PAPER J

REFLECTION IMAGING USING RICCATI EQUATIONS

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ABSTRACT

In investigating wave propagation in a stratified medium, the concept of reflection and transmission play important roles and often lead to a simplified and intuitive picture of wave propagation in inhomogeneous media. When we image properties of the medium, rather than calculating the wave field, using equations governing reflection function instead of the wave equation may have certain computational advantages. We derive the Riccati equations of an arbitrary incident angle directly from physical equations. By applying the single reflection approximation to the integral representation of the Riccati equations, we are able to reconstruct reflection coefficient profiles with Fourier transform techniques. We also derived a recursive relation with which we can map the reconstructed reflection coefficient profile onto a velocity profile.

INTRODUCTION

For the scalar field produced by a time harmonic point source in a stratified inhomogeneous medium, the reduced wave equation can be written as

$$\Delta p + k^2(z)p = -\delta(z - z_0)\delta(r) / 2\pi r \qquad (1)$$

If we consider a finite volume of the medium, we shall obtain the exact solution by three well known methods, which lead to three different representations of p. These are the method of normal modes, the method of Hankel transform, and the method of multiple scattering. These representations can be transformed into one another by using contour integration and residual evaluation, the binomial expansion and Poisson summation formula (Keller, 1977, Brekhovskikh, 1982). However, these representation are very complicated and are inconvenient for inverse problems. We know that when a plane wave falls on a boundary between two media of different properties, it is split into a transmitted wave proceeding into second medium and a reflected wave propagated back into the first medium. Very often we do not seek the field itself in the medium but some other characteristic quantity, such as the reflection coefficient. Therefore, instead of using the reduced wave equation, it may have some computational advantages to directly use original physical equations. These coupled first order differential equations can be combined to Riccati equations which are first order but often nonlinear differential equations. In the following sections we first derive the Riccati equations for an arbitrary incident angle and their integral representations, and then apply single reflection approximation to the integral representation to reconstruct the reflection coefficient profile with Fourier transform techniques. We also discussed upgoing and downgoing waves separation and the mapping from the reflection coefficient profile to the velocity profile.

RICCATI EQUATIONS OF ARBITRARY INCIDENT ANGLE

Consider that waves propagate in the region $z_0 < z < L$ characterized by a wave number which is positive and varying continuously with z. Let this region extend from z_0 to $-\infty$ with a constant value k_0, and from L to ∞ with another constant value k_1 for the wave number as shown in Figure 1.

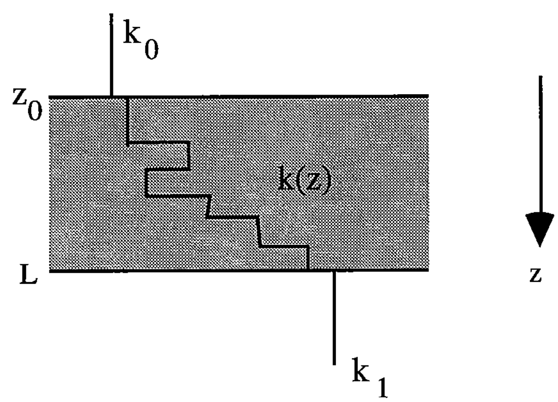

Fig.1. Wave number profile of inhomogeneous layer

Starting from the continuity equation and Euler equation:

$$\frac{\partial p}{\partial t} + c^2 \nabla \cdot v = 0, \tag{2}$$

and $$\frac{\partial v}{\partial t} + c^2 \nabla p = 0, \tag{3}$$

the following wave equations in the different regions can be derived as

$$(\Delta + k^2(z))p = 0, \qquad z_0 < z < L$$

$$(\Delta + k_0^2)p = 0, \qquad z < z_0$$

and $$(\Delta + k_1^2)p = 0. \qquad z > L$$

We know the solution to these wave equations corresponding to reflected and transmitted waves. However, instead of the wave equation, we like to arrive at the differential equations governing reflection function or transmission function directly from equation (2) and (3). We will see later this may result in certain computational advantages.

Assume the solution to above wave equation consists of upgoing and downgoing waves, then the pressure field and velocity field can be written as

$$p(x,z) = [u(z) + d(z)]e^{i\xi x}, \tag{4}$$

and $$v_z(x,z) = \frac{\beta(z)}{\omega}[-u(z) + d(z)]e^{i\xi x}, \tag{5}$$

where *u(z)* and *d(z)* represent upgoing and downgoing waves. The factor $e^{-i\omega t}$ is omitted. Notice that the expression (5) is true only under the condition of $(k(z))' << 1$ which implies the weak reflection of the medium (Harris, 1994). Let $\frac{\partial}{\partial t} = -i\omega$, then equations (2) and (3) become

$$\frac{\partial v_z(x,z)}{\partial z} = i\frac{\beta^2(z)}{\omega} p(x,z), \tag{2'}$$

and $$\frac{\partial p}{\partial z} = i\omega v_z(x,z). \tag{3'}$$

From equation (2') and (3') we arrive at

$$-u' + d' = i\beta(u+d) - \frac{\beta'}{\beta}(-u+d), \tag{6}$$

and

$$u' + d' = i\beta(-u+d). \tag{7}$$

Add equation (6) to equation (7) we obtain

$$d' = i\beta d - \frac{\beta'}{2\beta}(-u+d). \tag{8}$$

Subtracting equation (6) from equation (7) results in

$$u' = -i\beta u + \frac{\beta'}{2\beta}(-u+d). \tag{9}$$

By multiplying u to equation (8) and d to equation (9), we obtain

$$du' = -i\beta ud + \frac{\beta'}{2\beta}(-ud + d^2), \tag{10}$$

and $$ud' = i\beta ud - \frac{\beta'}{2\beta}(-u^2 + ud). \tag{11}$$

Subtracting equation (10) from (11) we have

$$\frac{du' - ud'}{d^2} = -2i\beta u / d + \frac{\beta'}{2\beta}(1 - u^2 / d^2),$$

or

$$R'(z) = -2i\beta(z)R(z) + \gamma(z)(1 - R^2(z)), \tag{12}$$

where $R(z) = u(z) / d(z)$ is defined as reflection function, the vertical wave number $\beta = k(z)\cos\theta(z)$ and the reflection coefficient over a fine layer or an effective interface

$\gamma = \frac{\beta'}{2\beta}$. Equation (12) is called Riccati equation governing reflection function which is a first order but nonlinear differential equation.

INTEGRAL REPRESENTATION OF REFLECTION FUNCTION

By noticeing that $\exp(2i\int_{z_0}^{z}\beta(z')dz')$ is an integrating factor for Riccati equation (12), the Riccati equation can be solved as the following:

$$\frac{d}{dz}[R(z)\exp(2i\int_{z_0}^{z}\beta(z')dz')] = \gamma(1-R^2)\exp(2i\int_{z_0}^{z}\beta(z')dz') \tag{12'}$$

where z_0 is a arbitrary reference point. With boundary condition $R(z)\to 0$ as $z\to\infty$, and integrating equation (12') from z to infinity, we obtain the integral representation of the reflection function

$$-R(z)\exp(2i\int_{z_0}^{z}\beta(z')dz') = \int_{z}^{\infty}\gamma(1-k^2(z))\exp(2i\int_{z_0}^{z}\beta(z')dz')dz$$

Let $\varphi(z) = 2\int_{z_0}^{z}\beta(z')dz'$. The above representation is simplified as

$$R(z)e^{i\varphi(z)} = -\int_{z}^{\infty}\gamma(z)(1-R^2(z))e^{i\varphi(z)}dz. \tag{13}$$

From the integral representation (13) we can see the physical significance of reflection function $R(z)e^{i\varphi(z)}$. It represents the ratio between complex amplitude of up and down waves and the phase variation due to propagation. At an effective interface, or over a fine layer, the reflection function degenerates to the form of the conventional reflection coefficient for a single interface, that is

$$-e^{-i\varphi(z)}\int_{z_n}^{z_{n+1}}\gamma(z)e^{i\varphi(z)}dz = -e^{-i\varphi(z)}\frac{\beta_{n+1}-\beta_n}{\beta_{n+1}+\beta_n}. \tag{14}$$

Note that the equation (13) is for the half space problem by which we mean that the incident field is illuminated at the top of the inhomogeneous layer as shown in figure 1. For the incident wave inside the layer, i.e. "the whole space problem", we can denote the reflection function results from contributions above and below the receiver depth as

$$R^{+}(z_g)e^{i\varphi(z_s,z_g)} = -\int_{z_g}^{\infty} \gamma(z')(1-R^2(z'))e^{i\varphi(z',z_s)}dz \tag{15}$$

$$-R^{-}(z_g)e^{i\varphi(z_s,z_g)} = -\int_{-\infty}^{z_g} \gamma(z')(1-R^2(z'))e^{i\varphi(z',z_s)}dz \tag{16}$$

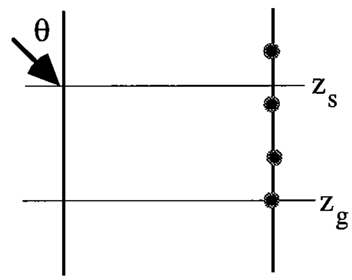

Fig. 2. Incident depth and receiver depth

For a given incident depth z_s and receiver depth z_g, see figure 2, equation (15) and (16) can be combined into one equation which is

$$R(z_g)e^{i\varphi(z_s,z_g)} = -\int_{-\infty}^{\infty} \gamma(z')(1-R^2(z))e^{i\varphi(z',z_s)}dz\,. \tag{17}$$

where we have used the same notation of "half space " for that of "whole space" without confusion.

UP AND DOWN WAVE SEPARATION

In order to get separated upgoing and downgoing waves from recorded data, we assume that all sources lies in the region $x < 0$ and that the medium is homogeneous for $x \geq 0$ (Devaney, 1986), the wave field can be expressed as a superposition of up and down plane waves

$$p(x,z,t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_z [U(k_z,\omega)e^{i(k\cdot r-\omega t)} + D(k_z,\omega)e^{-i(k\cdot r+\omega t)}], \tag{18}$$

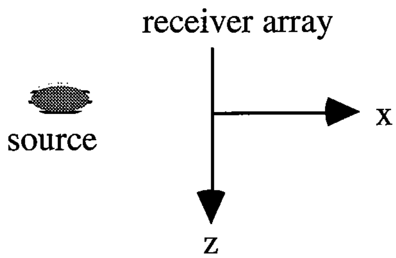

Fig. 3. Geometry of upgoing and downgoing wave separation.
The receiver array lies along the z-axis

where the wave vector $k = k_x\hat{x} + k_z\hat{z} = \sqrt{(\omega/v)^2 - k_z^2}\,\hat{x} + k_z\hat{z}$. Equation (18) expressed the wave field as a spectrum of up and down plane waves in the right half space. Amplitudes of *U* and *D* components in equation (13) can be determined from the total field recorded by the array *p(x=0,z,t)*. To show this we take the Fourier transform of p in z and t at *x=0*

$$U(k_z,\omega) + D(k_z,\omega) = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dt\, p(x=0,z,t) \tag{19}$$

Similarly, we take the Fourier transform of derivative of p in z direction:

$$ik_z[U(k_z,\omega) - D(k_z,\omega)] = \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dt \frac{\partial p(x=0,z,t)}{\partial z}. \tag{20}$$

From equation (19) and (20) we can solve for the spectrum *U* and *D*. The reflection function is then expressed as the ratio between the upgoing wave and downgoing wave:

$$R(z)e^{i\varphi(z)} = \frac{Ue^{ik\cdot r-i\omega t}}{De^{-ik\cdot r-i\omega t}} = \frac{U}{D}e^{2ik_z z} \tag{21}$$

REFLECTION IMAGING WITH SINGLE REFLECTION APPROXIMATION

Considering the phase factor in equation (17) consists of two parts, one is linear and the other is nonlinear, i.e.

$$\phi(z',z_s)=[\beta'+\beta''(z')]z'$$

where β' is the constant vertical wave number and $\beta''(z')$ is space variant local vertical wave number which can be considered as the cause of phase modulation in wave propagating in the medium. By neglecting $R^2(z)$ in the integrand of equation (17), i.e. only single reflection is taken into account, we have

$$\tilde{R}(\beta')=-\int_{-\infty}^{\infty}\gamma(z')e^{i[\beta'+\beta''(z')]z'}dz', \tag{22}$$

where $\tilde{R}(\beta')=R(z_g)e^{i\phi(z_s,z_g)}$. Since the wave number in equation (22) is spatially variant we call $\tilde{R}(\beta')$ the modified spectrum of $\gamma(z')$. In order to evaluate the function $\gamma(z')$ by the Fourier transform we look at the d.c. component of the spectrum

$$\tilde{R}(0)=-\int_{-\infty}^{\infty}\gamma(z')e^{i\beta''(z')z'}dz'. \tag{23}$$

If we multiply the complex conjugate of equation (23) to (22) we have

$$\tilde{R}(\beta')\tilde{R}^*(0)=-\int_{-\infty}^{\infty}\gamma(x)e^{-i\beta''(x)x}dx\int_{-\infty}^{\infty}\gamma(y)e^{i[\beta'+\beta''(y)]y}dy. \tag{24}$$

Under the condition that the $\gamma(z')$ is localized, i.e. the autocorrelation length of the γ function is small. Therefore equation (24) is reduced to

$$\tilde{R}(\beta)\tilde{R}(0)=-L\int_{-\infty}^{\infty}N(x)e^{i\beta'x}dx, \tag{25}$$

where $N(x) = \gamma(x)\gamma(x)$, and L is the length of integration. From equation (25) we can see that $\tilde{R}(k_s^{'})\tilde{R}^*(0)$ is proportional to spectrum of N(x). N(x) can be found by taking the inverse Fourier transform of $\tilde{R}(\beta')\tilde{R}^*(0)$.

VELOCITY RECOVERY FROM REFLECTION COEFFICIENTS

In the above section, we found the reflection coefficient γ which is a function both velocity and propagation angle, i.e.

$$\gamma(z) = \frac{\beta'(z)}{2\beta(z)}, \tag{26}$$

where $\beta(z) = k(z)\cos\theta(z)$. The velocity can be found by using equation (26) together with Snell's law. Integrating equation (26) over a layer, we have

$$\int_{\beta(z)}^{\beta(z+\Delta z)} \frac{d\beta}{\beta} = \int_{z}^{z+\Delta z} 2\gamma(z)dz,$$

or

$$\beta(z+\Delta z) = \beta(z)e^{2[\gamma(z+\Delta z)-\gamma(z)]}.$$

Therefore

$$v(z+\Delta z) = \frac{\cos\theta(z+\Delta z)}{\cos\theta(z)} v(z)e^{2[\gamma(z+\Delta z)-\gamma(z)]}. \tag{27}$$

Apply Snell's law

$$v(z+\Delta z) = \frac{\sin\theta(z+\Delta z)}{\sin\theta(z)} v(z) \tag{28}$$

to equation (27) we obtain the following recursive relation to recovery velocity

$$v(z+\Delta z) = v(z) / \sqrt{e^{4[\gamma(z+\Delta z)-\gamma(z)]}\cos^2\theta(z) + \sin^2\theta(z)}. \tag{29}$$

CONCLUSIONS

Directly from physical equations such as the continuity equation and Euler's equation we derived Riccati equation governing the reflection function and its integral representation for an arbitrary incident angle. With the reflection function and its integral representation to perform imaging has advantages over with wave equation. Because by using reflection function we can avoid to use the scattered field, as we did when we used the wave equation, which is difficult to obtain in practice.

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REFERENCES

Bellman R. and Vasudevan R. Wave propagation an invariant imbedding approach, D. Reidel Publishing Company, 1986

Brekhovskikh, L. and Lysaanov Yu., 1982. Fundamentals of Ocean Acoustics, Spring-Verlag Berlin Heidelberg, New York.

Devaney A. J., Aplane-wave decomposition for elastic wave fields applied to the separation of p-waves and s-waves in vector seismic data, Geophysics, 1986

Harris J. Lecture notes on wave phenomenon, 1994

Keller J. and Papadakis S. Wave propagation and underwater acoustics, Springer-Verlag, 1977