Normal Modes in Radially layered Media and Application to Attenuation Estimation

Youli Quan* and Xiaofei Chen**

ABSTRACT

We present an effective method to calculate the dispersion relation and eigen functions of normal modes for radially multi-layered media. These normal modes are known as the Stoneley wave, pseudo-Rayleigh waves, etc. in the acoustic logging problem. Based on this dispersion relation we derive a formula which describes the attenuation property of each mode in absorptive media. The stimulated acoustic logging technique can be used to stimulate or enhance a particular mode. We estimate the attenuation of the formation from the attenuation of the enhanced mode.

INTRODUCTION

The dispersion relation, which gives the phase velocities of normal modes, is very important for studying guided waves in acoustic logging. It has been studied by many authors, e.g., Boit (1952), Cheng and Toksöz (1981). In this work, the generalized reflection and transmission coefficients method (Chen et al., 1994; Chen, 1993; Luco and Apsel, 1983; Kennett, 1983) is developed to systematically and simultaneously determine the phase velocities and eigen functions of normal modes in radially layered media. The motivation of this work is from the stimulated mode acoustic logging method (SMAL) (Medlin and Schmitt, 1992). SMAL runs at a specific frequency to stimulate or enhance a given mode. If we know the relationship between the Q-value of a mode and the Q-value of the formation, we may estimate the formation Q-value from that mode. Cheng et al. (1982) gave a formula describing the attenuation of guided waves for a simple open borehole. In this work, a more general formula to calculate the attenuation of each normal mode for a cased borehole and other complicated boreholes is derived from the new dispersion relation. Finally, we show a numerical simulation of the SMAL method, and the attenuation estimation for the S wave.

* *STP, Stanford University*
** *Department of Earth Sciences, USC*

NORMAL MODES IN RADIALLY LAYERED MEDIA

The normal modes are the non-trivial solutions of the free elastodynamic equation under given boundary conditions. Starting from this point, we can derive an effective algorithm to determine the phase velocities and the corresponding eigen functions of the normal modes in radially layered media. In the acoustic logging problem, these normal modes are usually known as tube waves or guided waves.

The $k-\omega$ domain solutions of the free elastodynamic equation in a radially layered model are

$$\mathbf{U}^{(j)} = \mathbf{E}^{(j)}\mathbf{C}^{(j)}, \tag{1}$$

where j=1, 2, ..., N+1 are the index of layers, $\mathbf{U}^{(j)}$ are displacement and stress vectors, $\mathbf{E}^{(j)}$ are known matrices given in Chen et al. (1994), and $\mathbf{C}^{(j)} = [\mathbf{c}_-^{(j)}, \mathbf{c}_+^{(j)}]^{\mathrm{T}}$ are unknown coefficient vectors to be determined under given boundary conditions. Signs "+" and "-" refer to outgoing and incoming waves, respectively. $\mathbf{C}^{(j)}$ for each layer can be related by the generalized reflection/transmission matrices $\hat{\mathbf{R}}_{+-}^{(j)} / \hat{\mathbf{T}}_+^{(j)}$ defined in Chen et al. (1994) as follows,

$$\begin{cases} \mathbf{C}_+^{(j+1)} = \hat{\mathbf{T}}_+^{(j)}\mathbf{C}_+^{(j)}, & \text{(2a)} \\ \mathbf{C}_-^{(j)} = \hat{\mathbf{R}}_{+-}^{(j)}\mathbf{C}_+^{(j)} \quad . & \text{(2b)} \end{cases}$$

If the first layer is a fluid-filled borehole (the acoustic logging problem), $\mathbf{C}^{(1)} = [c_{p-}^{(1)}, c_{p+}^{(1)}]^{\mathrm{T}}$ and $\hat{\mathbf{R}}_{+-}^{(1)}$ becomes a scalar $\hat{R}_{+-}^{(1)}$. The non-singular condition at r=0 (the center of a borehole) implies

$$c_{p-}^{(1)} = c_{p+}^{(1)}. \tag{3}$$

For simplicity, we ignore the normalization factor used in Chen et al. (1994). From equation (2b) we get

$$(1 - \hat{R}_{+-}^{(1)})c_{p+}^{(1)} = 0. \tag{4}$$

In order to obtain a non-trivial solution we must have $c_{p+}^{(1)} \neq 0$, which leads to

$$1 - \hat{R}_{+-}^{(1)} = 0. \tag{5}$$

For a given angular frequency ω, only certain wave numbers k_n, n=1, 2, ..., $K(\omega)$, which are roots of equation (5), satisfy equation (5). These roots constitute the normal modes for certain boundary conditions. The phase velocities v_n of these modes are calculated by the relation

$$v_n = \frac{\omega}{k_n}, \qquad n=1, 2, ..., K(\omega). \tag{6}$$

Therefore, equation (5) is a dispersion relation whose roots correspond to the phase velocities of normal modes for a given frequency. This dispersion relation is very simple and can be easily solved.

Once we obtain the phase velocity of a normal mode, we can calculate its eigen function. According to equation (4), $c_{p+}^{(1)}$ can be any non-zero value. Without any loss of generality we set $c_{p+}^{(1)}(v_n)=1$. Then all other coefficients are obtained by the following relations:

$$c_{p-}^{(1)}(v_n) = \hat{R}_{+-}^{(1)}(v_n),$$

and

$$\begin{cases} \mathbf{c}_{+}^{(j+1)}(v_n) = \hat{\mathbf{T}}_{+}^{(j)}(v_n)\hat{\mathbf{T}}_{+}^{(j-1)}(v_n)...\hat{\mathbf{T}}_{+}^{(1)}(v_n) \\ \mathbf{c}_{-}^{(j+1)}(v_n) = \hat{\mathbf{R}}_{+-}^{(j+1)}(v_n)\mathbf{c}_{+}^{(j+1)}(v_n) \end{cases}, \quad j=1, 2,..., N. \tag{7}$$

Substituting these coefficients into equation (1), we obtain the corresponding eigen functions.

Q-VALUES OF NORMAL MODES

In the above discussion, we only consider the lossless case, i.e., the Q-value for all waves is infinite. In real media the Q-value is finite, i.e., there exists attenuation. For an absorptive medium, the attenuation can be introduced via the complex velocity

$$\tilde{v}(\omega) = v(\omega)[1 - \frac{i}{2Q}] \tag{8}$$

Once the real P and S velocities (α_j, β_j) for each layer are replaced by the complex velocities $(\tilde{\alpha}_j, \tilde{\beta}_j)$, the corresponding phase velocities are complex too (i.e., roots of equation (5) become complex numbers). We assume the complex phase velocity of the n^{th} mode to be

$$\tilde{v}_n(\omega) = v_n(\omega)[1 - \frac{i}{2Q_n}], \tag{9}$$

where $v_n(\omega)$ is a real phase velocity. Then, the dispersion equation can be written as

$$1 - \hat{R}_{+-}^{(1)}[\tilde{v}_n(\omega), \tilde{\mathbf{v}}_\mathbf{p}, \tilde{\mathbf{v}}_\mathbf{s}] = 0 \tag{10}$$

where

$$\tilde{\mathbf{v}}_\mathbf{p} = (\tilde{\alpha}_1, \tilde{\alpha}_2, ..., \tilde{\alpha}_{N+1})$$

and

$$\tilde{\mathbf{v}}_s = (\tilde{\beta}_1, \tilde{\beta}_2, \ldots, \tilde{\beta}_{N+1})$$

are complex P and S velocities for all layers.

Since $1/Q$ is small (usually less than 0.1), we expand $\hat{R}_{+-}^{(1)}$ about $1/Q$ as

$$\begin{aligned}\hat{R}_{+-}^{(1)}[\tilde{v}_n(\omega), \tilde{\mathbf{v}}_p, \tilde{\mathbf{v}}_s] = \hat{R}_{+-}^{(1)}[v_n(\omega), \mathbf{v}_p, \mathbf{v}_s] \\ + \sum_{j=1}^{N+1} \frac{\partial \hat{R}_{+-}^{(1)}[\tilde{v}_n(\omega), \tilde{\mathbf{v}}_p, \tilde{\mathbf{v}}_s]}{\partial \alpha_j} \Delta\alpha_j \\ + \sum_{j=1}^{N+1} \frac{\partial \hat{R}_{+-}^{(1)}[\tilde{v}_n(\omega), \tilde{\mathbf{v}}_p, \tilde{\mathbf{v}}_s]}{\partial \beta_j} \Delta\beta_j \\ + \frac{\partial \hat{R}_{+-}^{(1)}[\tilde{v}_n(\omega), \tilde{\mathbf{v}}_p, \tilde{\mathbf{v}}_s]}{\partial v_n} \Delta v_n + O(\frac{1}{Q^2}) ,\end{aligned} \tag{11}$$

where $\Delta\alpha_j = \tilde{\alpha}_j - \alpha_j = -i\alpha_j / 2Q_p$, $\Delta\beta_j = -i\beta_j / 2Q_S$ and $\Delta v_n = -iv_n / 2Q_n$. Since v_n is the phase velocity of lossless case corresponding to real velocities ($\mathbf{v}_p$, $\mathbf{v}_s$), we have

$$1 - \hat{R}_{+-}^{(1)}(\mathbf{v}_p, \mathbf{v}_s, v_n) = 0. \tag{12}$$

Substituting equation (11) into equation (10), neglecting the term $1/Q^2$ and other higher order terms, and using equation (12), we obtain the formula

$$\begin{aligned}\frac{1}{Q_n(\omega)} \cong -\sum_{j=1}^{N+1} \{ \frac{\partial \hat{R}_{+-}^{(1)}[\tilde{v}_n(\omega), \tilde{\mathbf{v}}_p, \tilde{\mathbf{v}}_s]}{\partial \alpha_j} \frac{\alpha_j}{v_n} \frac{1}{Q_{p_j}} + \\ \frac{\partial \hat{R}_{+-}^{(1)}[\tilde{v}_n(\omega), \tilde{\mathbf{v}}_p, \tilde{\mathbf{v}}_s]}{\partial \beta_j} \frac{\beta_j}{v_n} \frac{1}{Q_{S_j}} \} / \frac{\partial \hat{R}_{+-}^{(1)}[\tilde{v}_n(\omega), \tilde{\mathbf{v}}_p, \tilde{\mathbf{v}}_s]}{\partial v_n} ,\end{aligned} \tag{13}$$

which gives the relationship between the Q-value of n^{th} normal mode and the Q-values of P and S waves in all layers. Noticing that

$$-\frac{\partial \hat{R}_{+-}^{(1)}[\tilde{v}_n, \tilde{\mathbf{v}}_p, \tilde{\mathbf{v}}_s]}{\partial \alpha_j} \Big/ \frac{\partial \hat{R}_{+-}^{(1)}[\tilde{v}_n, \tilde{\mathbf{v}}_p, \tilde{\mathbf{v}}_s]}{\partial v_n} = \frac{\partial v_n}{\partial \alpha_j}$$

and

$$-\frac{\partial \hat{R}_{+-}^{(1)}[\tilde{v}_n, \tilde{\mathbf{v}}_p, \tilde{\mathbf{v}}_s]}{\partial \beta_j} \Big/ \frac{\partial \hat{R}_{+-}^{(1)}[\tilde{v}_n, \tilde{\mathbf{v}}_p, \tilde{\mathbf{v}}_s]}{\partial v_n} = \frac{\partial v_n}{\partial \beta_j} ,$$

we can write equation (13) in an alternative form:

$$\frac{1}{Q_n} \cong \sum_{j=1}^{N+1} \left\{ \frac{\partial v_n}{\partial \alpha_j} \frac{\alpha_j}{v_n} \frac{1}{Q_{P_j}} + \frac{\partial v_n}{\partial \beta_j} \frac{\beta_j}{v_n} \frac{1}{Q_{S_j}} \right\}. \tag{14}$$

For a simple fluid-filled open borehole, equation (14) becomes

$$\frac{1}{Q_n} \cong \frac{\partial v_n}{\partial \alpha_1} \frac{1}{Q_{P_1}} + \frac{\partial v_n}{\partial \alpha_2} \frac{1}{Q_{P_2}} + \frac{\partial v_n}{\partial \beta_2} \frac{\beta_2}{v_n} \frac{1}{Q_{S_2}}, \tag{15}$$

which is the same as in Cheng et al. (1982).

STIMULATED MODE ACOUSTIC LOGGING (SMAL)

The SMAL technique is a relatively new logging method. Some field and laboratory experiments have been done (see, e.g., Medlin and Schmitt, 1992). The SMAL method stimulates or enhances a specific mode or wave under an appropriate source frequency (or a very narrow frequency band). An excitation log obtained by running a broad band sweep reveals the wellbore response. From the response curve we find an appropriate resonant frequency to excite an interesting mode. In the following section one of the examples is on attenuation estimation from the enhanced pseudo-Rayleigh wave by SMAL. More simulations on SMAL, for example, attenuation estimation using the standing waves in a borehole, are in progress.

EXAMPLES

The dispersion relation [equation (5)], tube wave attenuation [equation (14)], and the attenuation estimation from the pseudo-Rayleigh wave are tested for three typical radially layered models (see Table 1).

Table 1. Model parameters

	Fluid	Casing or Invaded zone	Formation
Simple Borehole	r=12cm α=1.5km/s ρ=1.0	—	α=4.9km/s β=2.8km/s ρ=2.4
Cased Borehole	r=10cm α=1.5km/s ρ=1.0	r=11.5cm α=6.1km/s β=3.4km/s ρ=7.8	α=4.9km/s β=2.8km/s ρ=2.4
Invaded Borehole	r=12cm α=1.5km/s ρ=1.0	r=19cm α=3.8km/s β=2.5km/s ρ=2.2	α=4.9km/s β=2.8km/s ρ=2.4

Normal Modes

Let $d(f,k) = |1 - \hat{R}^{(1)}_{+-}(f,k)|$. A plot of $d(f, k)$ vs. (f, k) is helpful to see the root distribution, and what the normal mode looks like in the $k - \omega$ domain. Figures 1-3 show plots for the three models given in Table 1. In these plots, M1, M2, ... ,etc. indicate zeros or roots of $d(f, k)$, i.e., the normal modes. M1 is the Stoneley wave, and M2, M3, ..., etc. are pseudo-Rayleigh waves. R1, R2, etc. are resonances (Schmitt and Bouchon, 1985). These resonances are only local minima, not roots. The coordinates (f, k) corresponding to modes M1, M2,..., etc. are applied to calculate phase velocities of these modes using equation (6). In the next example we use the phase velocity of the fundamental pseudo-Rayleigh wave (M2) to find its Q-value.

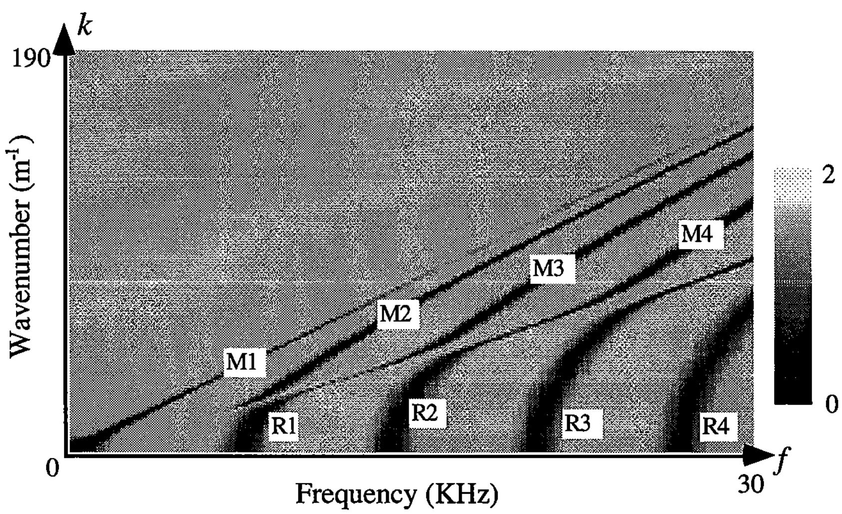

FIG. 1. A plot of the function $d(f,k) = |1 - \hat{R}^{(1)}_{+-}(f,k)|$ for the *simple* open borehole.

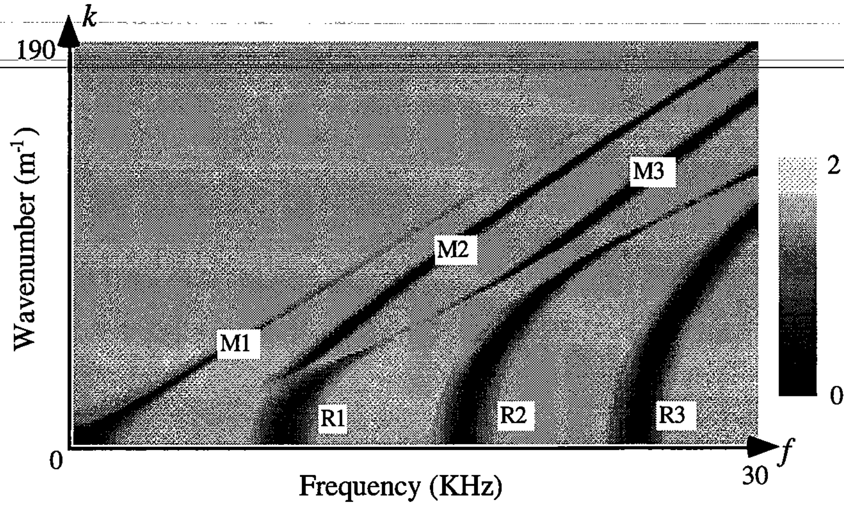

FIG. 2. A plot of the function $d(f,k) = |1 - \hat{R}^{(1)}_{+-}(f,k)|$ for the *cased* borehole.

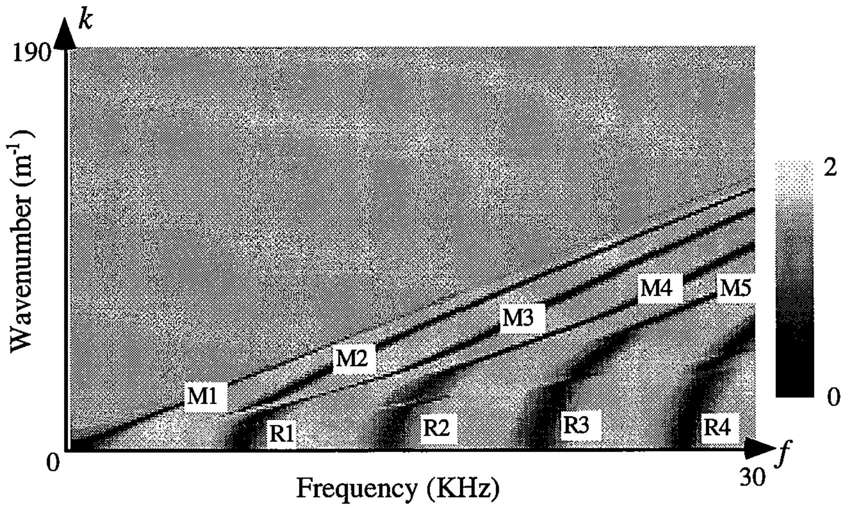

FIG. 3. A plot of function $d(f,k) = |1 - \hat{R}^{(1)}_{+-}(f,k)|$ for the borehole with an *invaded* zone.

<u>Attenuation of the Pseudo-Rayleigh Wave (MODE M2)</u>

The dependence of the Q_2 value of mode M2 on the Q_S-value of the S wave in the formation is shown in Figure 4 by the partition coefficient $(\partial v_n / \partial \beta)(\beta / v_n)$ [equation (14)]. It can be seen that this coefficient is sensitive to Q_S at the cutoff frequency. The next test shows how to estimate Q_S from Q_2 near the cutoff frequency for the simple borehole. Similarly, we can calculate the partition coefficients for the formation P wave and the fluid P wave. The partition coefficient for the formation P wave is negligible at all frequencies. The partition coefficient for the fluid P wave is small at the cutoff frequency, and becomes larger with increasing frequency.

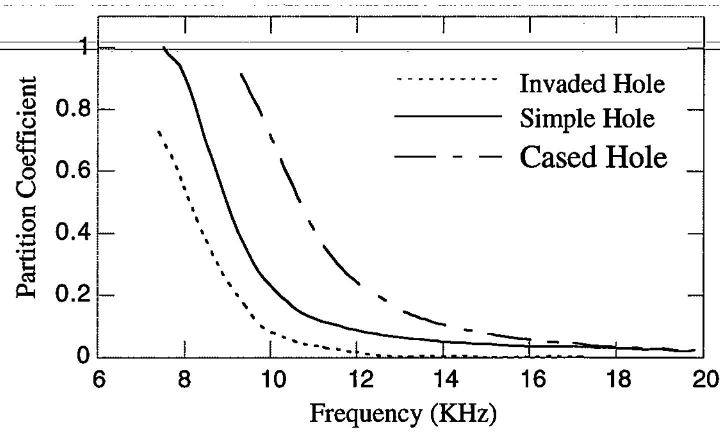

FIG. 4. Attenuation partition coefficients for the three borehole models.

Attenuation Estimation from the Pseudo-Rayleigh Wave

A linear sweep with the frequency band of 0.1-29 KHz is used as the source to calculate a synthetic excitation log (Figure 5) for the simple borehole model given in Table 1. Around f=9.5 Hz there is a peak on the response curve. This frequency is close to the cutoff frequency of the fundamental pseudo-Rayleigh wave. Then we run a short sweep 1.2 ms long with a frequency band of 9-9.5 KHz. This gives a SMAL log of the enhanced pseudo-Rayleigh wave, and is shown in Figure 6. Using the amplitude ratio method (Cheng et al., 1982), we find the estimated Q_{PR} =130 The given S wave Q_S is 80, and Q-values of other waves are infinite. From Figure 4 we find that the partition coefficient is about 0.45 for this case. Then, we get the estimated Q_S=60. The pseudo-Rayleigh wave is dominant in Figure 6. But other waves, e.g., the Stoneley wave, direct P and S waves are also in it, though they are very weak. This is one of the possible reasons why the estimated Q_S is smaller than the given value.

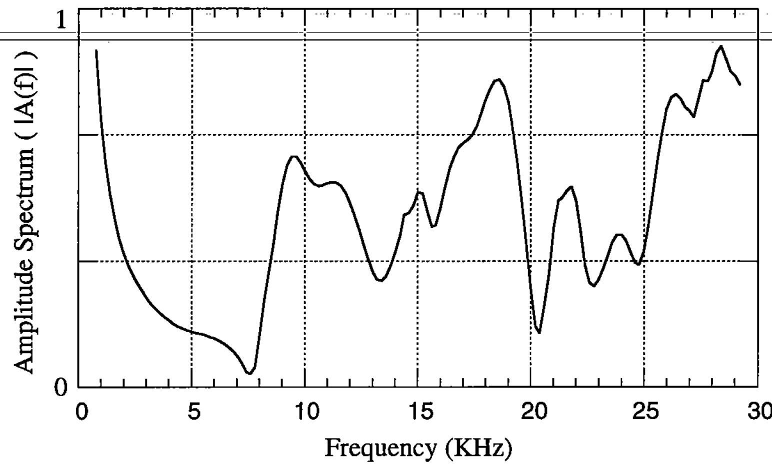

FIG. 5. Excitation log for the simple borehole.

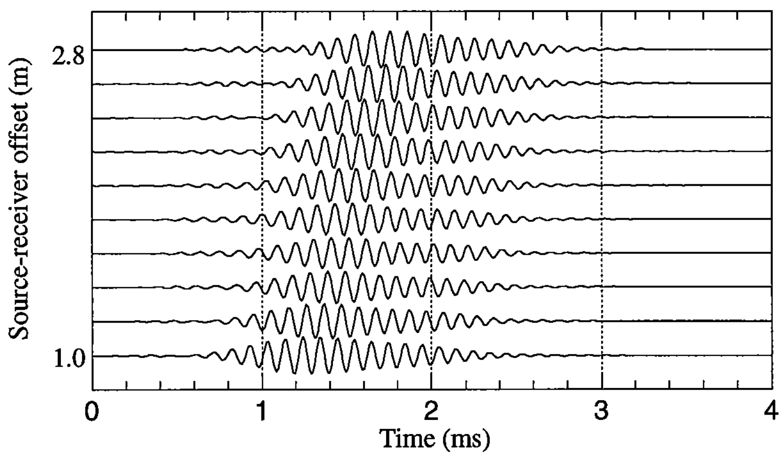

FIG. 6. The enhanced fundamental mode of the pseudo-Rayleigh wave near the cutoff frequency (9.5 KHz) for the simple borehole model.

CONCLUSIONS

A new method based on the generalized reflection and transmission coefficients is developed to calculate the phase velocities, eigen functions, and Q-values of normal modes in radially layered media. Numerical examples show that our algorithm is very efficient, and it can be applied to estimate the S-wave attenuation of the formation from the tube waves, even in a cased borehole.

REFERENCES

Boit, M. A., 1952, Propagation of elastic waves in a cylindrical bore containing a fluid: J. Appl. Physics, **23**, 997-1005.

Chen, X., Quan, Y, and Harris, J. M., 1994, Seismogram synthesis for radially multi-layered media using the generalized reflection/transmission coefficients method: Expanded Abstracts of the 64th SEG annual meeting.

Chen, X., 1993, A systematic and efficient method of computing normal modes for multilayered half-space: Geophys. J. Int, **115**, 391-409.

Cheng, C. H. and Toksöz, M. N., 1981, Elastic wave propagation in a fluid-filled borehole and synthetic acoustic logs: Geophysics, **46**, No. 7, 1042-1053.

Cheng, C. H., Toksöz, M. N. and Willis, M. E., 1982, Determination of In situ attenuation from full waveform acoustic logs: J. G. R., **87**, No. B7, 5477–5484.

Kennett, B. L. N, 1983, *Seismic wave propagation in stratified media*: Cambridge University Press, New York.

Luco, J. E. and Apsel, R. J., 1983, On the Green's function for a layered half-space, Part I: Bull. Seism. Soc. Am., **73**, 909-929.

Medlin, W. L., Schmitt, D. P., 1992, Acoustic logging based on wellbore resonance: SPE 24686, the 67th Annual SPE Technical. Conference.

Schmitt, D. P. and Bouchon, 1985, Full wave acoustic logging: synthetic microseismograms and frequency-wavenumber analysis: Geophysics **50**, No. 11, 1756-1778.