PAPER R

WAVEFORM TOMOGRAPHY FOR TWO PARAMETERS IN ELASTIC MEDIA

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ABSTRACT

In this paper, we developed one waveform tomography method for P velocity and density inversion in elastic media. Starting from the elastic wave equation, we derived one P wave equation which includes the scattered terms of P-wave to P-wave conversion and S-wave to P-wave conversion. By discretizing the scattering integral equation corresponding to this equation directly, we can obtain one equation corresponding to each point on the waveform. The result of this formulation is a very large system of algebraic equations that is solved using ART and SIRT. The results of computational simulation applied to cross-hole geometry show that our method is valid when the velocity and density perturbation is not very large. Next we will extend this method to heterogeneous media and apply it to real field data.

INTRODUCTION

Recently, the wave equation tomography methods based on acoustical wave equation have been applied to the real cross-hole data (Harris and Wang,1993). Therefore, these methods become more attractive in the inversion field. But, there are many complicated wave events contained in seismogram data, e.g., S wave, mode converted wave, Rayleigh waves, etc. , which can not be described by an acoustical equation. However, each of the different events provides useful information about the subsurface compressional and shear wave velocities and all these events can be modeled well by the elastic wave equation. Therefore, we should develop the inversion methods based on elastic wave equation to obtain more physical parameters of the media. In elastic wave equation inversion studies, one method is to find the parameters (P- and S-wave velocities and density) that minimize the square error between the wavefield computed using this model and the observed wavefield (Tarantola, 1984; Mora, 1986). But in this method, the seismogram must be calculated by solving the 2D elastic wave equation numerically in each iteration, therefore, the computation of this method is very costly. In order to simplify the inverse problem, some researchers considered a one dimensional inverse problems (Norton and Testard, 1988) and obtained a better inversion solution. However, the media of the earth usually are two dimensional, and in this case, this method is unvalid. Additionally, some only used the SH wave in their inversion method (Hooshyer and Weglein, 1986). In this case, the equation for inversion can be reduced to a scalar equation, but the SH wave is independent of P wave and SV wave, and therefore, the SH wave can only invert shear modulas and density. In order to obtain Lamé parameters and density in two dimensional elastic media effectively, we should develop the inversion methods based on the P wave equation.

In this paper, starting from the 2D isotropic inhomogeneous elastic wave equation, the scattering theory in weak inhomogenous media is studied. We also establish a scalar P wave equation which includes the scattered terms of P-wave to P-wave conversion and S-wave to P-wave conversion. From this equation, a scattering integral equation in the frequency domain is derived. Under the Born and geometrical optical approximation, we show how the time-domain scattered fields can be related to density and velocity perturbations through a generalized Radon transform. Therefore, time-domain P-wave scattered fields can be used as projection data to relate each point in the scattered waveform to the elastic parameters. The result of this formulation is a very large system of algebraic equations that is solved using ART and SIRT. The algorithm is tested on numerically simulated data generated for the cross-hole geometry of sources and receivers. The test results illustrate the ease of implementation and robustness of the method.

THEORY OF P WAVE SCATTERING IN WEAK INHOMOGENEOUS MEDIA

From 2D isotropic inhomogeneous elastic wave equation, we have

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = c_p^2 \nabla(\nabla \cdot \mathbf{u}) - c_s^2 \nabla \times (\nabla \times \mathbf{u}) + \frac{1}{\rho}[(\nabla\lambda)\nabla \cdot \mathbf{u} + (\nabla\mu)\cdot(\nabla\mathbf{u}) + (\nabla\mathbf{u})\cdot(\nabla\mu)] \quad (1)$$

where **u=u(x**,t), it is displacement field, c_p, c_s are velocity of P wave and S wave, μ, λ, ρ are shear modula, Lame parameter and density of elastic media.

Assuming that the background is homogeneous and its elastic parameters are $c_{p0}^2, c_{s0}^2, \lambda_0, \mu_0, \rho_0$, and corresponding perturbation parameters due to heterogeneities are δc_p^2, δc_s^2, $\delta\lambda$, $\delta\mu$, and $\delta\rho$, we have

$$c_p^2 = c_{p0}^2 + \delta c_p^2 \,, \qquad c_s^2 = c_{s0}^2 + \delta c_s^2 \,,$$

$$\lambda = \lambda_0 + \delta\lambda \,, \qquad \mu = \mu_0 + \delta\mu \,, \tag{2}$$

$$\text{and } \rho = \rho_0 + \delta\rho .$$

Putting equation(2) into equation (1), we have

$$\frac{\partial^2 \mathbf{u}}{\partial t^2} = c_{p0}^2 \nabla(\nabla \cdot \mathbf{u}) - c_{s0}^2 \nabla \times (\nabla \times \mathbf{u}) + \mathbf{B}, \tag{3}$$

where

$$\mathbf{B} = \delta c_p^2 \nabla(\nabla \cdot \mathbf{u}) - \delta c_s^2 \nabla \times (\nabla \times \mathbf{u}) + \frac{1}{\rho_0}[(\nabla \delta\lambda)(\nabla \mathbf{u}) + (\nabla \delta\mu) \cdot (\nabla \mathbf{u}) + (\nabla \mathbf{u}) \cdot (\nabla \delta\mu)] \,. \tag{4}$$

By defining

$$\Phi = \nabla \cdot \mathbf{u}(\mathbf{x}, \mathrm{t}) \tag{5}$$

and taking divergence of both sides of equation (3), we have

$$\frac{\partial^2 \Phi}{\partial t^2} - c_{p0}^2 \nabla^2 \Phi = \nabla \cdot \mathbf{B} \tag{6}$$

where

$$\nabla \cdot \mathbf{B} = \nabla \cdot [(2\delta c_p^2 + \frac{\delta\rho}{\rho_0} c_{p0}^2)] \nabla \Phi - (\delta c_p^2 + \frac{\delta\rho}{\rho_0} c_{p0}^2) \nabla^2 \Phi$$

$$-\nabla \cdot [(2\delta c_s^2 + c_{s0}^2 \frac{\delta\rho}{\rho_0}) \nabla \times (\nabla \times \mathbf{u})] \tag{7}$$

in which the second derivative of $\delta\lambda, \delta\mu$ and $\delta\rho$ with respect to space position are omitted due to the weak inhomogeneous media assumption.

By taking the Fourier transform over time t on both sides of equation (7), we derive the equation in frequency domain,

$$\nabla^2\tilde{\Phi} + k_{p0}^2\tilde{\Phi} = -\frac{1}{c_{p0}^2}\nabla\cdot\tilde{\mathbf{B}}, \tag{8}$$

where $\tilde{\Phi}$ and $\tilde{\mathbf{B}}$ are the Fourier transform of Φ and $\mathbf{B}$ with respect to time t.

By defining

$$b_1(\mathbf{x}) = 2\delta c_p^2 / c_{p0}^2 + \delta\rho / \rho_0 , \tag{9}$$

$$b_2(x) = \delta c_p^2 / c_{p0}^2 + \delta\rho / \rho_0 , \tag{10}$$

$$b_3(x) = 2\delta c_s^2 / c_{p0}^2 + c_{s0}^2 / c_{p0}^2 \frac{\delta\rho}{\rho_0}, \tag{11}$$

,

therefore, equation (8) can be recast as

$$\nabla^2\tilde{\Phi} + k_{p0}^2\tilde{\Phi} = -\nabla\cdot[b_1(\mathbf{x})\nabla\tilde{\Phi}] + b_2(x)\nabla^2\tilde{\Phi} + \nabla\cdot[b_3(x)\nabla\times(\nabla\times\mathbf{u})] \tag{12}$$

In the right hand side of above equation, there are three terms, the first two terms belong to the scattering sources of P to P wave, the third term comes from the S wave to P wave conversion due to the inhomogeneous perturbation in elastic media.

Let

$$\tilde{\Phi} = \tilde{\Phi}_{in} + \tilde{\Phi}_{sc} , \tag{13}$$

where $\tilde{\Phi}_{in}$ is the incident field in the background media, which satisfies

$$\nabla^2\tilde{\Phi}_{in} + k_{p0}^2\tilde{\Phi}_{in} = s(\omega)\delta(\mathbf{x}-\mathbf{x}_s) \ . \tag{14}$$

In this equation $s(\omega)$ is source function, $\mathbf{x}_s$ is the position of the source, and the scattering field $\tilde{\Phi}_{sc}$ can be expressed as

$$\begin{aligned}\tilde{\Phi}_{sc}(\mathbf{x},k_{p0}) = -S(\omega)\cdot\int_\Omega d\mathbf{x}'\{\nabla\cdot(b_1(\mathbf{x}')\nabla\tilde{\Phi}) - b_2(\mathbf{x}')\nabla^2\tilde{\Phi} \\ -\nabla\cdot[b_3(\mathbf{x}')\nabla\times(\nabla\times\mathbf{u})]\}\mathrm{G}(\mathbf{x},\mathbf{x}',\mathrm{k}_{p0})\end{aligned} \tag{15}$$

where $\mathrm{G}(\mathbf{x},\mathbf{x}',\mathrm{k}_{p0})$ is the Green function, which satisfies

$$\nabla^2 G_0 + k_{p0}^2 G_0 = \delta(\mathbf{x} - \mathbf{x}') \tag{16}$$

From equation (15), we know that the scattering fields are produced by three perturbation parameters. Next we will develop one waveform tomography method to invert for parameters $b_1(\mathbf{x})$ and $b_2(x)$. For $b_3(x)$ inversion, the S wave scattering problem should be consider, it will be studied in the future.

WAVEFORM TOMOGRAPHY METHOD FOR P VELOCITY AND DENSITY

In our imaging method, the P wave point source is used to illuminate the object region. Then, applying the Born approximation to equation (15), the total field $\tilde{\Phi}$ and $\mathbf{u}$ in the integral equation (15) can be replaced by incident field $\tilde{\Phi}_{in}$ and $\mathbf{u}_{in}$, where $\mathbf{u}_{in}$ is displacement field of incident P wave. Therefore, $\nabla \times \mathbf{u}$=0 in equation (15), and the third term is omitted. Notice that $\tilde{\Phi}_{in}$ satisfies equation (14),. Therefore, the second term in equation (15) can be simplified as

$$-\int dx' b_2(x')\nabla^2 \tilde{\Phi}_{in} G(\mathbf{x},\mathbf{x}',k_{p0}^2) = k_{p0}^2 \int dx' b_2(x')\tilde{\Phi}_{in} G(\mathbf{x},\mathbf{x}',k_{p0}^2)\ . \tag{17}$$

Under the Born approximation, equation (15) can be expressed as

$$\tilde{\Phi}_{sc}(\mathbf{x},k_{p0}) = -S(\omega)\cdot \int_{\Omega} d\mathbf{x}'\{\nabla\cdot(b_1(\mathbf{x}')\nabla\tilde{\Phi}_{in}) + k_{p0}^2 b_2(\mathbf{x}')\tilde{\Phi}_{in}\} G(\mathbf{x},\mathbf{x}',k_{p0}^2) \tag{18}$$

Applying the geometrical optical approximation, we have

$$\tilde{\Phi}_{sc}(\mathbf{x},k_{p0}) = -S(\omega)\cdot \int_{\Omega} d\mathbf{x}'\{\nabla\cdot(b_1(\mathbf{x}')\nabla A(\mathbf{x}',\mathbf{x}_s)e^{-i\omega T(\mathbf{x}',\mathbf{x}_s)})$$

$$+k_{p0}^2 b_2(\mathbf{x}')A(\mathbf{x}',\mathbf{x}_s)e^{-i\omega T(\mathbf{x}',\mathbf{x}_s)}\}\cdot A(\mathbf{x},\mathbf{x}')e^{-i\omega T(\mathbf{x},\mathbf{x}')}\cdot\frac{c_0}{i\omega}\ , \tag{19}$$

where the amplitude A and travel time T satisfy the transport equation and eikonal equation, respectively.

By using Green's theorem in the plane and assuming that b_1 and b_2 =0 on the boundary of the imaging region, we have

$$\tilde{\Phi}_{sc}(\mathbf{x},k_{p0}) = i\omega S(\omega)\int_{\Omega} d\mathbf{x}' a(\mathbf{x},\mathbf{x}',\mathbf{x}_s)(b_1(\mathbf{x}')\cdot\cos(\theta)+b_2(\mathbf{x}'))e^{-i\omega\tau(\mathbf{x},\mathbf{x}',\mathbf{x}_s)} \tag{20}$$

where

$$a(\mathbf{x},\mathbf{x}',\mathbf{x}_s) = A(\mathbf{x}',\mathbf{x}_s)A(\mathbf{x},\mathbf{x}')/c_0, \tag{21}$$

$$\tau(\mathbf{x},\mathbf{x}',\mathbf{x}_s) = T(\mathbf{x}',\mathbf{x}_s)+T(\mathbf{x},\mathbf{x}') . \tag{22}$$

and θ is the angle between $\nabla T(\mathbf{x}',\mathbf{x}_s)$ and $\nabla T(\mathbf{x},\mathbf{x}')$.

Taking the inverse Fourier transform of both sides of equation (22), we obtain the scattering field in time domain as

$$\Phi_{sc}(\mathbf{x},t) = W(t) * \int_{\Omega} d\mathbf{x}' a(\mathbf{x},\mathbf{x}',\mathbf{x}_s)(b_1(\mathbf{x}')\cdot\cos(\theta)+b_2(\mathbf{x}'))\delta(t-\tau(\mathbf{x},\mathbf{x}',\mathbf{x}_s))$$

$$= W(t) * \int_{I(\tau)} a(\mathbf{x},\mathbf{x}',\mathbf{x}_s)(b_1(\mathbf{x})\cos(\theta)+b_2(\mathbf{x}))ds \tag{23}$$

where $W(t)=S'(t)$, convolution is denoted by *. The above equation is called a generalized Radon transform. I(τ) is an isochronic plane with a fixed time τ, from which the perturbation parameters can be related to the waveform. For each point on the waveform, we can derive one equation for inversion. For each pair of source and receiver $(\mathbf{x},\mathbf{x}_s)$, all the isochronic lines can cover the object region. when the spatial position of the source or receiver is moved, the direction of set of isochronic lines belonging to the same pair of source and receiver $(\mathbf{x},\mathbf{x}_s)$ is changed. Therefore, the projection in different direction can be derived, and we can use the waveform as a projection to invert for the elastic parameters.

Considering k^{th} isochronic plane $I(t_k)$, equation (23) can be discetized as

$$d_k = W_k * \sum_m a(l_{km})\cdot(b_2(l_{km})\cos(\theta_{km})+b_1(l_{km}))\Delta s \qquad (1\le k\le K, 1\le m\le M) \tag{24}$$

where $d_k = \Phi_{sc}(\mathbf{x},\mathbf{x}_s,t_k)$, $W_k = W(t_k)$, $K = L\times S\times R$, L, S and R are the total number of sampling points on the waveform, source and receiver, l_{km} is the length of k^{th} isochronic line from the first point to m^{th}point , M is the total numbers of the integral step along m^{th} isochronic line, $a(l_{km})$, $b_1(l_{km})$ and $b_2(l_{km})$ are the values of $a(\mathbf{x},\mathbf{x}',\mathbf{x}_s)$, $b_1(\mathbf{x}')$ and $b_2(\mathbf{x}')$ at l_{km}, θ_{km} is the value of θ at point l_{km}, and Δs is the integral step. The object region is divided into $I\times J$ pixels. When the coordinate of l_{km} satisfies,

$i\Delta x \le x \le (i+1)\Delta x$, $i\Delta z \le z \le (i+1)\Delta z$, where Δx and Δz are the width of pixel in x and z direction, $1 \le i \le I$, $1 \le j \le J$.

By defining

$$\rho_{nkm} = \begin{cases} 1 & when \quad n = (j-1)\times J + i \\ 0 & when \quad n \ne (j-1)\times J + i \end{cases} \tag{25}$$

we have

$$b_1(l_{km}) = \sum_n \rho_{nkm} \cdot b_{1n} \qquad (1 \le n \le I \times J) \tag{26}$$

$$b_2(l_{km}) = \sum_n \rho_{nkm} \cdot b_{2n} \qquad (1 \le n \le I \times J), \tag{27}$$

where b_{1n} and b_{2n} are the values of $b_1(\mathbf{x}')$ and $b_2(\mathbf{x}')$ at nth pixel. Puting equations (25)-(27) into equation (24), we have

$$d_k = W_k * \sum_n (c_{1kn} \cdot b_{1n} + c_{2kn} \cdot b_{2n}) \tag{28}$$

where

$$c_{1kn} = \sum_m a(l_{km}) \cdot \rho_{nkm} \cdot \Delta s \tag{29}$$

$$c_{2kn} = \sum_m a(l_{km}) \cdot \rho_{nkm} \cdot \cos(\theta_{km}) \cdot \Delta s \tag{30}$$

Let

$$g_{kn} = \begin{cases} c_{1kn} & when \quad 1 \le n \le I \times J \\ c_{2kn} & when \quad I \times J \le n \le 2 \times I \times J \end{cases} \tag{31}$$

$$f_{kn} = \begin{cases} b_{1n} & when \quad 1 \le n \le I \times J \\ b_{2n} & when \quad I \times J \le n \le 2 \times I \times J \end{cases} \tag{32}$$

Then equation (28) can be written as

$$d_k = W_k * \sum_n^{2IJ} g_{kn} \cdot f_n \tag{33}$$

Let

$$h_{kn} = \sum W_{k-l} g_{ln} \qquad (1 \le l \le L) \tag{34}$$

then equation (33) can be written as

$$d_k = \sum_{n}^{2IJ} h_{kn} \cdot f_n \tag{35}$$

From equation (35), we can invert for P velocity and density using the waveform as the projection. In order to solve equation (35) effectively, the algorithms ART and SIRT are used to solve equation (35) to get the fast execution speed of the inversion. Then we have the following ART and SIRT iterative form:

$$f_n^{(q+1)} = f_n^{(q)} + \alpha \frac{h_{kn}}{\sum_n h_{kn}^2} (d_k - \sum_n h_{kn} \cdot f_n) \tag{36}$$

$$f_n^{(q+1)} = f_n^{(q)} + \beta \sum_k \frac{h_{kn}}{\sum_n h_{kn}^2} (d_k - \sum_n h_{kn} \cdot f_n) / z_n \tag{37}$$

where α, β are the weight factors, z_n is the non-zero numbers of h_{kn} $(1 \le k \le L \cdot S \cdot R)$.

By the above methods, the waveform can be used as the projection to invert for the parameters $b_1(x)$ and $b_2(x)$, and the P velocity and density can be obtained from them.

COMPUTATION SIMULATION

Now we apply the above method to the cross-well imaging system. At first, three sources are located on the surface to illuminate the imaging region, and for each source, 20 receivers located in the left well at x=-50m and 20 receivers located in right well at x=50m are used to receive the signal, respectively, Then 10 sources located in left well are used to illuminate the imaging region, and 20 receivers located in right well are used to receive the signal. The depth of the well is 200m. The source function we used is a ricker wavelet, the center frequency is 200 Hz. The imaging region is divided into 20 by 20 pixels, where he width of each pixel is 3 meters. Figure 1 and figure 2 are the P velocity and density perturbation models for the reconstruction test. The parameters of the background are: c_{p0} = 2500 m/s, ρ_0=2.3 g/cm^3 .The perturbation of P velocity and density are 10% with respect to the background values. Figure 3 and figure 4 are the inverted results of P reconstruction and density by SIRT. Figure 5 and figure 6 are the reconstruction results of P velocity and density by ART. From figure 4 and figure 6, we can see the vertical resolution is not high in this imaging geometry. This is because there are not enough projections in the vertical direction. From the reconstruction results, we can see that the tomography results derived by SIRT are better than that by ART.

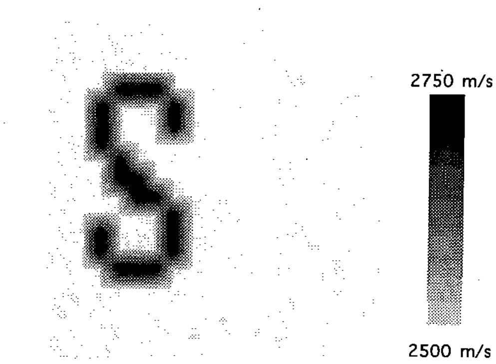

Figure 1. The synthetic P wave velocity model

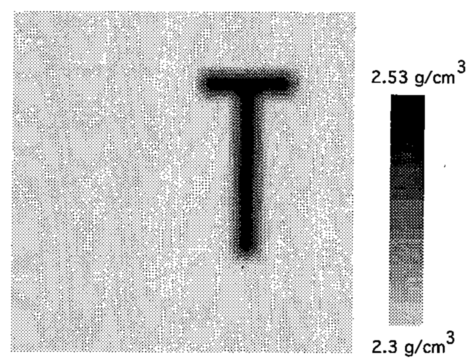

Figure 2. The synthetic density model

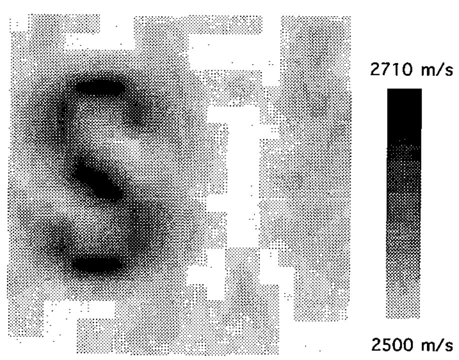

Figure 3. P velocity reconstructed result by SIRT method

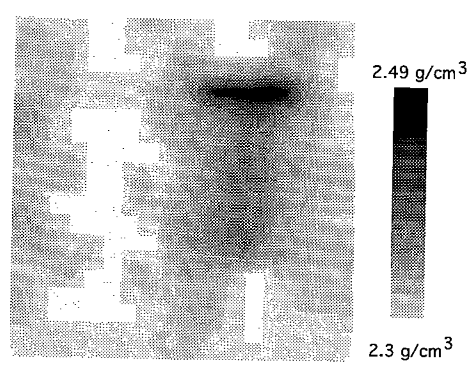

Figure 4. Density reconstructed result by SIRT

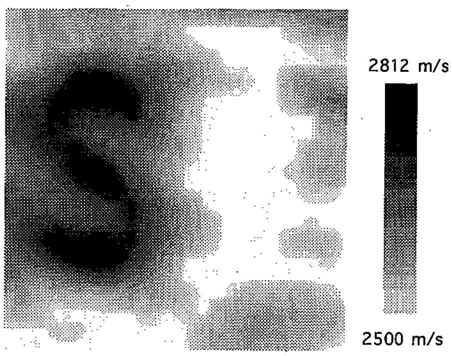

Figure 5. P velocity reconstructed result by ART method

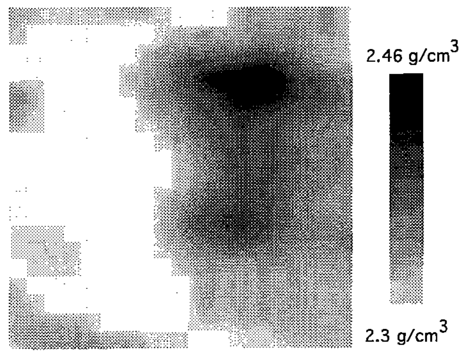

Figure 6. Density reconstructed result by ART

CONCLUSIONS

A new waveform tomography for P velocity and density in elastic media is put forward in this paper. Although the elastic wave equation inversion is a very complicated mathematics problem; this problem can be simplified immensely. Under the Born and geometrical optics approximation when starting from P wave equation, the waveform data can be related to the perturbation of P velocity and density by a generalized Radon transform. For each point on the waveform, discretizing this Radon transform can lead to one equation for these two parameters. Then back projection methods ART and SIRT are used to solve such a huge system. The numerical results show that this method is very robust to reconstruct P velocity and density. Although the Born approximation in real data application is limited, the main structure in the media can be estimated by use of it.

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