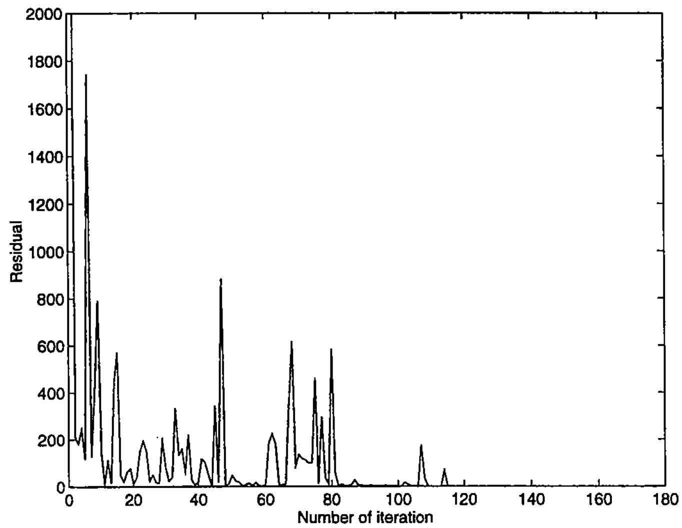

Figure 1.11: This is residual error of conjugate gradient iterations.

REFERENCE

Alford, R. M., Kelly, K. R., Boore, D. M., 1974, Accuracy of finite-difference modeling of the acoustic wave equation: Geophysics, **39**, 834–842.

Habashy, T.M., et al., Beyond the Born and Rytov approximations: a nonlinear approach to electromagnetic scattering: Journal of Geophysical research 10 Feb. vol.98, no.B2, p. 1759-75.

Torres-Verdin, C., et al, 1994, A two-step linear inversion approach for the reconstruction of two-dimensional distributions of electrical conductivity: Proc. of the SPIE, vol.2241, p. 40-54.

J. M. Harris, 1987, Diffraction tomography imaging with arrays of source and detectors: IEEE Trans. Geoscience Remote Sensing, vol. Ge-25, No. 4, 448-455, 1987

R. S. Wu and M. N. Toksoz, 1987, Diffraction tomography and multi-source holography applied to seismic imaging: Geophysics 52, 11-25 (1987)

Optical Society of America, 1994, The collected optics papers of Lord Rayleigh: Washington, DC: Optical Society of America, Part A-B