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DIFFRACTION TOMOGRAPHY USING MULTISCALE FOURIER TRANSFORMS

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ABSTRACT

To overcome the difficulty resulting from a strongly non-uniform medium, a variable background is chosen to maintain a weak contrast between the scatterers and the background so that the single scattering approximation is still valid. Assuming that the amplitude variation of the wave field, due to propagation, is less than the phases, the Green's function of variable background can be treated as the Green's function associated with a constant reference background modified by a phase distortion function. The spectrum of the scattering field is expressed as a planar integral of harmonic oscillators. Each oscillator possesses an amplitude consisting of the scattering potential and a nonlinear phase. The phase factor is further expanded into Fourier series. The inverse Fourier transform applied to the filtered spectrum of the measurements is equivalently applied to each harmonic component in the series which results in multiscale images. The complete image is obtained, via Mobius transform, with those multiscale images.

INTRODUCTION

The image reconstructed with the ray tomography has the resolution only on the scale of first Fresnel zone even for full aperture. When a higher resolution result is required, such as in reservoir imaging, other wave phenomena should also be utilized in addition to transmission traveltimes. The resolution of the reconstruction with diffraction tomography is about one wavelength. The existing diffraction tomographic inversions are mainly based on plane wave expansion and Fourier transform techniques for uniform background medium (Devaney, 1982, Harris, 1987, Wu and Toksoz, 1987). Such methods are simple to implement but do not work well when the background medium is strongly non-uniform. One way to overcome the problem of the strong inhomogeneity of the medium is to apply the distorted Born approximation (Devaney, et al. 1983). This consists of adopting a variable background to maintain a weak contrast between the perturbation and the background medium. However, under the distorted Born approximation, the difficulty is not only how to find the Green's function associated with the variable background but also the Green's function generally has little use for utilizing Fourier transform techniques to reconstruct images. This is why most proposed algorithms dealing with variable background are restricted to some special case. For example, the case of 1-D medium in which the problem is greatly simplified (Dickens, 1992, Huan, 1992).

By introducing a reference medium and the WKBJ Green's functions, and assuming the amplitude variations due to the variation of the background is less the phases, we reformulate the inverse scattering problem of an arbitrary 2-D host medium such that the spectrum of the scattered field is expressed as a planar integral of harmonic oscillators. The oscillator possesses an amplitude which consists of the scattering potential and a nonlinear phase. In order to utilize Fourier transform reconstruction techniques, the phase factor is expanded into Fourier series. The inverse Fourier transform applied to the spectrum of the measurements is equivalently applied to each harmonic component in the series which leads to the construction of the images with different scales. The complete image is combined, via Mobius inversion, from those components with different scales. The reconstruction algorithm is essentially the same as that of Fourier diffraction tomography for a constant host medium, except that we first construct the images with different scales and then combine them together.

SPECTRUM OF SCATTERED FIELD FROM A HARMONIC OSCILLATOR MODEL

For a variable background medium, the scattered field generated by an inhomogeneity perturbed over a variable background can be written as

$$u^{sc}(s,g) = \int o(r)u(r,s)\tilde{G}(g,r)dr \tag{1}$$

where $\tilde{G}$ is the Green's function of the background. With distorted Born approximation, the equation (1) is liberalized as

$$u^{sc}(s,g) = \int o(r)\tilde{G}(r,s)\tilde{G}(g,r)dr \tag{2}$$

Generally, it is difficult to find the Green's $\tilde{G}$ function associated with the variable background. By introducing the Green's functions of a constant reference background, we can rewrite equation (2) as

$$u^{sc}(s,g) = \int o(r) \frac{\tilde{G}(r,s)\tilde{G}(g,r)}{G(r,s)G(g,r)} G(r,s)G(g,r)dr, \tag{3}$$

where $G(r,s)$ and $G(g,r)$ are the Green's functions associating with the reference background. By replacing the Green's functions in $\frac{\tilde{G}(r,s)\tilde{G}(g,r)}{G(r,s)G(g,r)}$ with corresponding WKBJ forms,

$$\tilde{G}(r,r') = A(r,r')e^{-i\theta(r,r')} \text{ and } G(r,r') = \frac{1}{|r-r'|}e^{-ik\cdot|r-r'|},$$

and neglecting the variation of the amplitude due to the variation of the background, i.e., assuming

$$\frac{A(r,s)}{|r-s|} \approx 1, \text{ and } \frac{A(g,r)}{|g-r|} \approx 1,$$

we have

$$\frac{\tilde{G}(r,s)\tilde{G}(g,r)}{G(r,s)G(g,r)} \approx e^{i\phi(r)}, \tag{4}$$

where $\phi(r)$ is the phase distortion resulting from the "variation" of the background medium. Substituting (4) into equation (3) leads to

$$u^{sc}(s,g) = \int o(r)e^{i\phi(r)}G(r,s)G(g,r)dr. \tag{5}$$

We can see that equation (5) is the same as in the case of the uniform background medium, except that the integrand is modified by a phase distortion function. In a 2-D medium with a line source, the Green's function is the Hankel function of first kind and zero order, i.e.,

$$G(r,r') = \frac{i}{4}H_0^{(1)}(\bar{k}|r-r'|).$$

Taking the Fourier transform of equation (5) over s and g , which is decomposing the cylindrical wave into plane waves, we have

$$u^{sc}(k_s,k_g)4\gamma_s\gamma_g e^{-i(\gamma_s d_s+\gamma_g d_g)} = \int o(r)e^{-ik(r)\cdot r}dr, \tag{6}$$

where d_s, - d_g is the separation between source and receiver well, $\gamma_s = \sqrt{\bar{k}^2 - k_s^2}$ and $\gamma_g = \sqrt{\bar{k}^2 - k_g^2}$. $\bar{k} = (\bar{k}_x,\bar{k}_z) = (\gamma_g - \gamma_s, k_s + k_g)$ is the wave vector in the reference medium. The resultant wave vector $k(r) = \bar{k}(1 - \frac{\Delta v(r)}{\bar{v}})$, where $\Delta v(r)$ is the variation on the top of the reference medium $\bar{v}$. The equation (6) states that the spectrum of the scattered field is generated equivalently by an "harmonic oscillator" $o(r)e^{-ik(r)\cdot r}$ located at each image point. The oscillator possesses the amplitude o(r) and the nonlinear phase $\phi(r) = k(r)\cdot r$.

RECONSTRUCTION WITH MULTISCALE FOURIER TRANSFORMS

Equation (6) is not a conventional Fourier type integral, since the resultant wave vector $k(r)$ is spatially variant. We can not directly reconstruct the scattering potential function $o(r)$ via the inverse Fourier transform. One way to overcome this difficulty is to expand phase function $p(x,z) = e^{i[k_x(x,z)x + k_z(x,z)z]}$ into a Fourier series with $\bar{k}$ as the fundamental wave number, and then treat each harmonic separately. Notice that $p(x,z)$ is defined in the rectangle region $0 \le x \le L,\ 0 \le z \le H$. If we extend p(x,z) into a periodic odd function with periods of L and H in the horizontal and vertical directions respectively, i.e.,

$$odd(x,z) = \begin{cases} p(x,z) & 0 \le x \le L,\ 0 \le z \le H \\ -p(-x,z) & -L \le x < 0,\ 0 \le z \le H \\ -p(x,-z) & 0 \le x \le L,\ -\mathrm{H} \le z < 0 \\ p(-x,-z) & -L \le x < 0,\ -\mathrm{H} \le z < 0, \end{cases}$$

and

$$\begin{aligned} &odd(x+2L,z) = odd(x,z) \\ &odd(x,z+2H) = odd(x,z), \qquad x,z \in (-\infty,\infty) \end{aligned}$$

then the finite sine transform of $odd(x,z)$ can be written as

$$P(m,n) = \frac{4}{L \times H} \int_0^L \int_0^H p(x,z)\sin(\bar{k}_x mx)\sin(\bar{k}_z nz)dxdz,$$

where L and H are the width and height of the image domain respectively. With the above definitions, we can expand p(x,z) as

$$p(x,z) = \sum_{m,n=1}^{\infty} P(m,n)\sin(\bar{k}_x mx)\sin(\bar{k}_z nz). \tag{7}$$

Substituting expansion (7) into equation (6) and interchanging the order of the summation and integration we obtain

$$\begin{aligned} &u^{sc}(k_s,k_g)4\gamma_s\gamma_g e^{-i(\gamma_s d_s+\gamma_g d_g)} = \\ &\sum_{m,n=1}^{\infty} P(m,n)\int o(x,z)\sin(\bar{k}_x mx)\sin(\bar{k}_z nz)\}dxdz. \end{aligned} \tag{8}$$

From equation (8) we can see that the filtered spectrum of the measurement on the left side of the equation is related to a series of weighted multiresolution potential spectra on the right side. Taking the inverse Fourier transform to both sides of equation (8) we obtain

$$\begin{aligned} &FT^{-1}\{u^{sc}(k_s,k_g)4\gamma_s\gamma_g e^{-i(\gamma_s d_s+\gamma_g d_g)}|J(\bar{k};k_s,k_g)|\} = \\ &\sum_{n=1}^{\infty} \frac{P(m,n)}{-4}\{o(\frac{x}{m},\frac{z}{n}) - o(\frac{-x}{m},\frac{z}{n}) - o(\frac{x}{m},\frac{-z}{n}) + o(\frac{-x}{m},\frac{-z}{n})\}, \end{aligned} \tag{9}$$

where $J(\bar{k};k_s,k_g)$ is the Jacobean transformation from $(\bar{k}_x,\bar{k}_z)$ to (k_s,k_g). Notice that the average value of the image is not computed correctly by the finite algorithm, since it will evaluate the spectrum at the origin as zero. The D.C. component of the image is restored by computing it directly form the data as $4u^{sc}(0,0)e^{-i\bar{k}L} / L \times H$.

In the case of a constant background medium, equation (9) would be the reconstructed scattering potential function. Now, after the inverse Fourier transform is applied to equation (9), instead of the potential function itself, a summation of multiscale components of the potential function is reconstructed. The role of the harmonic indexes m and n is that of the scale lengths in the Wavelet transforms. Consequently, the component $o(\frac{x}{m},\frac{z}{n})$ is an image with a specific scale. With large scale length, i.e., small m and n,

$o(\frac{x}{m},\frac{z}{n})$ provides a global view, while small scales, i.e., large m and n, $o(\frac{x}{m},\frac{z}{n})$ provide increasingly detailed views of smaller subsets of the image. The remaining problem is to invert equation (9), i.e., to combine those multiscale images into a complete image.

MULTISCALE INVERSION VIA MOBIUS TRANSFORM

We want to invert the scattering potential function $o(x,z)$ using equation (9), which can be rewritten as

$$d(x,z) = \frac{-1}{4}\sum_{m,n=1}^{\infty}\frac{P(m,n)}{mn}o(\frac{x}{m},\frac{z}{n}), \tag{10}$$

where $d(x,z) = FT^{-1}\{u^{sc}(k_s,k_g)4\gamma_s\gamma_g e^{-i(\gamma_s d_s+\gamma_g d_g)}|J(\bar{k};k_s,k_g)|\}$. Losing no generality, we have assumed $o(x,z) = -o(-x,z) = -o(x,-z) = o(-x,-z)$ in equation (9). According to Mobius inversion theorem (Hardy, 1979, Chen, N. 1989), if

$$F(x) = \sum_{n=1}^{\infty} f(\frac{x}{n}) \tag{11}$$

then

$$f(x) = \sum_{n=1}^{\infty} \mu(n)F(\frac{x}{n}), \tag{12}$$

where Mobius function

$$\mu(n) = \begin{cases} 1 & \text{n = 1} \\ (-1)^r & \text{n include r distinct prime factors} \\ 0 & \text{otherwise.} \end{cases}$$

Applying Mobius transform (12) to equation (10) we obtain the complete image of the scattering potential

$$o(x,z) = \frac{-4}{P(1,1)}\sum_{m,n=1}^{\infty}\frac{\mu(m)\mu(n)}{mn}d(\frac{x}{m},\frac{z}{n}). \tag{13}$$

Note that $\left| d(\frac{x}{m},\frac{z}{n})/mn \right| \le \frac{const.}{mn}$. Therefore, the series (13) is absolutely convergent. Equation (13) can be explicitly written as

$$o(x,z) = \frac{-4}{P(1,1)} \sum_{m,n=1}^{\infty} \frac{\mu(m)\mu(n)}{mn} \times \int u^{sc}(k_s,k_g) 4\gamma_s\gamma_g e^{-i(\gamma_s d_s + \gamma_g d_g)} |J(\bar{k};k_s,k_g)| e^{i(\bar{k}_x \frac{x}{m} + \bar{k}_z \frac{z}{n})} dk_s dk_g \tag{14}$$

which states that the potential function is reconstructed by summing a series of multiscale inverse Fourier transforms of the filtered spectra of the measurement.

It is convenient to use $u^{sc}(k_s,k_g,\omega/n)$ instead of $u^{sc}(nk_s,nk_g,\omega)$ in the computation. It can be show that $u^{sc}(k_s,k_g,\omega/n) = u^{sc}(nk_s,nk_g,\omega)$. Thus,

$$o(x,z) = \frac{-4}{P(1,1)} \sum_{m,n=1}^{\infty} \frac{\mu(m)\mu(n)}{mn} \times n\int u^{sc}(nk_s,nk_g,\omega) 4\hat{\gamma}_s\hat{\gamma}_g e^{-i(\hat{\gamma}_s d_s + \hat{\gamma}_g d_g)} |J(\bar{k};nk_s,nk_g)| e^{i(\bar{k}_x \frac{x}{m} + \bar{k}_z z)} dk_s dk_g \,, \tag{15}$$

and

$$o(x,z) = \frac{-4}{P(1,1)} \sum_{m,n=1}^{\infty} \frac{\mu(m)\mu(n)}{mn} \times n\int u^{sc}(k_s,k_g,\frac{\omega}{n}) 4\hat{\gamma}_s\hat{\gamma}_g e^{-i(\hat{\gamma}_s d_s + \hat{\gamma}_g d_g)} |J(\bar{k};k_s,k_g)| e^{i(\bar{k}_x \frac{x}{m} + \bar{k}_z z)} dk_s dk_g \,. \tag{16}$$

CONCLUSIONS

We have presented a formulation of diffraction tomography for variable background medium which relates the filtered spectrum of the measurement to multiscale spectra of the scattering potential resulting from the Fourier expansion of the spatially variant phase function. The potential function is recovered from multiscale components via Mobius inversion. As well as the applicability to strongly non-uniform medium, the method can be easily implemented and is computationally efficient, since the algorithm is similar to what is used in a constant background medium.

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