PAPER A

FAST RAY-TRACING THROUGH UNDULATING ANISOTROPIC LAYERS

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ABSTRACT

A fast and robust scheme for two point raytracing through a 3-D piecewise homogeneous arbitrarily anisotropic medium is presented. These features are derived from a combination of the Newton-Raphson scheme and the continuation method. The algorithm performance can be useful in developing a fast traveltime inversion routine.

INTRODUCTION

In many geophysical settings it is desirable to use raytracing to get a first cut on a suitable anisotropic velocity model necessary for proper migration and/or inversion of seismic data. These data are often collected in a way such that not all source and receiver locations lie in a single vertical plane. Such a survey is called a three dimensional (3-D) survey. Modeling such a survey requires a 3-D modeling code unless the layers are all assumed to be flat (a 1-D medium) and the anisotropy is assumed to be transversely isotropic (at most) at every depth interval.

In a horizontally stratified medium each ray, from Snell's Law, has the same horizontal slowness throughout its entire length. This can be used to greatly simplifying two point raytracing in the medium. For 2-D anisotropy (e.g., transverse isotropy with a vertical symmetry axis, or orthorhombic with the vertical source-receiver plane a mirror plane of symmetry of the medium at every depth), rays remain in the source-receiver plane and the horizontal slowness, or p parameter, is the single ray parameter that must be found for any source-receiver pair (Musgrave, 1970). From this single p parameter the complete ray trajectory and the total traveltime are easily found. For 3-D anisotropy, even assuming flat layers, rays no longer lie in the source-receiver plane, yet each ray still has its unique horizontal slowness, now a horizontal vector which may be denoted by two horizontal slowness parameters, say p_x and p_y.

However, when it is necessary to allow for non-flat layer boundaries this extremely convenient property of a ray having a unique horizontal slowness is absent. Yet for the initial approach to velocity model building it is seldom desirable to resort to a full 3-D ray tracing code which can accommodate arbitrary elastic models. This is because 1) the input of an arbitrary model is awkward and time consuming, and 2) ray tracing through an arbitrary grid, even if the model is not too complicated, is relatively slow. The method presented here is a fast generalization of a flat layer code for those instances when several homogeneous layers will suffice, but when it is necessary for those layers to vary laterally, generally in three dimensions, but to still be recognizable as layers.

The method described here uses an iterative Newton-Raphson scheme (Press et al., 1990) to find the intersections of the desired ray with the non-flat layer boundaries. Our formulation is a generalization of the works of Keller & Perozzi (1983) and Docherty & Bleistein (1984) for 3-D arbitrarily anisotropic media. For this scheme to work in a robust fashion it is necessary to begin the iteration with a trial solution very close to the true solution. What we will use as a trial solution for an isolated ray is the flat layer solution. When that is not close enough we adapt a continuation method, allowing the layers to become non-flat in small increments, to make the Newton-Raphson scheme robust. Often, the objective is not to trace an isolated ray but to trace an entire suite of rays from a variety of nearby source locations to an array of closely spaced receivers. In that case, once a single ray is found through the laterally varying medium, its intersections with the layer boundaries will be used as a trial solution for the ray to the neighboring receiver, and so on. And in a similar fashion to cover all the source locations.

Thus, the assumptions for the fast 3-D ray tracing schemes presented here are as follows:

- homogeneous anisotropic elastic layers
- non-flat layer boundaries
- ray trajectory and traveltime from source to receiver in corresponding flat layered medium is known.

The ray trajectory is specified as the intersections (reflection or transmission points) of the ray with the layer boundaries since the ray traverses straight line segments between those points. Note that the corresponding flat layered medium consists of the same anisotropic layers as the non-flat layered medium but with layer boundaries consisting of horizontal planes which are best fitting to or most representative of the actual non-flat layer boundaries.

FORMULATION OF THE PROBLEM

We let our rays propagate through a set of homogeneous but arbitrarily anisotropic elastic layers separated by non-flat boundaries, the $i-$th boundary being specified by,

$$z = h_k + f_k(x,y) \tag{1}$$

where h_k is the depth of the corresponding best fitting flat layer boundary. Consider a particular ray consisting of $N+1$ straight ray segments, i.e., a ray which hits a layer boundary N times, with each segment a specified wave type. A ray consisting of only downward (or only upward) propagating segments would cross N different layer boundaries. In general, a ray would cross $M \leq N$ different layer boundaries. As an example of a ray trajectory for a pegleg multiple with $N=7$, even though only three different material layers are involved, see Figure 1. Note that once the source and receiver position are fixed, a ray may be specified by the function $j(i)$, $i=1,\dots,N+1$, where j denotes the layer in which the $i-$th ray segment is located. For this pegleg multiple example,

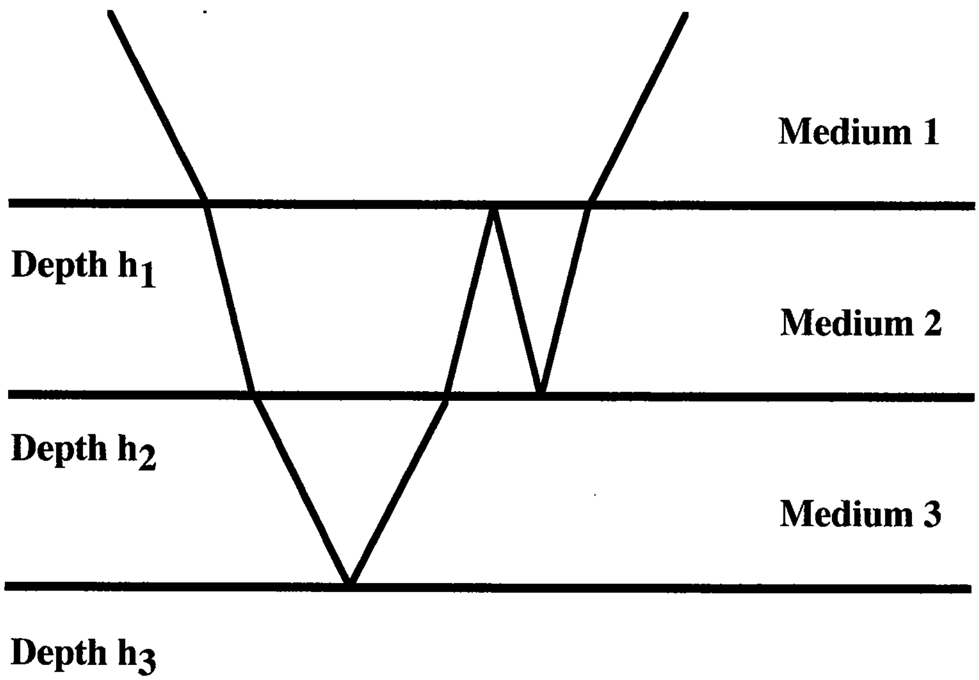
$$j(1)=1 \;\; j(2)=2 \;\; j(3)=3 \;\; j(4)=3 \;\; j(5)=2 \;\; j(6)=2 \;\; j(7)=2 \;\; j(8)=1.$$

Equivalently, the ray could also be specified by the function $k(i)$, $i = 1,...,N$, where k denotes the layer boundary crossed by the ray between the ith and $i+1$th ray segment. For this pegleg multiple example,

$$k(1)=1 \;\; k(2)=2 \;\; k(3)=3 \;\; k(4)=2 \;\; k(5)=1 \;\; k(6)=2 \;\; k(7)=1 \;\; .$$

The i-th ray segment in the $j(i)$th layer is a vector connecting its endpoints, $\mathbf{x}_i - \mathbf{x}_{i-1}$. Note that $\mathbf{x}_0$ is the source point and $\mathbf{x}_{N+1}$ is the receiver point. Each ray segment has an associated slowness vector $\mathbf{p}^{(i)}$ satisfying the slowness relation $F_{j(i)}(\mathbf{p}^{(i)}) = 0$. That ray segment is parallel to group velocity $\mathbf{v}_\mathbf{g}^{(i)}$ and thus we can write,

$$\mathbf{v}_\mathbf{g}^{(i)} \tau_i = \mathbf{x}_i - \mathbf{x}_{i-1} .$$



Layers: j (1, 2, 3, 4, 5, 6, 7, 8)=1, 2, 3, 3, 2, 2, 2, 1.

Intersections: k (1, 2, 3, 4, 5, 6, 7)=1, 2, 3, 2, 1, 2, 1.

Figure 1. Example of a pegleg multiple, $N = 7$.

Taking the dot product of the slowness with both sides of this equation and noting that in any anisotropic medium,

$$\mathbf{p} \cdot \mathbf{v}_\mathbf{g} \equiv 1,$$

(for example see Helbig and Schoenberg, 1987) we find that $\tau_i = \mathbf{p}^{(i)} \cdot (\mathbf{x}_i - \mathbf{x}_{i-1})$, or the traveltime for the entire ray,

$$\tau = \sum_{i=1}^{N+1} \mathbf{p}^{(i)} \cdot (\mathbf{x}_i - \mathbf{x}_{i-1}). \tag{2}$$

Note that the group velocity for the $i-$th ray segment, and hence the $i-$th ray segment itself, $\mathbf{x}_i - \mathbf{x}_{i-1}$, is perpendicular to the slowness surface $F_{j(i)}(\mathbf{p}^{(i)}) = 0$ of that $j(i)$th material layer at the value of the slowness vector of that $i-$th ray segment.

Fermat's principle states that the traveltime τ, as a function of ray trajectory over all trajectories neighboring the true trajectory, is minimum for the true ray trajectory (Arnold, 1989). Thus we can derive non-linear equations on the parameters specifying the traveltime, i.e.,

1. the N ray boundary intersections, RBIs, of the specified ray, x_i , y_i and from (1), $z_i = h_k + f_k(x_i, y_i)$, $i = 1,\ldots,N$
2. the $N+1$ slowness vectors, $\mathbf{p}^{(i)}$, for each of the ray segments, each constrained by $F_{j(i)}(\mathbf{p}^{(i)}) = 0$, the anisotropic dispersion relation of the $j(i)$th layer,

by minimizing,

$$\begin{aligned}
\tau = {} & \mathbf{p}^{(1)} \cdot \left(\mathbf{x}_{\mathbf{h}_1} - \mathbf{x}_{\mathbf{h}_S} + \mathbf{e}_\mathbf{z} \left[h_{k(1)} + f_{k(1)}(\mathbf{x}_{\mathbf{h}_1}) - z_S \right] \right) + \lambda_1 F_{j(1)}(\mathbf{p}^{(1)}) + \\
& \sum_{i=2}^{N} \left\{ \mathbf{p}^{(1)} \cdot \left(\mathbf{x}_{\mathbf{h}_i} - \mathbf{x}_{\mathbf{h}_{i-1}} + \mathbf{e}_\mathbf{z} \left[h_{k(i)} + f_{k(i)}(\mathbf{x}_{\mathbf{h}_i}) - h_{k(i-1)} + f_{k(i-1)}(\mathbf{x}_{\mathbf{h}_{i-1}}) \right] \right) + \lambda_i F_{j(i)}(\mathbf{p}^{(i)}) \right\} + \\
& \mathbf{p}^{(N+1)} \cdot \left(\mathbf{x}_{\mathbf{h}_R} - \mathbf{x}_{\mathbf{h}_N} + \mathbf{e}_\mathbf{z} \left[z_R - h_{k(N)} - f_{k(N)}(\mathbf{x}_{\mathbf{h}_N}) \right] \right) + \lambda_{N+1} F_{j(N+1)}(\mathbf{p}^{(N+1)})
\end{aligned} \tag{3}$$

with respect to $x_i, y_i, p_x^{(i)}, p_y^{(i)}, p_z^{(i)}$, and λ_i, where $\mathbf{x}_{\mathbf{h}_i} = \mathbf{e}_x x_i + \mathbf{e}_y y_i$ is the horizontal part of the position vector to the $i-$th RBI, subscript S denotes the source point, subscript R denotes the receiver point and the λ_i, $i = 1,\ldots,N+1$ are Lagrange multipliers.

THE NON-LINEAR EQUATIONS

To find extrema of traveltime, first set the partial derivatives, $\partial \tau / \partial x_i$ and $\partial \tau / \partial y_i$ to zero yielding,

$$\left.\begin{aligned}
p_x^{(i+1)} - p_x^{(i)} + \left(p_z^{(i+1)} - p_z^{(i)} \right) \frac{\partial f_{k(i)}}{\partial x_i} = 0 \\
p_y^{(i+1)} - p_y^{(i)} + \left(p_z^{(i+1)} - p_z^{(i)} \right) \frac{\partial f_{k(i)}}{\partial y_i} = 0
\end{aligned}\right\} i = 1,\ldots,N . \tag{4}$$

These are expressions of Snell's Law, which says the change of slowness is normal to the boundary. For the $i-$th intersection this direction is defined by the expression $\nabla g_{k(i)}(\mathbf{x}) = \nabla_\mathbf{h} f_{k(i)}(\mathbf{x}_\mathbf{h}) - \mathbf{e}_\mathbf{z}$. Thus the cross-product,

$$\left(\mathbf{p}^{(i+1)} - \mathbf{p}^{(i)} \right) \times \left[\nabla_\mathbf{h} f_{k(i)}(\mathbf{h}_i) - \mathbf{e}_\mathbf{z} \right]$$

vanishes. Equations (4) are the x and y components of this cross-product.

Setting $\partial\tau/\partial p_x^{(i)}, \partial\tau/\partial p_y^{(i)}$ and $\partial\tau/\partial p_z^{(i)}$, $i = 1,\ldots,N+1$ to zero gives the next set of equations. First, solve for the λ_i from the $\partial\tau/\partial p_z^{(i)} = 0$ equations. Substitution of the resulting values into the $\partial\tau/\partial p_x^{(i)} = 0$ and $\partial\tau/\partial p_y^{(i)} = 0$ equations yields,

$$
\begin{aligned}
&(x_1 - x_S)\frac{\partial F_{j(1)}}{\partial p_z^{(1)}} - \left[h_{k(1)} + f_{k(1)}(x_1,y_1) - z_S\right]\frac{\partial F_{j(1)}}{\partial p_x^{(1)}} = 0 \\
&(y_1 - y_S)\frac{\partial F_{j(1)}}{\partial p_z^{(1)}} - \left[h_{k(1)} + f_{k(1)}(x_1,y_1) - z_S\right]\frac{\partial F_{j(1)}}{\partial p_y^{(1)}} = 0 \\
&\left.\begin{aligned}
&(x_i - x_{i-1})\frac{\partial F_{j(i)}}{\partial p_z^{(i)}} - \left[h_{k(i)} + f_{k(i)}(x_i,y_i) - h_{k(i-1)} + f_{k(i-1)}(x_{i-1},y_{i-1})\right]\frac{\partial F_{j(i)}}{\partial p_x^{(i)}} = 0 \\
&(y_i - y_{i-1})\frac{\partial F_{j(i)}}{\partial p_z^{(i)}} - \left[h_{k(i)} + f_{k(i)}(x_i,y_i) - h_{k(i-1)} + f_{k(i-1)}(x_{i-1},y_{i-1})\right]\frac{\partial F_{j(i)}}{\partial p_y^{(i)}} = 0
\end{aligned}\right\} i = 2,\ldots,N \\
&(x_R - x_N)\frac{\partial F_{j(N+1)}}{\partial p_z^{(N+1)}} - \left[z_R - h_{k(1)} - f_{k(N)}(x_N,y_N)\right]\frac{\partial F_{j(N+1)}}{\partial p_x^{(N+1)}} = 0 \\
&(y_R - y_N)\frac{\partial F_{j(N+1)}}{\partial p_z^{N+(1)}} - \left[z_R - h_{k(1)} - f_{k(N)}(x_N,y_N)\right]\frac{\partial F_{j(N+1)}}{\partial p_y^{(N+1)}} = 0
\end{aligned}
\tag{5}
$$

Since in any layer the ray direction, given by $\mathbf{x}_i - \mathbf{x}_{i-1}$, is normal to the slowness surface, i.e., parallel to the gradient of $F_{j(i)}$ with respect to $\mathbf{p}^{(i)}$, the cross-product

$$(\mathbf{x}_i - \mathbf{x}_{i-1}) \times \nabla_{\mathbf{p}} F_{j(i)}(\mathbf{p}^{(i)})$$

vanishes. Equations (5) are the x and y components of this cross-product.

Finally, setting $\partial\tau/\partial\lambda_i$ to zero, yields the third set of equations,

$$F_{j(i)}(p_x^{(i)}, p_y^{(i)}, p_z^{(i)}) = 0, \quad i = 1,\ldots,N+1 \tag{6}$$

i.e., the slowness vectors must satisfy their respective dispersion relations.
Equations (4), (5) and (6) are $5N+3$ non-linear equations on the $5N+3$ unknowns x_i, y_i, $i = 1,\ldots,N$, and $p_x^{(i)}, p_x^{(i)}, p_x^{(i)}$, $i = 1,\ldots,N+1$.

This problem becomes two dimensional when, first, the medium is two dimensional (no variation with y and the source and receivers lie in a single plane perpendicular to the y-axis, call it the $y = 0$ plane and, secondly, the y-axis is a two-fold symmetry axis for all the material layers, i.e., the $y = 0$ plane is at least a mirror plane of symmetry for all the layers. In this case, rays stay in the $y = 0$ plane, the y component of slowness vanishes everywhere, and there is no conversion from in-plane waves (qP, qSv) to cross-plane waves (Sh) (Musgrave, 1970). In this case, the 2nd of equations (4) and the 2nd, 4th and 6th of equations (5) are identically satisfied, and (4), (5), and (6) become a set of $3N+3$ nonlinear equations on the $3N+3$ unknowns:

$$x_i, y_i, \ i = 1,...,N, \ \text{ and } \ p_x^{(i)}, p_x^{(i)}, p_x^{(i)} \ , \ i = 1,...,N+1$$

ITERATIVE TECHNIQUES TO SOLVE THE NON-LINEAR EQUATIONS

In either two or three dimensions the set of equations to be solved, consisting of (4), (5), and (6), is a square non-linear set of the form,

$$\Phi(\mathbf{u}) = \mathbf{0}, \tag{7}$$

where $\mathbf{u}$ is the solution vector consisting of the x and y coordinates of the RBI's and the slowness vectors. Let the zero'th solution be the solution for the corresponding flat layer problem, in which case the horizontal slowness is the same for all ray segments. Solving the system of equations (7) iteratively, at the j th iterate,

1. solve $\left(\nabla_{\mathbf{u}}\Phi|_{\mathbf{u}^j}\right)\Delta\mathbf{u} = -\Phi\left(\mathbf{u}^j\right)$ for $\Delta\mathbf{u}$, and then
2. update the solution vector according to $\mathbf{u}^{j+1} = \mathbf{u}^j + \Delta\mathbf{u}$.

This is the Newton-Raphson technique (Press et al., 1990) for the iterative solution of a set of non-linear equations.

If Newton-Raphson does not converge in a few iterations one can try solving it in steps. Denote the flat layer geometry by $\kappa = 0$ and the specified non-flat geometry by $\kappa = 1$. Between these cases lies a continuum of intermediary geometry's parameterized by κ, $0 \le \kappa \le 1$ for which the layer boundaries are given by,

$$z = h_k + \kappa f_k(x, y).$$

The non-linear system associated with κ is,

$$\Phi(\mathbf{u}_\kappa, \kappa) = \mathbf{0} \ ;$$

$\Phi(\mathbf{u}_\kappa, \kappa)$ may be written simply by replacing $f_k(x,y)$ in Φ by $\kappa f_k(x,y)$. We now try solving the non-flat case for some value of κ less than 1, say $1/n$, still using the $\kappa = 0$ solution as the initial trial solution. If Newton-Raphson now converges in a few iterations, then use a larger value for κ, say $2/n$, now with the solution for $\kappa = 1/n$ as the initial trial solution. The procedure continues until a solution for $\kappa = 1$ is realized, solving the desired two-point ray tracing problem. This is known as the continuation method (Keller & Perozzi, 1983; Docherty & Bleistein, 1985), and here continuation is being applied to a measure of the non-flatness of the layers. This procedure can be made quite robust and the number of iterations required at each value of κ may be kept quite low. This is because the zero'th trial solution (from the previous value of κ) may be chosen to be very close to the required solution at the present value of κ.

EXAMPLES

Often in seismic modeling, one requires traveltimes from a near surface source to a large number of closely spaced receivers, such as for modeling surface seismic or VSP traveltimes. The first ray found for flat layers could be the ray with vanishing horizontal slowness. When all the layers are at a minimum, up-down symmetric, i.e., the horizontal plane is at least a mirror plane of symmetry, that zero horizontal slowness ray is vertical. Now that ray may be used to find the ray to the same receiver location when the layers are no longer flat. If the lateral variation is large it may be necessary to use continuation to find that ray. After the first ray is found the other rays to neighboring receiver locations may be found by applying continuation to the receiver position. Thus the initial trial solution for the ray to a given receiver could naturally be the ray to the previous receiver. In this way ray trajectories and traveltimes can be rapidly built up to give almost continuous coverage, if desired.

As a three dimensional example we present the computation of rays for a walkway VSP common shot gather. The medium properties in the principal axes system are presented in the table below.

Layer	a_{11}	a_{12}	a_{13}	a_{22}	a_{23}	a_{33}	a_{44}	a_{55}	a_{66}
1	4.00	1.00	1.00	4.00	1.00	4.00	1.50	1.50	1.50
2	12.00	3.10	1.70	16.80	2.45	7.25	3.10	1.23	2.50
3	12.75	5.95	2.87	21.75	2.87	8.25	3.00	3.00	7.90
4	12.60	3.25	1.36	12.60	1.36	5.40	2.25	2.25	4.67

Table 1. Density normalized elastic moduli, at principal axes, for the model. The first layer is isotropic the second orthorhombic and the third and fourth layers are transversally isotropic.

The principal axis for the layers 2 and 4 were tilted. The orthorhombic medium in layer 2 was rotated by 45° around the Z axis and afterwards another 45° around the new X axis. The transversally isotropic medium at bottom half-space was rotated of 60° around the X axis. Figures 1 and 2 show qP rays at two different perspectives for a flat interfaces medium. Note that ray trajectories are not plane curves as they would be in the isotropic case. Figures 3 and 4 show two perspectives of the ray trajectories across a medium with the same elastic properties as before but with nonflat interfaces.

CONCLUSION

A fast ray tracing for non-flat homogeneous layers with arbitrary anisotropy was presented. The algorithm is very robust for qP-waves. It can lead to problems for quasi-shear waves around non-convex regions and near repeated roots of the slowness surface. The continuation approach can be applied to proceed from isotropic to anisotropic media and to extend results from a single source-receiver configuration to results for a full scale survey involving multiple sources and receivers. The performance of the algorithm makes it adequate for procedures that require the computation of a large number of ray trajectories, such as traveltime inversion routines.

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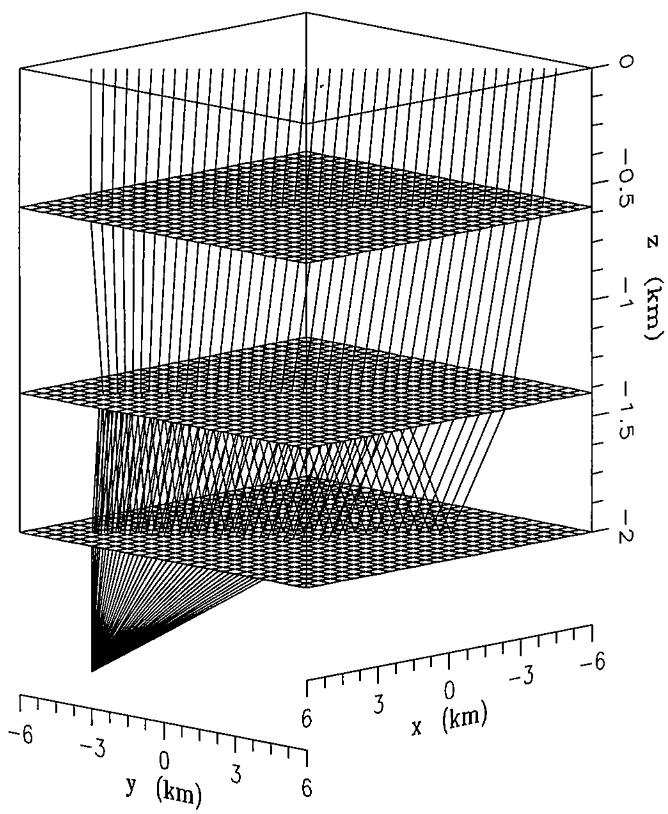

Figure 1. qP waves rays, including multiples, through a flat layered anisotropic media.

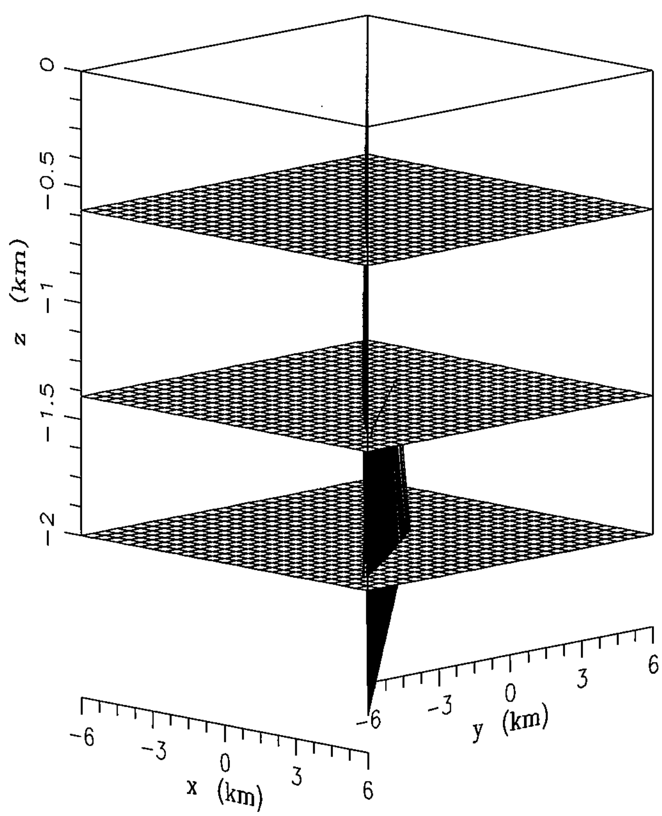

Figure 2. Another perpective for rays in figure 1. Notice that the rays are not plane curves as it should in the isotropic case.

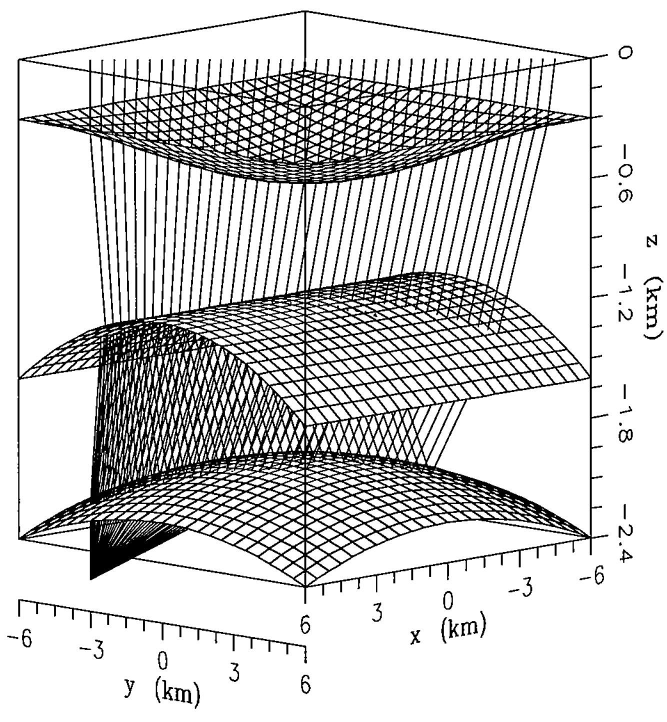

Figure 3. qP rays through a nonflat layered medium.

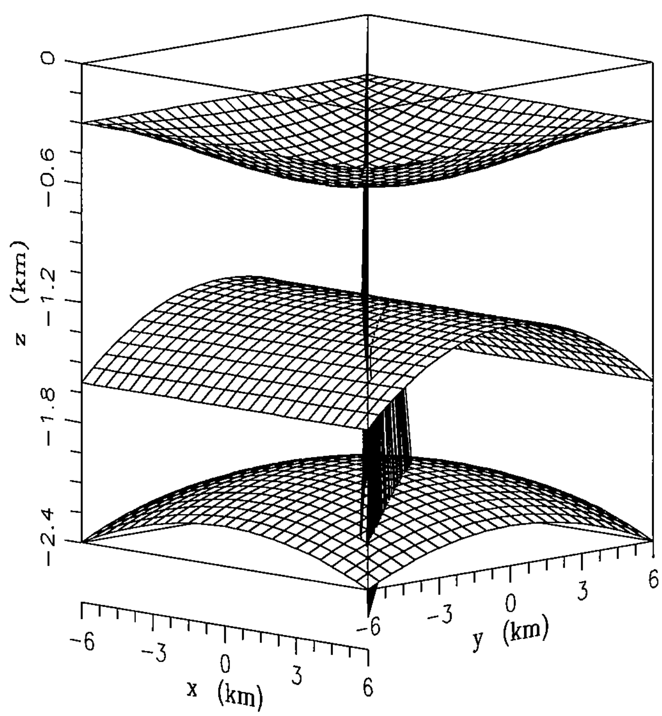

Figure 4. Another perspective for the rays in figure 3.