Paper O

DIFFRACTION TOMOGRAPHY OF THE RANDOM ELASTIC MEDIUM

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ABSTRACT

The diffraction tomography with backpropagation method is extended to the stochastic inversion of spatially random and time-independent elastic media. Within the accuracy of the first Born approximation, the spatial coherency of the field by such medium is related to the spatial coherency of the scattering potential and to its second moment respectively. By probing the random medium with multiple sources and receivers, the second order statistics of the medium can be recovered from the second order statistics of the perturbed field. For the forward modeling, the random distribution is characterized by an ellipsoidal autocorrelation function in the medium properties.

INTRODUCTION

It is important to determine the correlation properties of the randomly distributed heterogeneity or scattering centers from measurements of the scattered field in the reservoir characterization. From a typical sonic well-log we can see that the log curve is oscillating with a considerable number of phase-extreme and the phase periods are not constant. Comparing with other logs in the nearby locations, only relatively small number of phases can be correlated. The overwhelming majority of phases, their location, magnitudes, periods and number vary rapidly form well to well. This indicates that the real medium is a random one or, more precisely, the medium is consists of two components: regular and random components. The random heterogeneity of real media can be described in terms of random functions of position. The wave propagation can be studied with corresponding stochastic differential equations.

Despite extensive studies of the forward problem of wave propagation in random medium (Chernov, 1960; Tatarsky, 1961; Uscinski, 1979) progresses in inverse problem has been slow. There have been some efforts to determine the second-order moments of random scattering potentials, but these investigations have been confined to specific cases. In this study, we investigate the inverse scattering problem of the elastic random medium with a two dimensional model where the large scale inhomogeneities are represented by a homogeneous medium and small scale inhomogeneities are randomly distributed inside the homogeneous medium. The random distribution is characterized by an ellipsoidal autocorrelation function of the medium properties. Solutions of the problems are constructed as in the case of deterministic diffraction tomography. Since the scattering centers are randomly distributed in the medium, we describe it by the first and second moment. As we will show later, the first moment of the fluctuation of the elastic parameters gives the way that the incident wave is attenuated on passing through the medium. The second moment is the autocorrelation or crosscorrelation function of the fluctuation of the elastic parameters that are measures of the size and sharp of a typical irregularity in the medium. The resolved statistical quantities or combinations can be used in some other interpretation or simulation modalities.

SCATTERING FROM DETERMINISTIC ELASTIC MEDIA

We will first summarize the main formulas that we will need later, relating to scattering of elastic waves by deterministic media (Aki and Richards, 1980). The governing equation for an inhomogeneous elastic medium is

$$\rho \ddot{u}_i = (\lambda \nabla \cdot u)_{,i} + (\mu(u_{i,j} + u_{j,i}))_{,j}, \tag{1}$$

where u_i is the displacement in the i^{th} direction, λ and μ are Lamé parameters, and ρ is the density. Defining

$$\begin{aligned} \lambda &= \lambda_0 + \delta\lambda(r) \\ \mu &= \mu_0 + \delta\mu(r) \\ \rho &= \rho_0 + \delta\rho(r), \end{aligned} \tag{2}$$

where δ represents the perturbation to the homogeneous background. Substituting (2) into (1) results in

$$\begin{aligned} &\rho_0 \ddot{u}_i - (\lambda_0 + \mu_0)(\nabla \cdot u)_{,i} - \mu_0 \nabla^2 u_i = \\ &-\delta\rho \ddot{u}_i + (\delta\lambda + \delta\mu)(\nabla \cdot u)_{,i} + \delta\mu \nabla^2 u_i + (\delta\lambda)_{,i} \nabla \cdot u + (\delta\lambda)_{,j}(u_{i,j} + u_{j,i})_{,j}. \end{aligned} \tag{3}$$

The displacement in equation (3) can be represented as a superposition of the incident and scattered displacements, i.e.,

$$u_i = u_i^0 + u_i^{sc}. \tag{4}$$

Combining (3) and (4) we have

$$\begin{aligned} &\rho_0 \ddot{u}_i^{sc} - (\lambda_0 + \mu_0)(\nabla \cdot u^{sc})_{,i} - \mu_0 \nabla^2 u_i^{sc} = Q_i \\ &Q_i = -\delta\rho \ddot{u}_i + (\delta\lambda + \delta\mu)(\nabla \cdot u)_{,i} + \delta\mu \nabla^2 u_i + (\delta\lambda)_{,i} \nabla \cdot u + (\delta\lambda)_{,j}(u_{i,j} + u_{j,i})_{,j}, \end{aligned} \tag{5}$$

where Q_i is a secondary source. Using Green's theorem one obtain

$$u_i^{sc} = \int Q_j(r) *_t G_{ij}(r) d^2 r \tag{6}$$

where $*_t$ denotes convolution in time. Assuming the fluctuation of the physical parameters are small and applying the Born approximation, we have the integral representation

$$u_{lj}^{sc}(r_g, r_s, \omega) = -\int [-\omega^2 \delta\rho G_{li}^s G_{ij}^g + \delta\lambda G_{lk,k}^s G_{ij,i}^g + \delta\mu (G_{li,k}^s + G_{lk,i}^s) G_{ij,k}^g] dxdz, \tag{7}$$

in the frequency domain, where $G_{ij}^g = G_{ij}(r_g, r)$ and $G_{ij}^s = G_{ij}(r, r_s)$ are Green's functions. The displacement for a source in the l-direction and measurements in the j-direction can be represented as superposition of four wave types, i.e., P-to-P, P-to-S, S-to-P and S-to-S. Therefore

$$u_{ij}^{s} = u_{ij}^{PP} + u_{ij}^{PS} + u_{ij}^{SP} + u_{ij}^{SS}. \tag{8}$$

This representation is a direct consequence of the Green's function representation. The two dimensional Green's function for a homogeneous background is discussed in Appendix A. Here we give the results in the transform domain. For P-SV waves

$$\tilde{G}(r,k_s) = \frac{i}{2\rho\omega^2}\begin{bmatrix}(k_x^s)^2 & k_z^{\alpha s}k_x^s \\ k_z^{\alpha s}k_x^s & (k_z^{\alpha s})^2\end{bmatrix}\frac{e^{i\gamma_s^{\alpha}x_s - ik^{\alpha}\hat{s}\cdot r}}{\gamma_s^{\alpha}} + \frac{i}{2\rho\omega^2}\begin{bmatrix}(k_z^{\beta s})^2 & -k_z^{\beta s}k_x^s \\ -k_z^{\beta s}k_x^s & (k_x^s)^2\end{bmatrix}\frac{e^{i\gamma_s^{\beta}x_s - ik^{\beta}\hat{s}\cdot r}}{\gamma_s^{\beta}}. \tag{9}$$

SCATTERING FROM SPATIALLY RANDOM ELASTIC MEDIUM

If the medium is no longer deterministic, the fluctuations of the elastic parameters become, for each frequency, random function of the position vector r. These quantities must be then be characterized by appropriate statistical assembles (Uscinski, 1979). We assuming the regular part of the elastic parameters is constant and the perturbations are random functions of spatial variables, i.e.,

$$\begin{aligned} &\delta\lambda(r,\gamma) << \lambda_0 && <\delta\lambda(r,\gamma)>= 0 \\ &\delta\mu(r,\gamma) << \mu_0 && <\delta\mu(r,\gamma)>= 0 \\ &\delta\rho(r,\gamma) << \rho_0 && <\delta\rho(r,\gamma)>= 0 \end{aligned} \tag{10}$$

For a spatially homogeneous process, the correlation function depends only on the coordinate differences $r = r_2 - r_1$, i.e., $N_{12} = N_{12}(r)$. For $r = 0$, the function N_{12} achieves its maximum N_{11}, equal to the mean square fluctuations. The correlation coefficient N is defined as the ratio of the correlation function N_{12} to the mean square fluctuation, i.e., $N = N_{12} / N_{11}$. As the distance between the points is increased, the correlation coefficient decreases from its maximum value of unity and becomes small compared to unity at a correlation distance. In other words, the statistical dependence between the fluctuations disappears. If the medium is not spatially homogeneous, the correlation function will depend not only on the coordinate difference but also on the coordinates themselves. However we shall consider only the statistically homogeneous case.

The forward problem of the scattering is investigated extensively. We consider the inverse problems of determine the two constituent factors of the two-point spatial correlation function of the scattering potential, i.e., its second moment and its degree of spatial correlation with the measured fluctuations of the scattered fields and their two-point correlation's. The stationary assumption is sufficient to write the autocorrelation of the medium as a function of the difference coordinates x_d and z_d:

$$R(x_d,z_d) =< p(x+x_d,z+z_d)p(x,z)>, \tag{11}$$

where $p(x,z)$ represents any one of the three elastic parameters. Obviously, it is convenient to calculate the statistical quantity in the transform domain. Substituting the appropriate parts of the Green's function in transform domain into (7) for each of the four waves, results in

$$\tilde{u}_{lj}^{pp}(k_s,k_g,\omega)=\frac{-1}{4\rho_0^2}\int[\frac{\delta\lambda}{\alpha_0^4}+\frac{\delta\rho}{\alpha_0^2}(\hat{i}\cdot\hat{g})+\frac{2\delta\mu}{\alpha_0^2}(\hat{i}\cdot\hat{g})^2]k_l^s k_j^g S^p R^p dxdz \quad (12)$$

$$\tilde{u}_{lj}^{ps}(k_s,k_g,\omega)=\frac{-1}{4\rho_0^2}\int[\frac{\delta\rho}{\alpha_0\beta_0}|\hat{i}\times\hat{g}|+\frac{2\delta\mu}{\alpha_0^2\beta_0^2}|\hat{i}\times\hat{g}|(\hat{i}\cdot\hat{g})]k_l^s k_J^g S^p R^s dxdz \quad (13)$$

$$\tilde{u}_{lj}^{sp}(k_s,k_g,\omega)=\frac{1}{4\rho_0^2}\int[\frac{\delta\rho}{\alpha_0\beta_0}|\hat{i}\times\hat{g}|+\frac{2\delta\mu}{\alpha_0^2\beta_0^2}|\hat{i}\times\hat{g}|(\hat{i}\cdot\hat{g})]k_L^s k_j^g S^s R^p dxdz \quad (14)$$

$$\tilde{u}_{lj}^{ss}(k_s,k_g,\omega)=\frac{-1}{4\rho_0^2}\int[\frac{\delta\rho}{\beta_0^2}(\hat{i}\cdot\hat{g})+\frac{\delta\mu}{\beta_0^4}((\hat{i}\cdot\hat{g})-|\hat{i}\times\hat{g}|)]k_L^s k_J^g S^s R^s dxdz \quad (15)$$

where

$$S^p=\frac{e^{i(\gamma_s^\alpha x_s+k^\alpha\hat{i}\cdot r)}}{\gamma_s^\alpha},$$

$$S^s=\frac{e^{i(\gamma_s^\beta x_s+k^\beta\hat{i}\cdot r)}}{\gamma_s^\beta},$$

$$R^p=\frac{e^{i(\gamma_g^\alpha x_g-k^\alpha\hat{i}\cdot r)}}{\gamma_s^\alpha},$$

and

$$R^s=\frac{e^{i(\gamma_g^\beta x_g-k^\beta\hat{i}\cdot r)}}{\gamma_g^\beta}$$

In (12)-(15), for the P-to-S and S-to-P modes, $k_J=k_L=k_z$ for $j=l=x$ and $k_J=k_L=-k_x$ for j=l=z. The convention for capital indexes given above holds for the S-to-S mode. Notice that

$$\begin{aligned}&\hat{i}\cdot\hat{g}=1-(\hat{g}-\hat{i})/2=1-|K|^2/2k^2 && (\hat{i}\cdot\hat{g})^2=1+|K|^4/4k^4-|K|^2/2k^2\\ &|\hat{i}\times\hat{g}|=|K|\sqrt{1-|K|^2/4} && |\hat{i}\times\hat{g}|^2=|K|^2-|K|^4/4\end{aligned} \quad (16)$$

These operators represent the coverage's related to the observation geometry and the polarization, and (12) to (15) are all in the form of the Fourier transform. The spectral of the random perturbation of the l, r and m can be written as

$$[\frac{\delta\lambda(K^{pp})}{\alpha_0^4}+\frac{\delta\rho(K^{pp})}{\alpha_0^2}(\hat{i}\cdot\hat{g})+\frac{2\delta\mu(K^{pp})}{\alpha_0^2}(\hat{i}\cdot\hat{g})^2]k_l^s k_j^g=-4\rho_0^2\gamma_g^\alpha\gamma_s^\alpha e^{-i(\gamma_g^\alpha x_g+\gamma_s^\alpha x_s)}\tilde{u}_{lj}^{pp}$$

(17)

$$[\frac{\delta\rho(K^{ps})}{\alpha_0\beta_0}|\hat{i}\times\hat{g}|+\frac{2\delta\mu(K^{ps})}{\alpha_0^2\beta_0^2}|\hat{i}\times\hat{g}|(\hat{i}\cdot\hat{g})]k_l^s k_J^g = -4\rho_0^2\gamma_s^\alpha\gamma_g^\beta e^{-i(\gamma_g^\beta x_g+\gamma_s^\alpha x_s)}\tilde{u}_{lj}^{ps}$$

(18)

$$[\frac{\delta\rho(K^{sp})}{\alpha_0\beta_0}|\hat{i}\times\hat{g}|+\frac{2\delta\mu(K^{sp})}{\alpha_0^2\beta_0^2}|\hat{i}\times\hat{g}|(\hat{i}\cdot\hat{g})]k_L^s k_j^g = 4\rho_0^2\gamma_s^\beta\gamma_g^\alpha e^{-i(\gamma_s^\beta x_s+\gamma_g^\alpha x_g)}\tilde{u}_{lj}^{sp}$$

(19)

$$[\frac{\delta\rho(K^{ss})}{\beta_0^2}(\hat{i}\cdot\hat{g})+\frac{\delta\mu(K^{ss})}{\beta_0^4}((\hat{i}\cdot\hat{g})-|\hat{i}\times\hat{g}|)]k_L^s k_J^g = -4\rho_0^2\gamma_s^\beta\gamma_g^\beta e^{-i(\gamma_g^\beta x_g+\gamma_s^\beta x_s)}\tilde{u}_{lj}^{ps}$$

(20)

where

$$K^{pp} = k^\alpha\hat{g} - k^\alpha\hat{i},$$

$$K^{ps} = k^\beta\hat{g} - k^\alpha\hat{i},$$

$$K^{sp} = k^\alpha\hat{g} - k^\beta\hat{i},$$

and

$$K^{ss} = k^\beta\hat{g} - k^\beta\hat{i}.$$

These equations indicate the spectrum relation between the physical parameters and the observed scattered field of four different modes. We define the power spectrum as the Fourier transform of the auto-correlation or cross-correlation. The auto-spectra or cross-spectra of the fluctuation of the elastic parameters is related to the power spectrum of the scattered fields, i.e.,

$$< m_{pp}^* m_{pp} >= (4\rho_0^2\gamma_g^\alpha\gamma_s^\alpha)^2 < \tilde{u}_{lj}^{pp*}\tilde{u}_{lj}^{pp} >, \quad (21)$$

$$< m_{ps}^* m_{ps} >= (4\rho_0^2\gamma_g^\beta\gamma_s^\alpha)^2 < \tilde{u}_{lj}^{ps*}\tilde{u}_{lj}^{ps} >, \quad (22)$$

$$< m_{sp}^* m_{sp} >= (4\rho_0^2\gamma_g^\alpha\gamma_s^\beta)^2 < \tilde{u}_{lj}^{sp*}\tilde{u}_{lj}^{sp} >, \quad (23)$$

and

$$< m_{ss}^* m_{ss} >= (4\rho_0^2\gamma_g^\beta\gamma_s^\beta)^2 < \tilde{u}_{lj}^{ss*}\tilde{u}_{lj}^{ss} >. \quad (24)$$

In expression (21) to (24), the random functions characterizing the medium appear by means of function m_{ij}. The mean value of the product $< m_{ij}{}^* m_{ij} >$ is combination of the auto-correlation and cross-correlation spectral of the physical parameters of the random medium. For instance,

$$< m_{pp}^* m_{pp} >= \frac{< \delta\lambda^* \delta\lambda >}{\alpha_0^8} + \frac{< \delta\rho^* \delta\rho >}{\alpha_0^4} (\hat{i} \cdot \hat{g})^2 + 4 \frac{< \delta\mu^* \delta\mu >}{\alpha_0^8} (\hat{i} \cdot \hat{g})^4 +$$
$$\frac{< \delta\lambda^* \delta\rho >}{\alpha_0^6} + 2 \frac{< \delta\lambda^* \delta\mu >}{\alpha_0^8} (\hat{i} \cdot \hat{g})^2 + \frac{< \delta\rho^* \delta\lambda >}{\alpha_0^6} (\hat{i} \cdot \hat{g}) +$$
$$2 \frac{< \delta\rho^* \delta\mu >}{\alpha_0^6} (\hat{i} \cdot \hat{g})^3 + 2 \frac{< \delta\mu^* \delta\lambda >}{\alpha_0^6} (\hat{i} \cdot \hat{g})^2 + 2 \frac{< \delta\mu^* \delta\rho >}{\alpha_0^6} (\hat{i} \cdot \hat{g})^3 \tag{25}$$

We can invert individual correlation functions by using the scattering power spectrum at nine distinct frequencies, i.e.,

$$\begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{19} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{29} \\ \cdot & \cdot & & & & & & & \cdot \\ \cdot & \cdot & & & & & & & \cdot \\ \cdot & \cdot & & & & & & & \cdot \\ \cdot & \cdot & & & & & & & \cdot \\ \cdot & \cdot & & & & & & & \cdot \\ \cdot & \cdot & & & & & & & \cdot \\ a_{91} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & a_{99} \end{bmatrix} \begin{bmatrix} < \delta\lambda^* \delta\lambda > \\ < \delta\rho^* \delta\rho > \\ < \delta\mu^* \delta\mu > \\ < \delta\lambda^* \delta\rho > \\ < \delta\lambda^* \delta\mu > \\ < \delta\rho^* \delta\lambda > \\ < \delta\rho^* \delta\mu > \\ < \delta\mu^* \delta\lambda > \\ < \delta\mu^* \delta\rho > \end{bmatrix} = \begin{bmatrix} D(\omega_1) \\ D(\omega_2) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ D(\omega_9) \end{bmatrix} \tag{26}$$

where the coefficients are

$$a_{i1} = 1/\alpha_0^4 \qquad a_{i2} = (\hat{i} \cdot \hat{g})^2 \qquad a_{i3} = 4(\hat{i} \cdot \hat{g})^2 / \alpha_0^4$$
$$a_{i4} = 1/\alpha_0^2 \qquad a_{i5} = 2(\hat{i} \cdot \hat{g})^2 / \alpha_0^4 \qquad a_{i6} = (\hat{i} \cdot \hat{g}) / \alpha_0^2$$
$$a_{i7} = 2(\hat{i} \cdot \hat{g})^3 / \alpha_0^2 \qquad a_{i8} = 2(\hat{i} \cdot \hat{g})^2 / \alpha_0^2 \qquad a_{i9} = 2(\hat{i} \cdot \hat{g})^3 / \alpha_0^2,$$

and the subscript $i = 1,...,9$ represent nine distinct temporal frequencies. Notice that the operator $(\hat{i} \cdot \hat{g})$ is a function of the frequency. Therefore, the autopower spectrum and crosspower spectrum of the fluctuation of the individual physical parameter can be solved using system (26) and backpropagated to the spatial domain as in the case of deterministic diffraction tomography.

INITIAL RESULTS OF THE NUMERICAL EXPERIMENT

With figure 1, we show the random velocity field with an exponential autocorrelation function $f(x,z) = \exp(-\sqrt{(x/a)^2 + (z/b)^2})$, where a and b is the horizontal and vertical autocorrelation length, respectively (see Appendix B). The inhomogeneities of the medium is described as isotropic or elongated in a direction parallel to either of the two Cartesian directions.

A random velocity field is shown in figure 2. λ is defined as the average wavelength occurred in regions of the medium where the fluctuations of elastic parameters are uncorrelated. $r_0 = b / a$ is defined as the aspect ratio. The weak scattering can be interpreted as $\lambda >> r_0$ which is used to gauge the wavefield modeling.

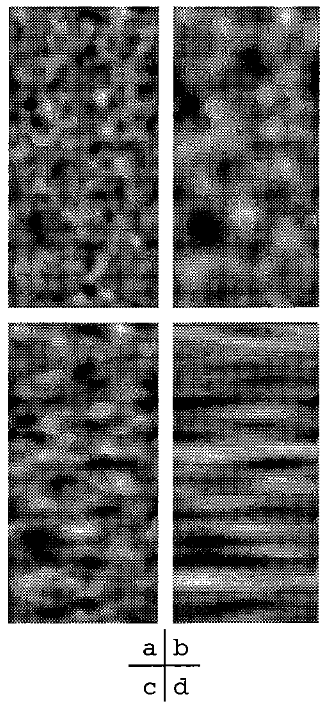

Figure 1. The random velocity field with different vertical and horizontal correlation length a's and b's. a) an isotropic model with a = b = 1; b) a = b = 2 ; c) a elliptical model with a = 2, b = 1; and d) a = 6, b = 1.

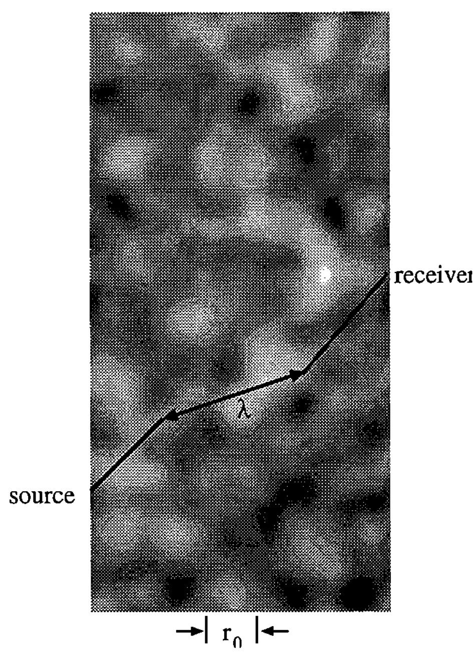

Figure 2. The weak scattering can be interpreted as $r_0 >> \lambda$, where λ is the average wavelength of the incident wave in the medium and r_0 is the aspect ratio (r_0 =b/a).

Two vertical slices of the velocity field are compared with the sonic well logs from McElroy near offset wells is shown in figure 3. We can see that the real medium has similar random characteristics.

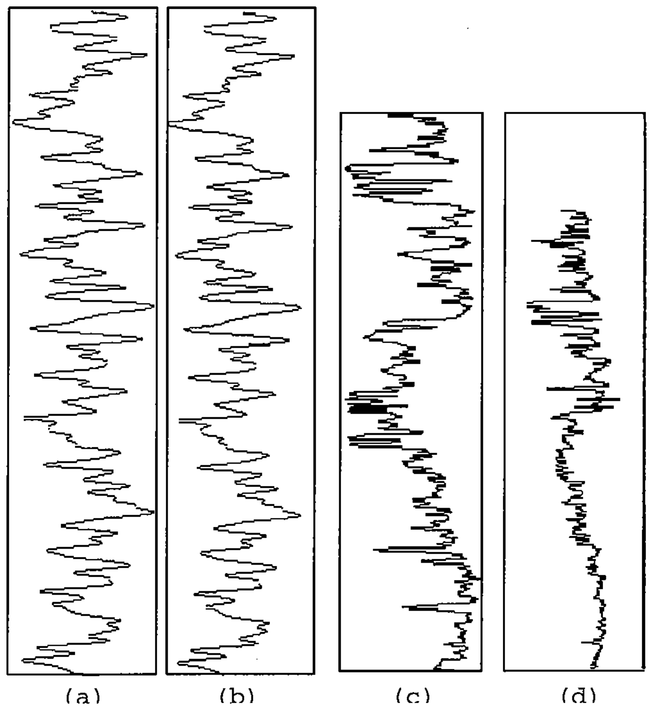

Figure 3. a) Two vertical slices of the random velocity field are compared with the sonic logs from McElroy near offset wells: a) a vertical slice of the velocity field at x=0; b) a vertical slice of the velocity field at x=100; c) McElroy 1068 log; and d) McElroy 1080 log.

In figure 4, the horizontal and vertical component seismograms obtained with finite difference scheme are shown. The autocorrelation lengths a and b is set 5m and 1m respectively. The grid size is chosen as $\Delta x = \Delta z = 0.15m$. The other elastic and statistical parameters are:

Central frequency of the source	$f_c = 400Hz$
P-wave velocity	$v_p = 3500m / \sec$
P-wave variance	$\sigma_p = 10\%$
P and s velocity ratio	$v_p / v_s = 1.65$
Variance of the ratio	$\sigma_{p/s} = 2\%$

From figure 4, we can see many weak scattering events after primary arrives in the seismogram that are due to small scale inhomogeneities.

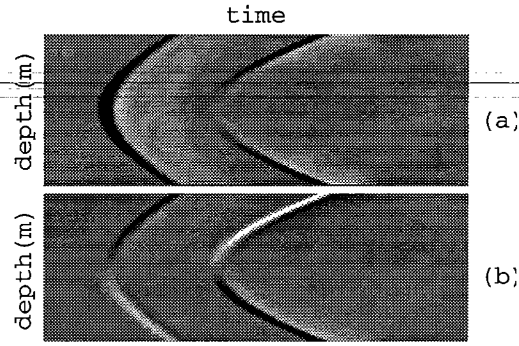

Figure 4. Finite difference simulation of the elastic wave propagation in random medium: a) horizontal component and b) vertical component.

For the real data, we use a similar a wavefield separation and processing scheme described in (Lazaratos, 1993), i.e.,

1) **2.5D Correction**

2) **Direct arrive removal**
   pick direct arrive
   align direct arrival
   subtract direct arrival
   dealign data

3) **Scattering enhancement and wove modes separation with F-K filters**

Using the above process flow, such as pp separation and enhancement, we processed a receiver gather of the McRlroy near offset data set which is shown in figure 5. We can see that there are many chaotic pp reflection/scattering features in the seismogram. Notice that the first p-wave arrival and firs s-wave arrive and thereafter have been eliminated. In figure 6, we show the power spectra of the processed data set at three distinct frequencies.

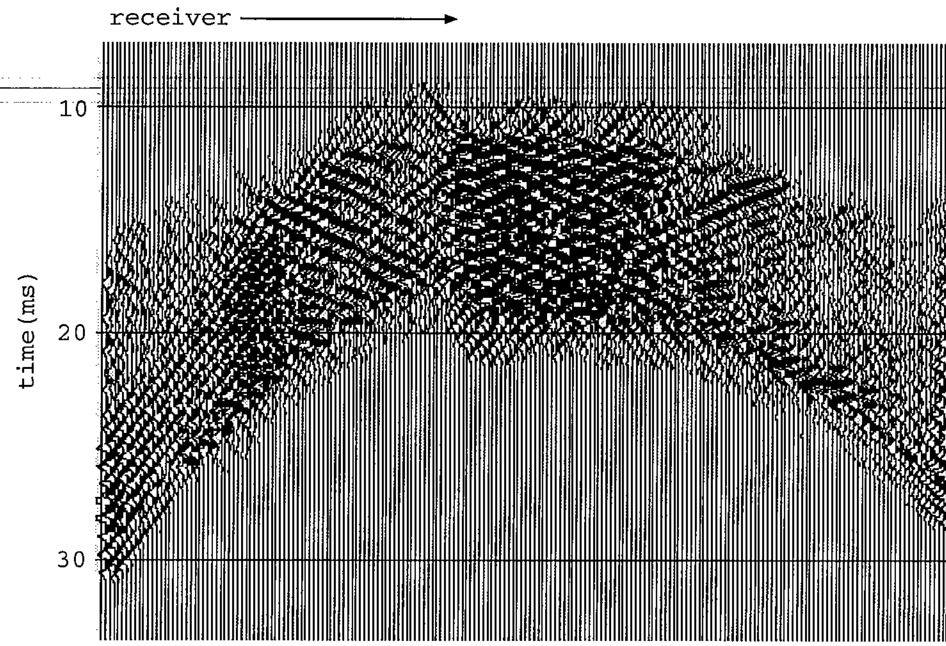

Figure 5. The processed wavefield from McElroy near-offset data set. The first p-arrival and first s-arrival and thereafter have been eliminated.

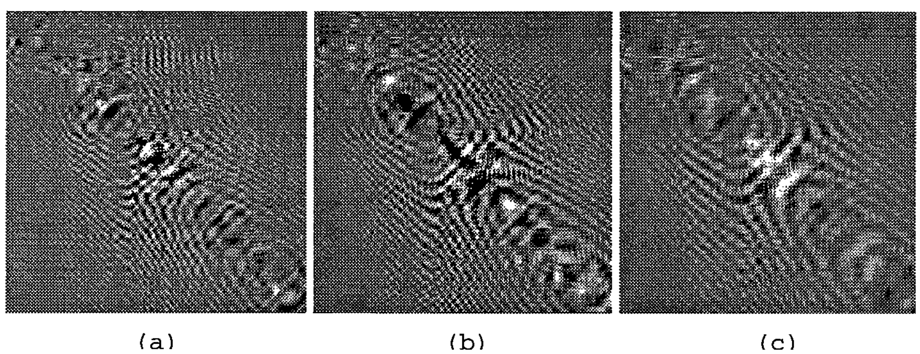

Figure 6. The spectra of the processed wavefield at three distinct frequencies: a), b) and c) is the spectrum at 1100 Hz; 1200 Hz; and 1600 Hz, respectively.

We invert the power spectra and get an autocorrelation function using diffraction tomographic technique discussed above. From figure 7, we can see that the auto correlation obtained is strongly aisotropical, i.e., the horizontal and vertical correlation lengths are different. This is expected since the other studies suggest the geology structure at the McElroy set are basically one dimensional. Note that the correlation function is not of a particular elastic parameters but of a m_{pp} function described in equation (21).

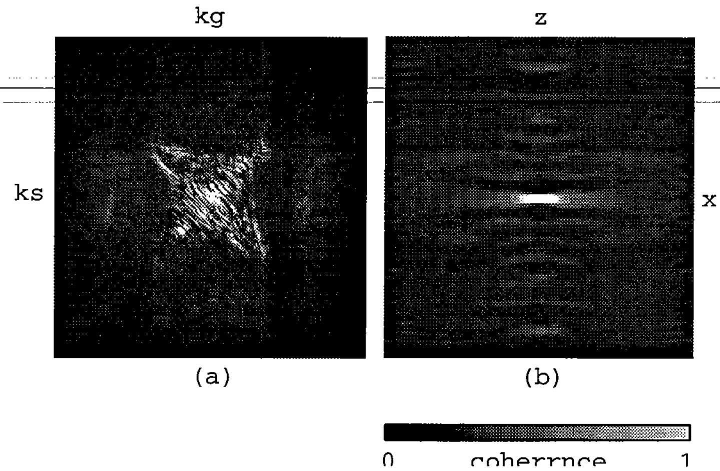

Figure 7. Diffraction tomographic inversion of the random medium: a) The spatial autopower spectrum of the processed wavefield of McElroynear-offset data set; and b) Inverted autocorrelation function of the m_{pp} function.

CONCLUSIONS

This is an on going research and the results shown are primary. One of the advantages of the method is that using resolved statistical quantities, such as the autopower and crosspower spectra. We can calculate the coherency between any two of the spectral that is a quantitative measure of the amount of linear relationship between two physical parameters. The may facilitate the study of the porosity or permeability of the medium or other simulation modalities. For the perturbation theory discussed above to be applicable it is necessary that the scattered field is small in comparison with the incident.

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APPENDIX A

P-SV WAVE GREEN'S FUNCTION

The equation of motion in an elastic, isotropic medium is

$$\rho\ddot{u}_i-(\lambda u_{k,k})_{,i}-(\mu(u_{i,j}+u_{j,i}))_{,j}=F_i. \qquad \text{A-1}$$

For a two dimensional homogeneous background medium, in the x and z direction, we have

$$\rho\ddot{u}_x-\lambda(u_{x,xx}+u_{z,xz})-2\mu u_{x,xx}-\mu(u_{x,zz}+u_{z,xz})=F_x, \qquad \text{A-2}$$

and

$$\rho\ddot{u}_z-\lambda(u_{x,xz}+u_{z,zz})-2\mu u_{z,zz}-\mu(u_{x,xz}+u_{z,zz})=F_z. \qquad \text{A-3}$$

The displacements can be represented in terms of potentials (Aki and Richards, 1980) as

$$\begin{bmatrix}u_x\\u_z\end{bmatrix}=\begin{bmatrix}\phi_{,x}\\\phi_{,z}\end{bmatrix}+\begin{bmatrix}-\psi_{,z}\\\psi_{,x}\end{bmatrix} \quad \text{and} \quad \begin{bmatrix}F_x\\F_z\end{bmatrix}=\begin{bmatrix}\Phi_{,x}\\\Phi_{,z}\end{bmatrix}+\begin{bmatrix}-\Psi_{,z}\\\Psi_{,x}\end{bmatrix} \qquad \text{A-4}$$

where the potentials satisfy

$$\ddot{\phi}-\alpha^2\nabla^2\phi=\Phi/\rho \quad \text{and} \quad \ddot{\psi}-\beta^2\nabla^2\psi=\Psi/\rho, \qquad \text{A-5}$$

with $\alpha=\sqrt{(\lambda+2\mu)/\rho}$ being the P-wave velocity and $\beta=\sqrt{\mu/\rho}$ the S-wave velocity of the medium. The upgoing and downgoing wavefields for the potentials in the Fourier transform domain are:

$$\phi^d=e^{ik_z^\alpha z} \quad \text{and} \quad \psi^d=e^{ik_z^\beta z} \quad \text{for } z>z_s, \qquad \text{A-6}$$

$$\phi^u=e^{-ik_z^\alpha z} \quad \text{and} \quad \psi^u=e^{-ik_z^\beta z} \quad \text{for } z<z_s. \qquad \text{A-7}$$

Rewriting the explicit form of A-2 and A-3 in the frequency domain

$$\begin{bmatrix}(\lambda+2\mu)\partial_x^2+\mu\partial_z^2+\rho\omega^2 & (\lambda+\mu)\partial_x\partial_z\\(\lambda+\mu)\partial_x\partial_z & (\lambda+2\mu)\partial_z^2+\mu\partial_x^2+\rho\omega^2\end{bmatrix}\begin{bmatrix}G_{xx} & G_{xz}\\G_{zx} & G_{zz}\end{bmatrix} \qquad \text{A-8}$$

$$=-\begin{bmatrix}\delta(x-x_s)\delta(z-z_s) & 0\\0 & \delta(x-x_s)\delta(z-z_s)\end{bmatrix}.$$

Taking the Fourier transform over x, A-8 becomes

$$\begin{bmatrix} -(\lambda+2\mu)k_x^2+\mu\partial_z^2+\rho\omega^2 & i(\lambda+\mu)k_x\partial_z \\ i(\lambda+\mu)k_x\partial_z & (\lambda+2\mu)\partial_z^2-\mu k_x^2+\rho\omega^2 \end{bmatrix}\begin{bmatrix} \tilde{G}_{xx} & \tilde{G}_{xz} \\ \tilde{G}_{zx} & \tilde{G}_{zz} \end{bmatrix}$$

A-9

$$= -\begin{bmatrix} \delta(z-z_s) & 0 \\ 0 & \delta(z-z_s) \end{bmatrix} e^{-ik_x x_s}.$$

Using the potential and A-4, the solution of A-9 can be given as

$$\tilde{G}^d = \begin{bmatrix} ic_1k_x\phi^d - c_3\partial_z\psi^d & ic_5k_x\phi^d - c_7\partial_z\psi^d \\ c_1\partial_z\phi^d + ic_3k_x\psi^d & c_5\partial_z\phi^d + ic_7k_x\psi^d \end{bmatrix} \text{ for } z > z_s, \qquad \text{A-10}$$

and

$$\tilde{G}^u = \begin{bmatrix} ic_2k_x\phi^u - c_4\partial_z\psi^u & ic_6k_x\phi^u - c_8\partial_z\psi^u \\ c_2\partial_z\phi^u + ic_4k_x\psi^u & c_6\partial_z\phi^u + ic_8k_x\psi^u \end{bmatrix} \text{ for } z < z_s. \qquad \text{A-11}$$

Applying the conditions

$$\tilde{G}^d = \tilde{G}^u \qquad \text{and} \qquad \partial_z\tilde{G}^d - \partial_z\tilde{G}^u = -\begin{bmatrix} 1/\mu & 0 \\ 0 & 1/(\lambda+2\mu) \end{bmatrix} e^{-ik_x x_s}$$

the coefficients c_1 to c_8 is solved for

$$c_1 = c_8 = k_x / 2\rho\omega^2 k_z^\alpha$$

$$c_4 = -c_3 = c_5 = -c_6 = 1/2\rho\omega^2$$

$$c_7 = c_8 = k_x / 2\rho\omega^2 k_z^\beta.$$

Substituting the coefficients back we obtain

$$\tilde{G} = \begin{bmatrix} k_x^2 & sgn(z-z_s)k_xk_z^\alpha \\ sgn(z-z_s)k_xk_z^\alpha & (k_z^\alpha)^2 \end{bmatrix} \frac{ie^{ik_z^\alpha|z-z_s|-ik_xx_s}}{2\rho\omega^2k_z^\alpha}$$

A-12

$$+\begin{bmatrix} (k_z^\beta)^2 & sgn(z_s-z)k_xk_z^\beta \\ sgn(z_s-z)k_xk_z^\beta & k_x^2 \end{bmatrix} \frac{ie^{ik_z^\beta|z-z_s|-ik_xx_s}}{2\rho\omega^2k_z^\beta},$$

or equivalently

$$\tilde{G} = \begin{bmatrix} \frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x\partial z} \\ \frac{\partial^2}{\partial x\partial z} & \frac{\partial^2}{\partial z^2} \end{bmatrix} \frac{ie^{ik_z^\alpha|z-z_s|-ik_xx_s}}{2\rho\omega^2k_z^\alpha} + \begin{bmatrix} \frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial x\partial z} \\ -\frac{\partial^2}{\partial x\partial z} & \frac{\partial^2}{\partial z^2} \end{bmatrix} \frac{ie^{ik_z^\beta|z-z_s|-ik_xx_s}}{2\rho\omega^2k_z^\beta}. \qquad \text{A-13}$$

Taking the inverse Fourier transform on A-13 we arrive

$$G=\begin{bmatrix}\frac{\partial^2}{\partial x^2} & \frac{\partial^2}{\partial x\partial z}\\ \frac{\partial^2}{\partial x\partial z} & \frac{\partial^2}{\partial z^2}\end{bmatrix}\int \frac{ie^{ik_z^\alpha|z-z_s|-ik_x x_s}}{2\rho\omega^2 k_z^\alpha}dk_x+\begin{bmatrix}\frac{\partial^2}{\partial x^2} & -\frac{\partial^2}{\partial x\partial z}\\ -\frac{\partial^2}{\partial x\partial z} & \frac{\partial^2}{\partial z^2}\end{bmatrix}\int \frac{ie^{ik_z^\beta|z-z_s|-ik_x x_s}}{2\rho\omega^2 k_z^\beta}dk_x .$$

A-14

The individual components of the plane wave decomposition of the Green's function along the source line can be written as

$$\tilde{G}(r,k_s)=\frac{i}{2\rho\omega^2}\begin{bmatrix}(k_x^s)^2 & k_z^{\alpha s}k_x^s\\ k_z^{\alpha s}k_x^s & (k_z^{\alpha s})^2\end{bmatrix}\frac{e^{i\gamma_s^\alpha x_s-ik^\alpha\hat{s}\cdot r}}{\gamma_s^\alpha}+\frac{i}{2\rho\omega^2}\begin{bmatrix}(k_z^{\beta s})^2 & -k_z^{\beta s}k_x^s\\ -k_z^{\beta s}k_x^s & (k_x^s)^2\end{bmatrix}\frac{e^{i\gamma_s^\beta x_s-ik^\beta\hat{s}\cdot r}}{\gamma_s^\beta},$$

A-15

where $k^\alpha=\omega/\alpha$, $k^\beta=\omega/\beta$ and $k_z^{\alpha s}$, $k_z^{\beta s}$ are the vertical wavenumbers over the source line for P and S waves respectively. The decomposition along the receiver line can be obtained by replacing the s's with g's in A-15.

APPENDIX B

TWO DIMENSIONAL RANDOM MEDIUM MODELING

In this appendix, we present a scheme of generating a two dimensional random velocity field following Ikelle (1992). Assuming the random velocity field has a exponential autocorrelation function:

$$auto(x,z) = \exp(-\sqrt{x^2 / a^2 + z^2 / b^2}), \qquad \text{B-1}$$

where a and b are the autocorrelation lengths. The correlation function is considered ellipsoidal because the variables a and z have different scaling factor a and b. A distribution of uncorrelated random number $\theta \subset [0,2\pi)$ is defined as the phase of the random velocity field. In the Fourier transform domain, the random velocity field has the form of

$$Rand(k_x,k_z) = \sqrt{auto(k_x,k_z)} \cdot \exp[i\theta(k_x,k_z)] \qquad \text{B-2}$$

Obviously, the velocity field

$$rand(x,z) = FT_{2d}\{Rand(k_x,k_z)\} \qquad \text{B-3}$$

has a autocorrelation function described above.

In the implantation, we make sure that the first two statistical moments are correct. To satisfy the finite difference's condition of stability and dispersion, the limits to p-wave velocity and s-wave velocity variations are imposed. These limits are large enough so that statistical parameters do not change.