DIFFRACTION TOMOGRAPHY OF STRONGLY SCATTERING MEDIA PART II: USING FOURIER SERIES EXPANSION

Guan Y. Wang

ABSTRACT

To overcome the difficulty resulting from a strongly scattering medium, the total field in the integral equation is expanded as a Fourier series. The coefficients of the expansion are determined by the boundary values and the orthogonality of the Fourier series. The inverse Fourier transform applied to the filtered spectrum of the measurements is equivalently applied to each harmonic component in the series that results in multiscale images. The complete image is obtained, via Mobius transform, with those multiscale images.

INTRODUCTION

In anothe paper in this volume (Paper M, Diffraction tomography of Strongly scattering media part I: Using Phase Extrapolation,) referenced as paper I, the unknown total field in the integral equation is replaced by the extrapolation of the field at the boundary using the Fourier phase shift technique. The method works well for models where there are a few small scatterers. In this kind of model, multiple scattering can be strong but the propagation speed of the scattering fields is not significantly influenced by the inhomogeneities. Therefore, the field extrapolation is treated with a global dispersion relation. However, when the scatterers are more numerous or the scale of the scale of scatterers are larger both the field and propagation of the field are significantly modified by the presence of the scatterers. In this case, the scheme of phase shift extrapolation failed. The resultant image is distorted by the incorrect extrapolation.

In this study, the unknown total field is expanded as a Fourier series to avoid using the dispersion relation that is unknown or very complicated. The coefficients of the expansion are determined by the boundary values and the orthogonality of the Fourier series. The filtered spectrum of the measurement is related to multiscale spectra of the scattering potential resulting from the Fourier expansion of the imhomogeneous wave. The object function is recovered from multiscale components via Mobius inversion. As well as the applicability to strongly non-uniform medium, the method can be easily implemented and is computationally efficient, since the algorithm is similar to what is used in a constant background medium.

SPECTRUM OF THESCATTERED FIELD

Following the notation in the Paper I, the spectrum representation of the observed scattered field is written as

$$u^{sc}(k_g,k_s)=\frac{-ie^{i\gamma_g x_g}}{2\gamma_g}\int v(x,z)e^{-i\gamma_g x+ik_g z}u(x,z|k_s)dxdz, \quad (1)$$

where the total field in the integrand is unknown and $\gamma_s=\sqrt{k_0^2-k_s{}^2}$ .x_g is the separation between source and receiver well,

The total field in the integrand is known, but equation (1) is not a conventional Fourier type integral since the wave vector *k(r)* is spatially variant. We can not directly reconstruct the scattering potential function *v(r)* via the inverse Fourier transform. One way to overcome this difficulty is to expand the unknown inhomogenous field in terms of homogeneous field. That is,

$$u(x,z|k_s)=\sum_{n=-\infty}^{\infty}c_n e^{in\gamma_s(x-x_g)+ink_s z}. \quad (2)$$

Comparing with the back extrapolation approach discussed in the Paper I

$$u(x,z|k_s)=\int u(x_g,k_g|k_s)e^{i\gamma_g(x-x_g)+ik_g z}dk_g$$

in which a globule dispersion relation is used and the relation is true only for homogeneous medium. The equation (2) states that the spectrum of the scattered field is generated equivalently by a series of multiple harmonic components. In other words, a field can be decomposed into harmonic components. The amplitude of each component is determined by the boundary value and the orthogrity of the Fourier base function. Notice that the wave vector in (2) consists with a series of multiplication of a homogeneous wave number. It is possible to reconstruct the scattering potential image with Fourier transforms.

At the observation well, Equation (2) can be expressed as

$$u(x_g,z|k_s)=\sum_{n=-\infty}^{\infty}c_n e^{ink_s z}, \quad (3)$$

where the c_n is determined by virtue of the othogonality of the Fourier base function, i.e.,

$$c_n=\int u(x_g,z|k_s)e^{ink_s z}dz=u(x_g,nk_s|k_s). \quad (4)$$

Substituting (4) in to (1) and changing the order of the summation and integration, one obtain

$$u^{sc}(k_g,k_s)=\frac{-i}{2\gamma_g}\sum_{n=-\infty}^{\infty}\frac{e^{i(\gamma_g-n\gamma_g)x_g}}{u(nk_s,k_s)}\int o(x,z)e^{i(n\gamma_s-\gamma_g)x+i(k_s+k_g)z}dxdz \quad (5)$$

MULTI-SCALE RECONSTRUCTION

From equation (5) we can see that the filtered spectrum of the measurement on the left side of the equation is related to a series of weighted multiresolution potential spectra on the right side, i.e.,

$$2i\gamma_g u^{sc}(k_g,k_s)=\sum_{n=-\infty}^{\infty}\frac{e^{i(\gamma_g-n\gamma_s)x_g}}{u(nk_s,k_s)}V(n\gamma_s-\gamma_g,nk_s+k_g) \tag{6}$$

The notation can be simplified as

$$D_{k_g}(k_s)=\sum_{n=-\infty}^{\infty}O_{k_g}(nk_s), \tag{7}$$

where $D_{k_g}(k_s)=2i\gamma_g u^{sc}(k_s,k_g)$ and $O_{k_g}(k_s)=\frac{e^{i(\gamma_g-n\gamma_s)x_g}}{u(nk_s,k_s)}V(n\gamma_s-\gamma_g,nk_s+k_g)$.

The equation (7) is rewritten as

$$\begin{aligned}D_{k_g}(k_s)&=\sum_{n=-\infty}^{\infty}O_{k_g}(nk_s)\\&=\sum_{n=1}^{\infty}[O_{k_g}(nk_s)-O_{k_g}(-nk_s)]\end{aligned} \tag{8}$$

where $n = 0$ term is dropped since the average value of the image is not computed correctly by the finite algorithm, since it will evaluate the spectrum at the origin as zero. The D.C. component of the image is restored by computing it directly form the data as $4u^{sc}(0,0)e^{-i\bar{k}L}/L\times H$, where LH is image area.

Applying the Mobius inversion to (8), we obtain

$$O_{k_g}(k_s)+O_{k_g}(-k_s)=\sum_{n=1}^{\infty}\mu(n)D_{k_g}(nk_s). \tag{9}$$

or

$$\frac{e^{i(\gamma_g-\gamma_s)x_g}}{u(k_s,k_s)}V(\gamma_s-\gamma_g,k_s+k_g)+\frac{e^{i(\gamma_g+\gamma_s)x_g}}{u(-k_s,k_s)}V(-\gamma_s-\gamma_g,-k_s+k_g)=\sum_{n=1}^{\infty}\mu(n)D_{k_g}(nk_s) \tag{10}$$

Notice that the first term on the left hand of (10) is ford scattering projection of potential function weighted by the spectrum of the forward field and the second term is the back scattering projection of the potential function weighted by the spectrum of the backward field. Unlike the situation in which the character of the angular dependence of the scattering amplitude in the Born approximation is determined by the wave energy. If the energy is sufficiently small the scattering is found to be isotropic and independent of energy. Otherwise, the scattering is sharply anisotropic and is principally in the forward direction. Since the total field including multiple scattering that is no longer only in the forward direction.

For the back scattering data, we can have similar results by repeating the above derivation, i.e.,

$$\frac{1}{u(k_s,k_s)}V(\gamma_s-\gamma_g,k_s+k_g)+\frac{1}{u(-k_s,k_s)}V(-\gamma_s-\gamma_g,-k_s+k_g)=\sum_{n=1}^{\infty}\mu(n)B_{k_g}(nk_s). \quad (11)$$

If we make the measurements at both of the source and receiver wells, we could separate forward and backward scattering projections by solve the 2 by 2 system consisting of equation (10) and (11). Then the scattering potential can be reconstructed via inverse Fourier transform. However, only crosswell data are available in practice. By noticing the complementary nature of the coverage of the forward and backward projections, we can design a filter to eliminate either one of them. For instance, a filter is designed to filt out backward projection $V(-\gamma_s-\gamma_g,-k_s+k_g)$, i.e.,

$$V(\gamma_s-\gamma_g,k_s+k_g)=\frac{e^{i(\gamma_g-\gamma_s)x_g}}{u(k_s,k_s)}f(k_g,k_s)\sum_{n=1}^{\infty}\mu(n)D_{k_g}(nk_s), \quad (12)$$

where the filter

$$f(k_g,k_s)=\begin{cases}1 & (\gamma_s-\gamma_g)^2+(k_s+k_g\pm k_0)^2\le k_0^2\\ 0\end{cases}$$

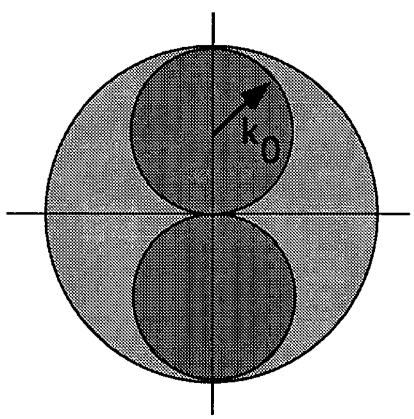

Figure 1. The coverage of the spectrum. The dark shadow is for forward scattering coverage and lighter shadow is for backscattering coverage.

In (12), we see that while the potential spectrum function is different from the result from the Born approximation but their theoretic coverage is the same. The potential function is then reconstructed via inverse Fourier transform. By doing so, the resolution of the reconstruction is degreased since the projection coverage is smaller after filtering operation. However, the non linearity due to multiple scattering in strongly scattering medium is still included in (12). Taking the inverse Fourier transform to both sides of equation () one obtain the reconstructed image, i.e.,

$$v(x,z)=\sum_{n=1}^{\infty}\mu(n)FT^{-1}\{\frac{e^{i(\gamma_g-\gamma_s)x_g}}{u(k_s,k_s)}f(k_g,k_s)D_{k_g}(nk_s)\} \quad (13)$$

In the case of the Born approximation, equation (13) would be one single term. Now, after the inverse Fourier transform is applied to equation (13) a summation of multiscale components of the potential image is reconstructed. The role of the harmonic indexes n is that of the scale lengths in the wavelet transforms. Consequently, a component $FT^{-1}\{\frac{e^{i(\gamma_g-\gamma_s)x_g}}{u(k_s,k_s)} f(k_g,k_s)D_{k_g}(nk_s)\}$ is an image with a specific scale. With large scale length, i.e., large n, $FT^{-1}\{\frac{e^{i(\gamma_g-\gamma_s)x_g}}{u(k_s,k_s)} f(k_g,k_s)D_{k_g}(nk_s)\}$ provides a global view. While small scales, i.e., small n, the component provides detailed views of smaller subsets of the image. The remaining problem is to combine those multiscale images into a complete image. The inverse Fourier transform is explicitly written as following:

$$FT^{-1}\{\frac{e^{i(\gamma_g-\gamma_s)x_g}}{u(k_s,k_s)} f(k_g,k_s)D_{k_g}(nk_s)\}$$
$$= \int \frac{e^{i(\gamma_g-\gamma_s)x_g}}{u(k_s,k_s)} f(k_g,k_s)D_{k_g}(nk_s)e^{i[(\gamma_s-\gamma_g)x+(k_s+k_g)z]}|J|dk_g dk_s \qquad (14)$$

where $|J|=\frac{|\gamma_g k_s + \gamma_s k_g|}{\gamma_s\gamma_g}$ is the Jacobean transformation from the coordinates (K_x, K_z) to the coordinates (k_s, k_g). Notice that in (13) as n is increased the data $D_g(nk_s)$ provide the components with higher and higher frequency. After certain n, $D_g(nk_s)$ is replaced as zero to guarantee the resolution of the image is within the physical limitation. In another word, series (13) has only limit number of terms. This is not surprising because of the sinusoidal nature of propagating wave. A few terms of (2) are sufficient to reproduce the unknown total field.

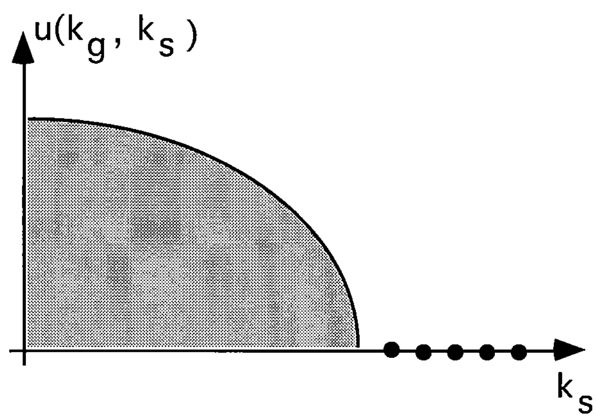

Figure 2. The spectrum response function is cut off at high frequencies.

CONCLUSIONS

We have presented a formulation of diffraction tomography, for strongly scattering medium, which relates the filtered spectrum of the measurement to multiscale spectra of the scattering potential resulting from the Fourier expansion of the imhomogeneous wave. The potential function is recovered from multiscale components via Mobius inversion. As well as the applicability to strongly non-uniform medium, the method can be easily implemented and is computationally efficient, since the algorithm is similar to what is used in a constant background medium.

REFERENCE

Chen, Nan-xian, 1989, Modified Mobius Inverse Formula and its application in physics, Physical review letters, 64, 11.

Hardy, G. H. and Wright, E. M., 1979, An introduction to the theory of numbers: Oxford University Press, Oxford, 5th ed.

Harris, J. M. 1987, Diffraction tomography with arrays of discrete sources and receivers: IEEE trans. Geosci. remote, GE-25(4), 448-455.

APPENDIX A

MOBIUS INVERSION THEOREM

According to Mobius inversion theorem (Hardy, 1979, Chen, N. 1989), if

$$F(x) = \sum_{n=1}^{\infty} f(nx) \tag{A-1}$$

then

$$f(x) = \sum_{n=1}^{\infty} \mu(n) F(nx), \tag{A-2}$$

provide $\sum_{m,n} |f(mnx)|$ converges.

where Mobius function

$$\mu(n) = \begin{cases} 1 & \text{n = 1} \\ (-1)^r & \text{n include r distinct prime factors} \\ 0 & \text{otherwise.} \end{cases}$$

Prove :

$$\begin{aligned} F(x) &= \sum_{n=1}^{\infty} \mu(n) f(nx) \\ &= \sum_{n=1}^{\infty} \mu(n) \sum_{m=1}^{\infty} F(mnx) \\ &= \sum_{k=1}^{\infty} \sum_{mn=k}^{\infty} \mu(n) F(mnx) \\ &= F(x) \end{aligned}$$

since $\sum_{mn=k} \mu(n) = \sum_{n|k} \mu(n) = \delta_{k,1}$.