PAPER I

THE MOMENT METHOD UTILIZING GREEN'S FUNCTIONS OF STRATIFIED MEDIA: SCATTERING SIMULATIONS

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ABSTRACT

An efficient numerical solution to the two-dimensional scattering problem is achieved by decomposing the original 2-D problem into a layered medium and relatively small scatterers embedded in it. The moment method is applied to solve the scattering from the small scatterers with the Green's function of a layered medium. The layered Green's function is calculated with Fourier transforms. The computational examples show that the proposed method is more efficient than that of directly applying the moment method, especially when large scale problems are involved.

INTRODUCTION

Many imaging problems are carried out in the frequency domain and often require intensive forward modeling computation. Thus, there is a need to be able to efficiently calculate the wave field directly in the frequency domain. Furthermore, for large spatial scale problems, the costs of the finite difference methods increase dramatically, so that it is limited in many real applications. Two-dimensional scattering from inhomogeneous bodies in an unbound uniform medium has been studied extensively with the moment method (Richmod, 1965). However, the method is effective only for small scatters and uniform host medium. The objective of this paper is to develop an efficient numerical solution to the 2-D scattering problem for the scatters embedded in a layered background medium. We calculate layered Green's function first and then apply it to the moment method. The efficiency is achieved by only discretizing the scatters embedded in the layered background.

The paper is organized into three sections. In the first section, the approach of the analysis is outlined in terms of the supposition principle. Section two is a brief review of the moment method. Section three is an analysis of the computation of the Green's function for layered medium. Finally, we employ the layered Green's function in the moment method to show the effectiveness of the proposed method. In the end of each of the sections, we show some numerical examples to ensure that the implementation of the algorithm is valid.

THE APPROACH OF THE ANALYSIS

For reservoir imaging problems, the low spatial frequency components can often be obtained via prior geological information, or traveltime tomography or other means as indicated in Figure 1.

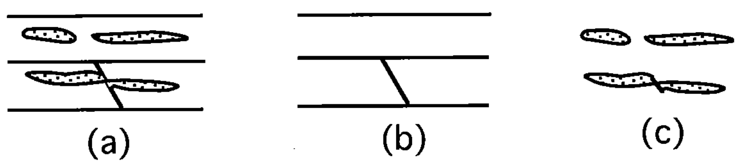

Fig. 1 For a common reservoir imaging problem, the velocity structure illustrated in (a), can be decomposed into low components (b) and high spatial frequency components (c).

In order to image high frequency components, it is necessary to be able to efficiently calculate the wave field of the slowly varying background. Since the background variation is relatively simple, in many practical situations, it can be described by stratified structure with some additional local features, as shown in Figure 2. Therefore, the problem can be analyzed separately as a layered medium with small scatterers embedded in it.

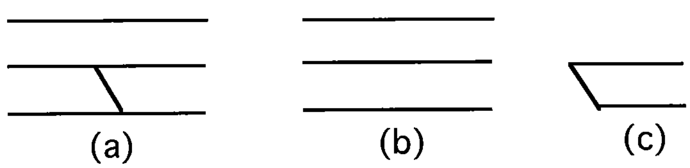

Fig. 2 A 2-D structure (a) can be separate as a 1-D background and a relatively small and isolated 2-D body.

For a two-dimensional scalar Homholtz equation

$$\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+k^2(x,z)\}u=-\delta(x-x')\delta(z-z'), \tag{1}$$

the corresponding integral equation can be written as

$$u(s,g)=u^i(s,g)-\int u(r',s)f(r')G^0(g,r')d^2r', \tag{2}$$

where $G^0(g,r')$ is the Green's function for a uniform background, and *f(r')* is the scattering potential relative to that background. The same problem can also written as

$$u(s,g)=u^l(s,g)-\int u(r',s)e(r')G^l(g,r')d^2r', \tag{3}$$

where $G^l(g,r')$ is the Green's function for layered background medium. The function *e(r')* is the scattering potential relative to the layered background. For most realistic situations, the distribution of the function *e(r')* is more isolated and weaker than that of *f(r')*. Consequently, solving equation (3) with the moment method is much easier than solving equation (2), since a relatively small area needs to be discretized. However, we have to solve following differential equation in order to obtain the Green's function for the layered background:

$$\{\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}+k_l^2(z)\}G^l=-\delta(x-x')\delta(z-z'). \tag{4}$$

Fortunately, we can almost solve the above equations analytically with the Fourier transform. We will discuss this in more detail in the following sections.

THE MOMENT METHOD

The moment method is often applied in the calculation of the scattered field from a two dimensional inhomogeneity (Bath, 1982, Chew, 1990). The scattered field is described by following equation:

$$u(r)=u^i-\int u(r')e(r')G(r,r')d^2r'. \tag{5}$$

The method is straightforward and efficient when the size of the scatterer is small. The relatively small and isolated region we separated from the layered background is divided into N square cells. Then, the scattering potential and wave field are represented as a summation of basis functions over the N cells, i.e.,

$$e(r) = \sum_{j,k} e(r_{jk}) b_{jk}(r)$$

$$u(r) = \sum_{j,k} a_{jk} b_{jk}(r)$$

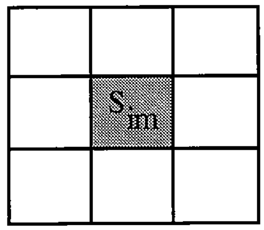

Fig. 3. Scattering potential is divided up into N square cells

where $b_{jk}(r)$ is the basis function, $e(r_{jk})$ and a_{jk} represent the coefficients describing the scattering potential and wave field over the basis function. We choose the same pulse basis function $b_{jk}(r)$ to discretize equation (5), where

$$b_{jk}(r) = \begin{cases} 1 & r \in s_{jk} \\ 0 & r \notin s_{jk} \end{cases} .$$

Applying the point-matching procedure, equation (5) can be written as a linear algebraic system

$$u_{jk} + \sum_{i,m} g(j,k,i,m) f(r_{im}) u_{im} = G(r_{jk}), \qquad (6)$$

where $g(j,k,i,m) = \int_{s_{im}} G(r_{jk}) d^2 r$.

We implemented the above algorithm for the homogeneous background. With the model shown in Figure 4, the scattered and total field both in the time and frequency domains are calculated. Similar to finite difference method, the dimension of the cell is chosen approximately as one tenth wavelength at the lowest velocity in the calculations. The results are displayed in Figure 5 and 6 and it is obvious that the forward scattering is stronger in the forward direction, as expected.

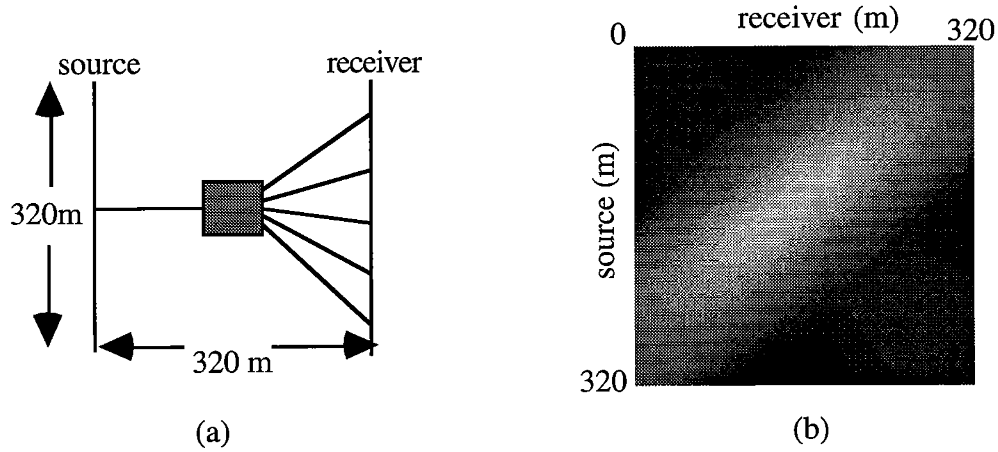

Fig. 4. (a) Forward scattering model for computation with moment method.
(b) The amplitude of the calculated scattered field (Frequency= 400Hz).

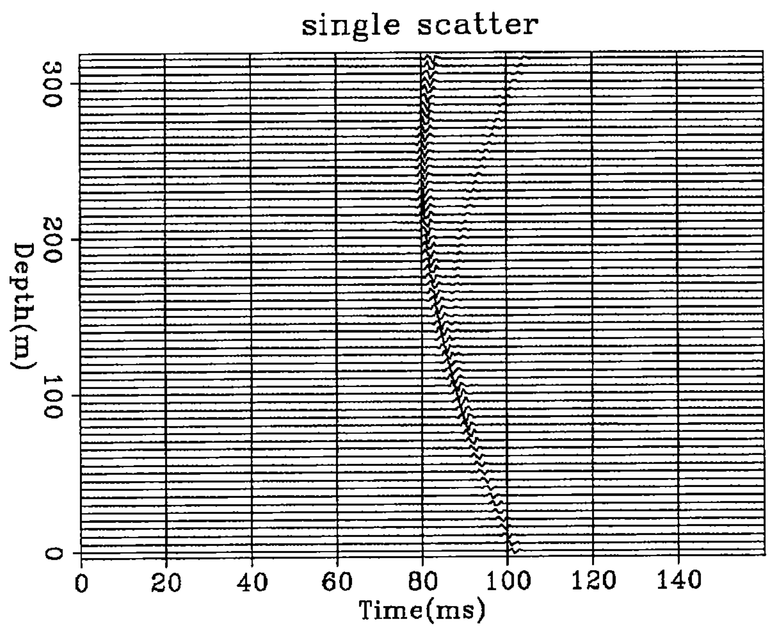

Fig. 5 Forward scattering modeling with moment method.
Total field time signal with a source at the depth of 250m.

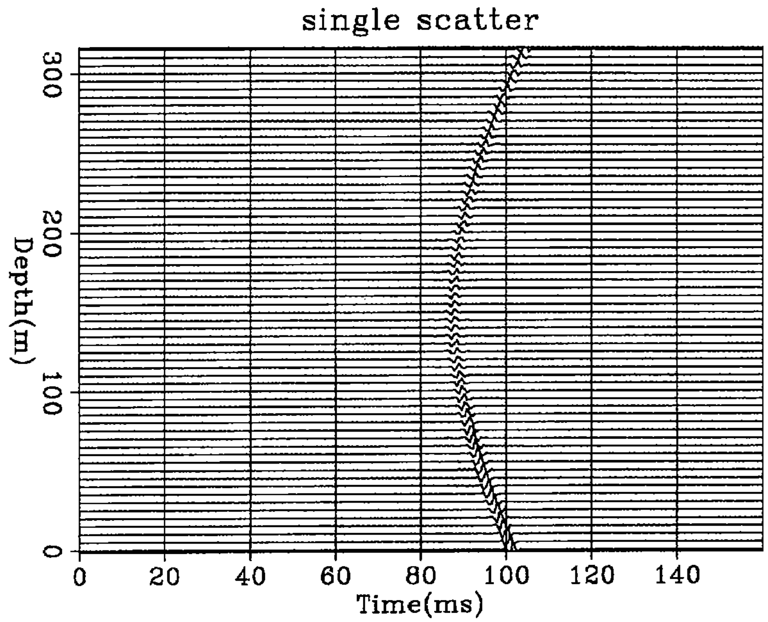

Fig. 6 Forward scattering modeling with moment method.
Time domain scattered field which is stronger in forward direction.

GREEN'S FUNCTIONS FOR THE LAYERED MEDIUM

We calculate the Green's function of the stratified medium through the Fourier transform (Brekhovskikh, 1982). Since the medium is 1-D, we take the Fourier transform of the Holmholtz equation (4) along the horizontal direction and obtain

$$[\frac{\partial^2}{\partial z^2} + k^2(z) - \xi^2]G(\xi,z,z') = -\delta(z-z')e^{i\xi x'}, \qquad (7)$$

where ξ is horizontal wave number. The solution of this equation $G(\xi,z,z')$ is the one-dimensional Green's function and satisfies appropriate boundary conditions. Recall that $G(\xi,z,z')$ is continuous everywhere in the interval of definition and dG/dz is continuous everywhere in the interval except at $z=z'$. At the source depth z',

$$\lim_{\varepsilon\to 0}\int_{z'-\varepsilon}^{z'+\varepsilon} \delta(z-z')dz = 1,$$

and
$$\lim_{\varepsilon\to 0}\int_{z'-\varepsilon}^{z'+\varepsilon} \gamma^2(z)U(\xi,z,z')dz = 0,$$

which is true, if $\gamma(z)$ and $G(\xi,z,z')$ are continuous. Here, $\gamma(z)$ is vertical wave number. From the above analysis, we can see that the delta function creates an artificial layer or interface for our problem. The eigen-equation of equation (7) is

$$[\frac{\partial^2}{\partial z^2}+\gamma^2(z)]\psi=0. \tag{8}$$

The solution to equation (7) can be constructed with two linearly independent solutions of the eigen equation (8), i.e.,

$$G=\begin{cases} a\psi_1 & z\le z' \\ b\psi_2 & z\ge z' \end{cases}.$$

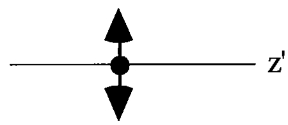

Fig. 7 at the source depth the derivative of the Green's function is discontinuous

With the constrains of the conditions discussed above, we have

$$a\psi_1(z')=b\psi_2(z')$$

and $$a\psi'_1(z')-b\psi'_2(z')=-e^{ik_x x'}.$$

Therefore, the unknown coefficients a and b can be solved. Notice that the solutions ψ_1 and ψ_2 are for arbitrary stratified medium. For a layered medium, ψ_1 and ψ_2 are plane waves in each layer, i.e.,

$$\begin{aligned} &Ae^{-i\gamma_1(z-z_1)} \\ &Be^{i\gamma_2(z-z_1)}+Fe^{-i\gamma_2(z-z')} \\ &Ee^{i\gamma_2(z-z')}+Ce^{-i\gamma_2(z-z_2)} \\ &De^{-i\gamma_2(z-z_2)}. \end{aligned} \tag{8}$$

The conditions at interface are:

i. The wave field is continuous.
ii. The derivative of the wave field is continuous (for constant density).
iii. The wave field at the source depth is continuous.
iv. The derivative of the wave field across the source depth is discontinuous.

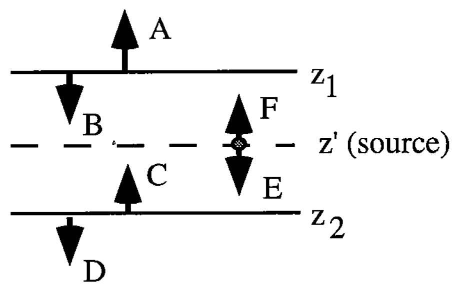

Fig. 8, In each layer the solution to the eigen-equation is a plane wave. Source depths are treated as artificial interfaces.

With these conditions we can solve for the unknown coefficients of the plane waves in each layer. The Green's function of the layered medium is obtained by taking the inverse Fourier transform of $G(\xi,z,z')$.

The computation of the Sommerfeld-type integral encountered in the spectral representation of Green's function has a well-known difficulty of their numerical evaluation, namely the oscillatory behavior of the integrand due to the function $e^{i\xi(x-x')}$ and the presence of singularities contributed by $G(\xi,z,z')$. These include poles and branch points that result from the dispersion relation

$$\gamma^2 = \frac{\omega^2}{v^2} - \xi^2.$$

It is possible to leave the pole out and take the Cauchy principle value, a common procedure for dealing with improper integrals. However, there is no need to proceed in this way, because all propagating wave systems are naturally dampened and any amount of dampening takes the poles and moves it off the real axis. The integral then becomes proper and can be evaluated without ambiguity. After including a small amount of attenuation, the dispersion relation becomes

$$\gamma^2 = \frac{(\omega + i\varepsilon)^2}{v^2} - (\xi_r + i\xi_i)^2. \tag{9}$$

From $\text{Im}\,\gamma = 0$ we have

$$\gamma_r \gamma_i = -2\xi_r \xi_i + 2\varepsilon \frac{\omega}{v^2} = 0.$$

Therefore, the curves on which the branch points lie are

$$\xi_r \xi_i = \varepsilon \frac{\omega}{v^2} > 0, \qquad (10)$$

as shown in Figure (10). Since z is positive, we can not have $\text{Im}\,\gamma < 0$ on the physical sheet of the complex function $e^{i\gamma z}$, since that would lead to an exponential solution which is unnatural.

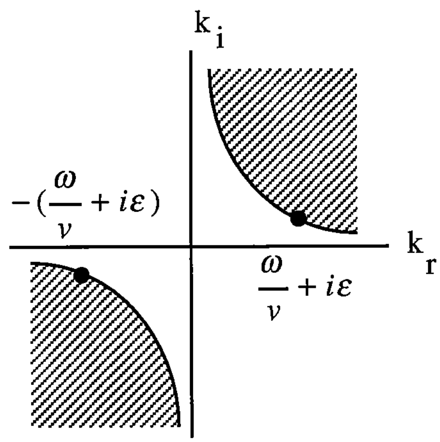

Fig. 9, Branch cut and branch points

We implemented the above algorithm of Green's function for layered medium. The results are shown in Figures 10, and 11. In figure 10 (b) the pattern of the frequency response agrees with those results calculated using the finite difference method. The reflection and transmission events in the figure 11 are consistent with those of events from theoretical analysis. With the verification of Green's function for layered background we are ready to apply the solution obtained using the Green's function, to the moment method to compute the scattering form 2-D structures.

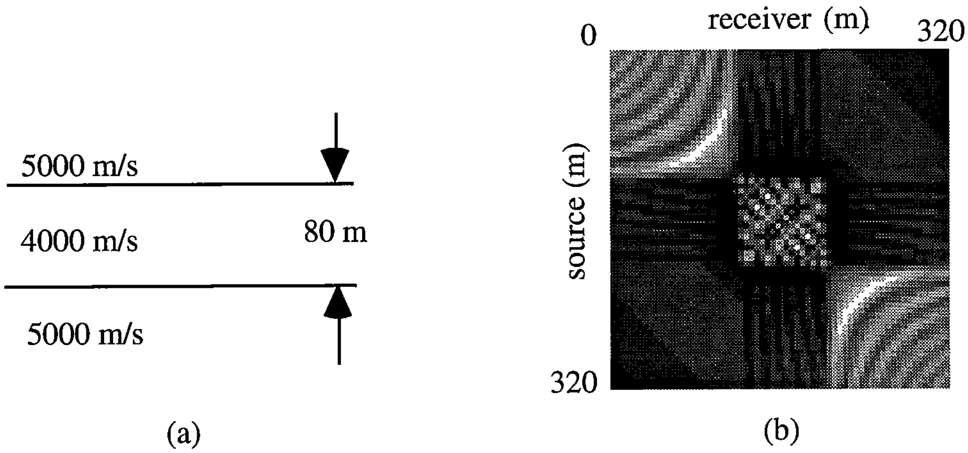

Fig. 10. (a) Three layer model. (b) The amplitude of the frequency response of the calculated Green's function (Frequency=400 Hz).

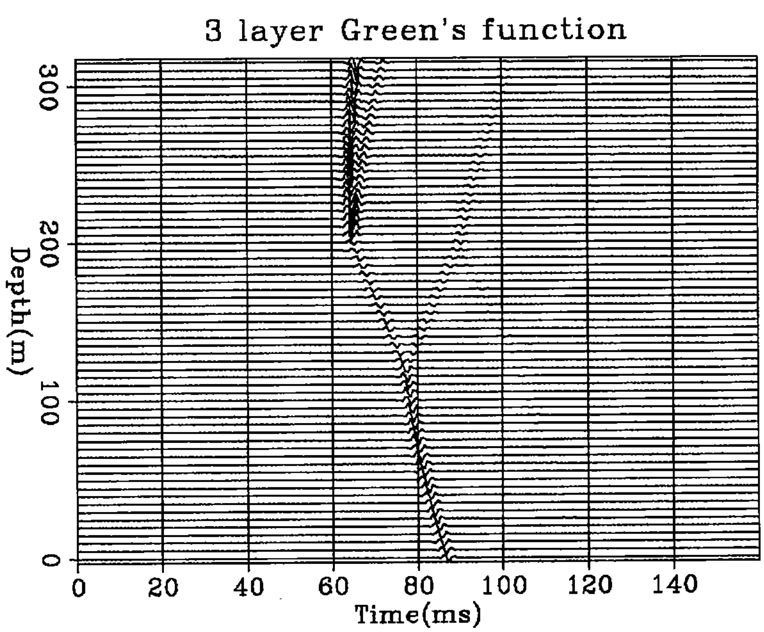

Fig. 11. The time domain Green's function of three layer model.
The reflection and transmission are consistent with theoretical analysis.

SAMPLE RESULTS OF SCATTERING FROM 2-D STRUCTURE

On inserting the proper representation for $G^l(r,r')$ into equation (6), which is rewritten here as

$$u_{jk} + \sum_{i,m} g^l(j,k,i,m) f(r_{im}) u_{im} = G^l(r_{jk}),$$

where $g^l(j,k,i,m) = \int_{s_{im}} G^l(r_{jk}) d^2 r$, and $G^l(r_{jk})$ is Green's function for layered background, we can calculate the scattering field from a 2d model as indicated in Figure 12 (a). The amplitude of the frequency response with the sample model is shown in Figure 12 (b) and the corresponding scattered field in the time domain is shown in Figure 13.

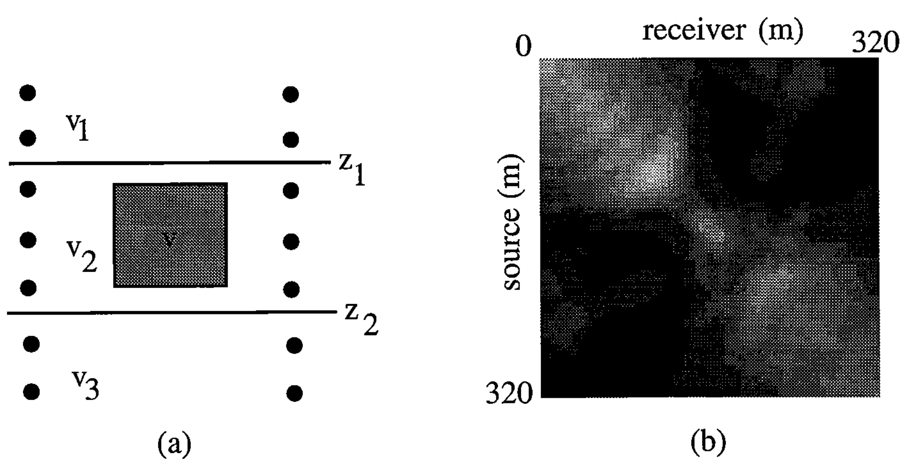

Fig. 12 (a) 2-D model forward scattering model, v1=4000, v2=3000 v3=3500 and v=3600 (m/s). (b) The amplitude of the frequency response at frequency=500 Hz

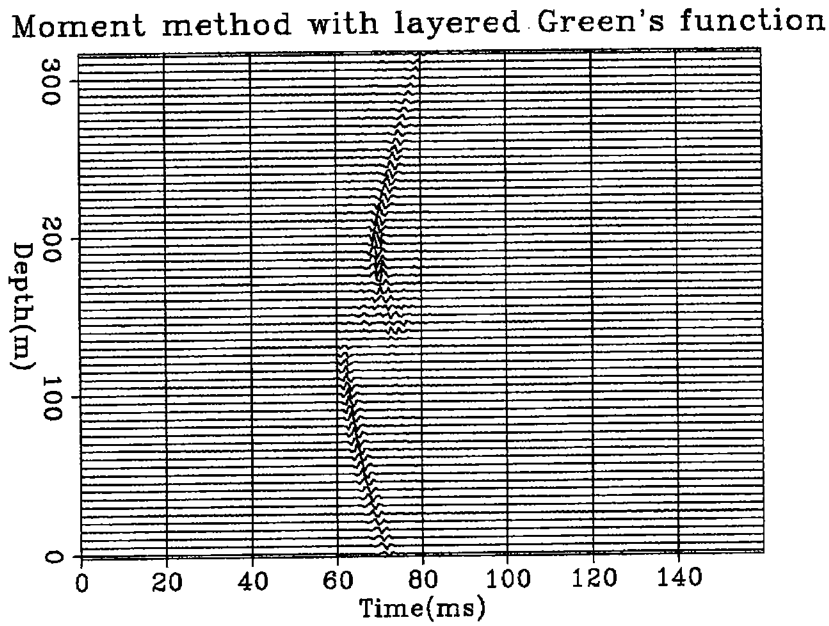

Fig. 13 Scattered field from the model in figure 12 using the moment method with layered Green's function.

CONCLUSIONS

With the layered Green's function, the calculations using the moment of method become much more efficient in some situations, because only relatively small scatterers needed to be discretized. The layered Green's function can be calculated with a Fourier transform technique. This forward scattering calculation provides a useful tool for migration or inversion in the frequency domain.

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