PAPER P

NONLINEAR MULTI-FREQUENCY WAVE EQUATION INVERSION

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ABSTRACT

A method of nonlinear wave equation inversion using multiple frequencies is developed. The method sequentially inverts low to high signal frequencies of the wave field. First, the low frequencies are used to reconstruct the low spatial wavenumbers of the heterogeneities. Then, higher frequencies are used to reconstruct higher spatial wavenumbers. A subspace method is used to minimize the misfit between the observed *total* wave field and the corresponding theoretical synthetic wave field. The gradient of the misfit function, the rate of change of the gradient, and a regularization factor are used to find the optimum search direction to the solution of inverse problem. It is not necessary to assume weak scattering, e.g., Born or Rytov, nor is it necessary to remove the incident field before inversion as required in linear diffraction tomography. These advantages make this method more flexible than linear diffraction tomography. Tests on synthetic data show that the method is effective in reconstructing velocities with up to 15% variation and that the inversion is stable and more precise than single frequency wave equation methods. These advantages are available with not much penalty of computation effort or time.

INTRODUCTION

Transmission travel time methods based on ray equation are the most often used methods in seismic inversion, because they are robust in field data applications (Dines and Lytle 1979; Wong. Hurley and West, 1983). Yet these methods can only give the low wavenumber components of the heterogeneous media. In order to derive higher resolution, some aspects of the full wave field, e.g., reflections, must be used.

In wave equation tomography, there are two types of methods: one is analytic inversion method, another is non-linear iterative method. In the analytic method, the Born or Rytov weak scattering approximation is made in order to derive the analytic inversion formula.

The perturbation of velocity to be inverted cannot be very large (Devaney, 1984, Harris, 1987, Wu, et al, 1987). Non-linear iterative wave equation tomography methods are better for large contract velocity (Torantola, 1987, Pratt, 1990, Yin, et al, 1993). Nonlinear methods have three main parts: (1) the forward calculation, (2) calculation of the Frechét derivative, and (3) the method of updating the model. In addition, we can implement the inversion in time domain or in frequency domain. In time domain, Torantola (1984, 1987, Mora, 1987) applied a back-propagation method to the difference field computed between the observed waveform and calculated waveform to invert for the media elastic parameters. Since the full waveform is used in this method, reconstruction is improved, but it requires a large amount of memory and computer time to implement

Inversions using the full waveform can also be implemented in frequency domain also. We can select different frequencies according to the resolution requirement. We expect the low frequencies to provide the low spatial wavenumber components of the media, and the high frequencies to provide the high wavenumbers. We can invert for a broad range of velocity scales from multi-frequency wave fields, thereby creating an inversion that is stable over the entire range while raising the resolution of the reconstruction step by step with frequency. In addition, scattering is strongest for heterogeneities comparable in size to the wavelength; therefore, each frequency of the multi-frequency decomposition is maximally sensitive to different scales of the heterogeneity spectrum.

In this paper, we describe a nonlinear, multi-frequency, and multi-grid wave equation inversion method based on the 2-D acoustical wave equation. In our method, the low frequency components in the waveform are first used to invert for the low wavenumber components of the media. A bi-linear interpolation method is used to interpolate the low frequency reconstruction to smaller cells or a finer grid. Then, higher frequency components of the waveform are used to invert the velocity on the finer grid, and so on until the velocities at the smallest grid are reconstructed.

When solving the nonlinear inverse problem, our goal is to reduce the value of the misfit function which describes the mismatch between the observed total wave field data and the corresponding synthetically calculated data below a threshold determined by the errors in the observations. We use the moment method to calculate the synthetic fields. In this way, our method does not require the removal of the incident wave field before inversion. This is important in field applications for empirically we do not know the incident field. The objective function for inversion is quadratic, thus its first and second derivatives are required to minimize it. In this work, we use the subspace method (Skilling, 1984; Kennet and Williamson, 1987) to minimize the quadratic misfit function involving second derivatives to the model and to avoid the inversion of large matrices. At each step in the iteration we use a local quadratic approximation to the misfit function and three directions are used to find a path towards the minimum. These three directions are the gradient of the misfit function, a regularization term, and the direction of the rate of change of the gradient. To prevent unreasonable behavior of the model parameters, some form of regularization condition on the model is imposed. We applied our method to the crosswell geometry to conduct a forward and inverse simulation. The results show that our multi-frequency and multi-grid method gives a better reconstruction for velocity than the single frequency method without much penalty in computation time.

THE ACOUSTICAL SCATTERING SOLUTION

In order to implement the nonlinear wave equation inversion, we should have a good and robust method of forward modeling for waves in inhomogeneous media. Starting from the wave equation in the frequency domain,

$$\nabla^2 U(\mathbf{r},\omega)+\frac{\omega^2}{V^2}U(\mathbf{r},\omega)=-S(\omega)\delta(\mathbf{r}-\mathbf{r}_s) \quad (1)$$

where **r**=(x, z) is a spatial position in the imaging region, $\mathbf{r}_s=(x_s,z_s)$ is the source position, ω is an angular frequency and $U(r,\omega)$ is the pressure field in the imaging region, and V is the velocity field. Taking $U(\mathbf{r},\omega)=U^{in}(\mathbf{r},\omega)+U^{sc}(\mathbf{r},\omega)$, $V^{-2}(\mathbf{r})=V_0^{-2}(\mathbf{r})-m(\mathbf{r})V_0^{-2}(\mathbf{r})$, where $V_0(\mathbf{r})$ is the velocity of the background, we derive an integral equation corresponding to Eqn. (1) as follows:

$$U(\mathbf{r},\omega)=U^{in}-\int_{\Omega}k_0^2(\mathbf{r}')\mathrm{U}(\mathbf{r},\omega)m(\mathbf{r})\mathrm{G}(\mathbf{r}_g,\mathbf{r}')d\mathbf{r}' \quad (2)$$

where Ω is the image area, $\mathbf{r}_g=(x_g,y_g)$ and $k_0=\omega/V_0(\mathbf{r})$. We divide the image region into N pixels, then, $\Omega=\Omega_1\cap\Omega_2\cap\cdots\cap\Omega_N(\Omega_i\cup\Omega_j=0,i=j)$. Assuming m_i is the value of m(**r**) at pixel Ω_i. By the moment method, Eqn. (2) can be discretized to give

$$\mathbf{F}\cdot\mathbf{a}=\mathbf{b} \quad (3)$$

where

$$\mathrm{F}_{ij}=\delta_{ij}+\iint_{\Omega_i}k_0^2G(|\mathbf{r}-\mathbf{r}'|,n_b)m(\mathbf{r}')d\mathbf{r}' \quad (4)$$

and **a** is the unknown column vector whose elements are the discretized total field U_i, **b** is a vector whose elements are the discretized value of the incident field U_i^{in} in the imaging region. The solution of this equation solves the forward modeling problem. By the same method, we can also obtain the Green's function $G(\mathbf{r}_g,\mathbf{r})$ by putting a point source at the receiver position $\mathbf{r}_g$ and applying the reciprocity principle. Because $U(\mathbf{r})$ is dependent on $m(\mathbf{r})$, Eqn. (2) is a non-linear system in $m(\mathbf{r})$. In order to derive an analytic inversion solution, various linearized methods based on the Born approximation have been derived (Wu and Toksoz, 1987, Harris, 1987, and Harris and Wang, 1993). Next, we will solve the inverse problem by using a non-linear iterative method.

NON-LINEAR ITERATIVE INVERSION

Suppose that the observed wave field is $U^0(\mathbf{r}_s,\mathbf{r}_s,\omega_l)$ $(1 \le s \le S, 1 \le g \le G, 1 \le l \le L)$. We wish to use these observed data to determine a model M=$(m_1,m_2,\cdots,m_N)$ from which the corresponding calculated wave field matches the observed wave field. Therefore, we should have a measure of fit, that is, we should establish a measure for assessing the degree of mismatch between the observed data and synthetic data. We choose the squared L2 norm of the observed and calculated data, that is, our goal is to minimize the function

$$J(M) = \tfrac{1}{2}\sum_{s}\sum_{g}\sum_{l}\|U^o(r_s,r_g,\omega_l) - U^c(r_s,r_g,\omega_l)\|^2 = \text{minimum} \qquad (5)$$

where U^o and U^c are measured field data and calculated field data, respectively. The wave equation inverse problem in crosswell geometry is not only nonlinear, but extremely non unique, that is, ill-posed. In order to derive a stable solution of the inverse problem and prevent unrealistic behavior of the model parameters, we impose some form of regularization condition to Eqn. (5) as follows:

$$Q(\mathbf{M}) = \tfrac{1}{2}\sum_{s}^{S}\sum_{g}^{G}\sum_{l}^{L}\|U^o(r_s,r_g,\omega_l) - U^c(r_s,r_g,\omega_l)\|^2 + \lambda H(\mathbf{M}) = \text{minimum} \qquad (6)$$

where H(M) is the regularization term, e.g., a smoothing operator. Then, we use steepest gradient method, conjugate gradients. Eqn. (6) is quadratic function, therefore, not only the first derivative, but also the second derivative are required. A subspace method (Kennet, etc., 1987, Skilling, et al, 1987) in which the Hessian matrix is used will be utilized to solve equation (6) in our paper.

In the subspace method, the current model is updated by a small model perturbation which is determined in a small subspace, that is

$$\mathbf{M}^{(q+1)} = \mathbf{M}^{(q)} + \Delta \mathbf{M}^{(q)} \tag{7}$$

$$\Delta \mathbf{M}^{(q)} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 \tag{8}$$

where $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are the basis of the subspace, and they are

$$(\mathbf{e}_1)_i = \frac{\partial J}{\partial m_i},\ (\mathbf{e}_2)_i = \frac{\partial H}{\partial m_i},\ (\mathbf{e}_3)_i = \sum_k \frac{\partial^2 J}{\partial m_i \partial m_k}\frac{\partial Q}{\partial m_k} \tag{9}$$

and x_1, x_2 and x_3 can be determined by the subspace method (Skilling,1984, Kennet, et al, 1987).

We can see that one of main points of the inversion step is the calculation of the Frechét derivative and the second derivative of the quadratic function. After the wave fields are calculated in the current background media, the total wave field at receiver array can be expressed as

$$U^c(\mathbf{r}_s,\mathbf{r}_g,\omega) = U^{in}(\mathbf{r}_s,\mathbf{r}_g,\omega) + \sum_j A_{ji} m_i \ \ (j = 1,\ldots, G\cdot S) \tag{10}$$

where

$$A_{ji} = -\iint_{\Omega_i} k_0^2 U^c(\mathbf{r}_s,\mathbf{r}',\omega) G(\mathbf{r}_g,\mathbf{r}',\omega) d\mathbf{r} \tag{11}$$

Therefore, the Frechét derivative for inversion and the second derivative of the objective function can be easily expressed as

$$\frac{\partial J}{\partial m_i} = \sum_l^{SGL} R(A_{li})\cdot\left[R(U_l^c) - R(U_l^o)\right] + I(A_{li})\cdot\left[I(U_l^c) - I(U_l^o)\right] \tag{12}$$

$$\frac{\partial^2 J}{\partial m_i \partial m_k} = \sum_l^{SGL} [R(A_{li})\cdot R(A_{lk}) + I(A_{li})\cdot R(A_{lk})] \tag{13}$$

where R and I means taking real part and imaginary part. S, G, and L are the total number of the discrete sources, receivers, and frequencies.

Because analytical formulas for the Frechét derivative can be obtained for our numerical method, we use them to invert the model by non-linear iterative methods of Eqns. (7) - (8). In addition, when we reconstruct the velocity, three overlapping grids are used to implement the reconstruction. The velocities at the largest grid spacing are reconstructed first, then the velocities at the middle grid, and finally at the smallest grid. Therefore, the following problem should be solved by subspace method:

$$Q(\mathbf{M}_1) = \tfrac{1}{2}\sum_{s}^{S}\sum_{g}^{G}\sum_{l}^{L_1} \| U^o(r_s, r_g, \omega_l) - U^c(r_s, r_g, \omega_l) \|_2 + \lambda H(\mathbf{M}_1) = \text{minimum} \quad (14)$$

$$Q(\mathbf{M}_2) = \tfrac{1}{2}\sum_{s}^{S}\sum_{g}^{G}\sum_{l}^{L_2} \| U^o(r_s, r_g, \omega_l) - U^c(r_s, r_g, \omega_l) \|_2 + \lambda H(\mathbf{M}_2) = \text{minimum} \quad (15)$$

$$Q(\mathbf{M}_3) = \tfrac{1}{2}\sum_{s}^{S}\sum_{g}^{G}\sum_{l}^{L_2} \| U^o(r_s, r_g, \omega_l) - U^c(r_s, r_g, \omega_l) \|_2 + \lambda H(\mathbf{M}_3) = \text{minimum} \quad (16)$$

where $\mathbf{M}_1, \mathbf{M}_2$ and $\mathbf{M}_3$ are the model vectors for the first, the second grid and third grid, respectively. L_1, L_2 and L_3 are the numbers of frequencies used in the first, second and third grids. After the velocities on smallest grid are reconstructed, the inversion stops.

INVERSION WITH SYNTHETIC DATA

Fig. 1 is a model for the crosswell geometry. The velocities in each region is shown. The minimum velocity is 5000 m/s, the maximum 5750 m/s, and the range of the variation is 15%. In the forward calculation, the image region is divided into 20×50 pixels, where the width of each pixel is 2 m. We select six frequencies from 100 Hz to 225 Hz in steps of $\Delta f = 25$ Hz. The forward wave field is produced by Eqn. (10). In the inversion step, the image region is divided into three grids. The number of pixels in the first, second and third grid are 10×25, 14×35 and 20×50, respectively. Two frequencies of 100 Hz to 125 Hz were used for the first grid, four frequencies of 100 to 175 Hz for the second grid, and six frequencies of 100 Hz to 225 Hz for the third grid. Fig. 2 gives the reconstruction results using multiple frequencies and the multiple grids with the non-linear wave equation method. The three frequency bands used 12, 8, and 6 iterations, respectively. Fig. 3 is the reconstruction result using single frequency of 225 Hz for 20×50 pixels. Comparing Figs. 2 and 3, we can see that the velocity in each region of Fig. 2 is more homogeneous than that of Fig. 3, that is the multi-frequency results are more accurate with fewer artifacts.

CONCLUSIONS

In this paper we have presented a non-linear multi-frequency inversion method of wave equation tomography. Our goal is to reduce the value of the misfit function which describes the mismatch between the observed total wave field data and the corresponding calculated data below a threshold determined by the errors in the observations. This goal is implemented by a subspace method. This method is a powerful technique which does not have the limitations of linear diffraction tomography. It does not require the removal of the incident field before inversion. When multi-frequency wave equation is applied to crosswell imaging, the precision of reconstruction is better than the single frequency method and the time for inversion is not significantly increased.

ACKNOWLEDGMENTS

This work is supported by the Seismic Tomography Project of Stanford University, a research consortium sponsored by companies of the oil and gas industry.

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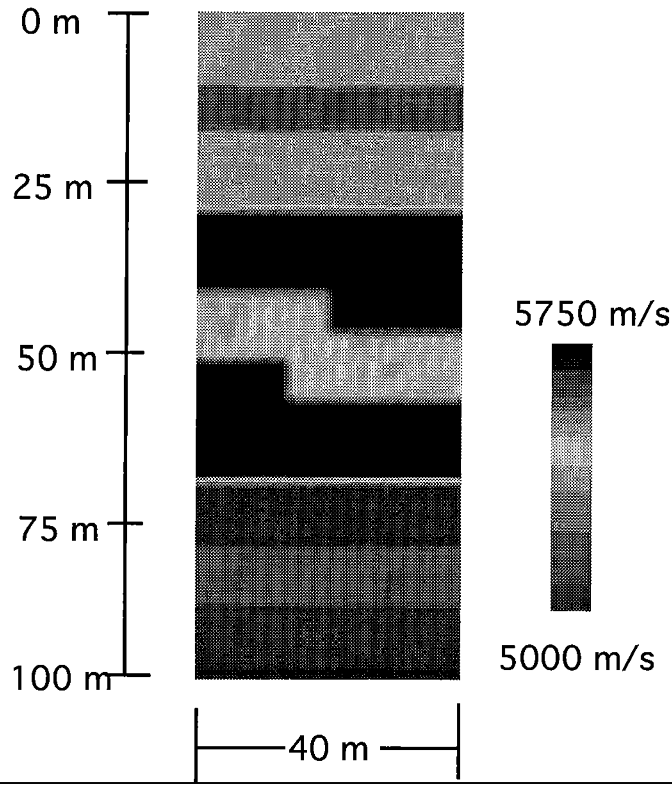

Fig. 1. The synthetic velocity model.

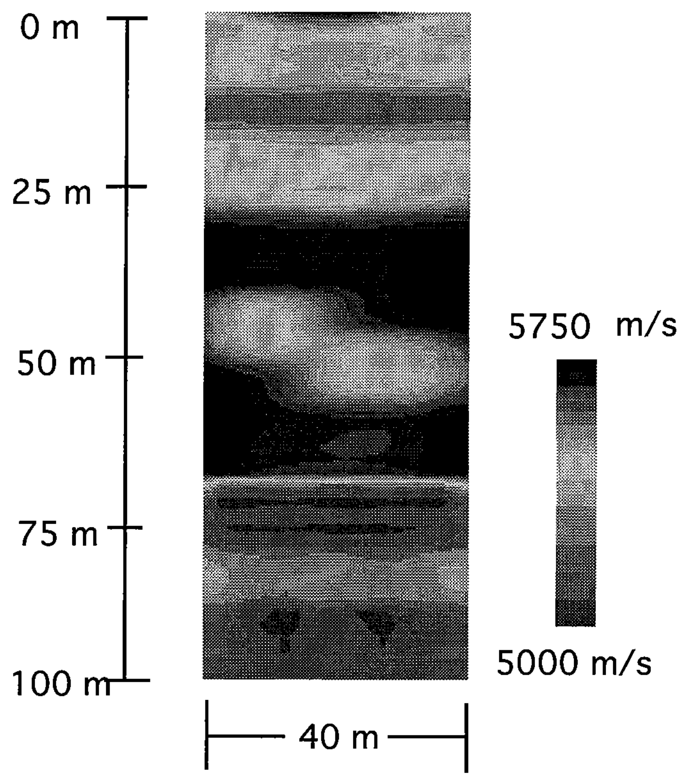

Fig. 2. Multiple frequency (six frequencies from 100- 225 Hz) inversion result derived by nonlinear wave equation inversion tomography.

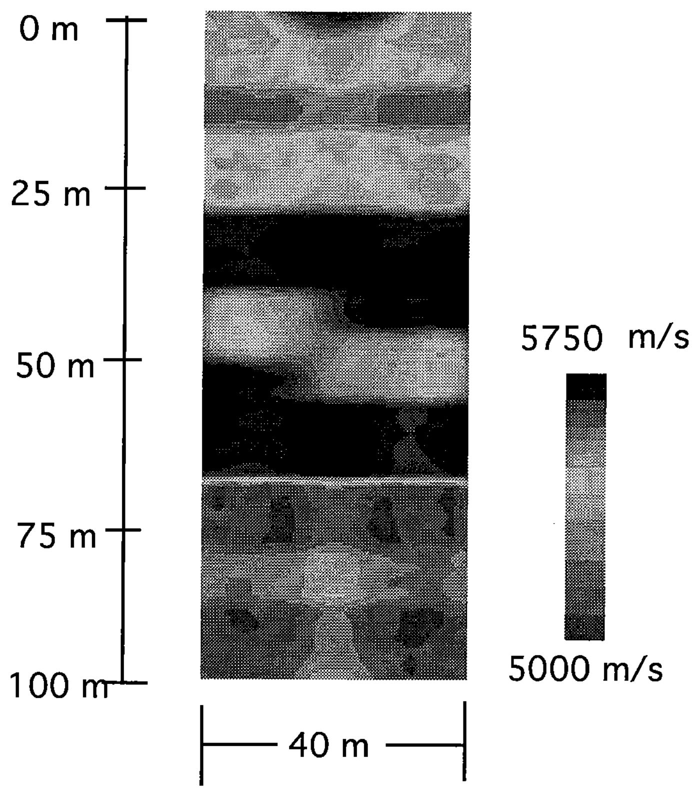

Fig. 3. Single frequency inversion result derived by nonlinear wave equation tomography after 25 iterations.