PAPER O

DIFFRACTION TOMOGRAPHY IN VARIABLE BACKGROUND MEDIUM

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ABSTRACT

Inhomogeneities embodied in a variable background is a more practical model in subsurface imaging. By introducing a reference field of a uniform medium we reformulate the inverse scattering problem such that the scattering is generated in the uniform medium with a composite potential which consists of the impedance and the original acoustic potential. The impedance is defined as the ratio between the fields of the reference and the variable background. An inversion method is established based on this modified scattering model. The algorithm is the same as that of Fourier diffraction tomography in a constant host medium. The approach however, differs from the conventional diffraction tomography in that instead of the scattering field itself, the image is reconstructed from a holographic field which is the multiplication of the scattering and its conjugate background fields. The background field is a field generated by a unit acoustic potential in a medium otherwise identical to that generating scattering field.

INTRODUCTION

The existing diffraction tomographic inversions are mainly based on plane wave expansion and Fourier transform techniques (Mueller et al., 1979, Devaney, 1982, Harris, 1987, Wu and Toksoz, 1987). Such methods are simple to implement but do not work well when the background medium is strongly non-uniform. One way to overcome the problem of the strong inhomogeneity of the medium is to apply the distorted Born approximation (Devaney, et. al, 1983), which means that the Green's function of a variable host medium is adopted to maintain a weak contrast between the potential and the background. However, under the distorted Born approximation, the difficulty is not only how to find the Green's function associating with the variable background but also the Green's function however founded generally can at the best be decomposed into space-variant "plane waves" which have little use in terms of utilizing Fourier transform techniques. This is why most of proposed algorithms dealing with variable background either from data space or model space are restricted to some special case, for example in a 1-D medium in which the problem is greatly simplified (Pai, 1990, Dickens, 1992, and Huan, 1992).

In this study, we re-exam the widely used inverse scattering model. By introducing a reference field of a uniform medium, we reformulate the inverse scattering problem such that the scattering is generated in that uniform medium with a composite potential which consists of the impedance and the original acoustic potential. The impedance is defined as the ratio between the fields of the reference and the variable background. The background field is a field generated by a unit acoustic potential in a medium otherwise identical to that generating scattering field. The measured scattering field is the interference of the scattering and background fields at the receiver locations. It includes the amplitude and phase as does a hologram. The information about the potential or object function is encoded in the interference pattern and can be read out by using a replica of the background field according to the principle of diffraction. The holographic field is constructed by multiplying the conjugate background field to the scattering field. Applying the holographic field to the modified scattering model and assuming the potential function is localized, we established a Fourier diffraction tomographic inversion method. The algorithm is the same as that of Fourier diffraction tomography in a constant host medium. The approach however, differs from the conventional diffraction tomography in that instead of the scattering field itself, the image is reconstructed from a holographic field.

BASIC EQUATION OF INVERSE SCATTERING

The total acoustic field can be written as

$$u(x,\omega) = u^{i}(x,\omega) + u^{sc}(x,\omega) \tag{1}$$

which satisfies a Helmholtz equation

$$\Delta u + k^2 u = -q(x,\omega) - q_c(x,\omega) \tag{2}$$

where the inhomogeneity $q(x,\omega)$ is due to the primary source, and $q_c(x,\omega)$ denotes secondary source. Obviously, $q(x,\omega)$=0, for x $\notin$ Q, and $q_c(x,\omega)$=0 for x $\notin$ V.$\cup$ S . See Figure 1.

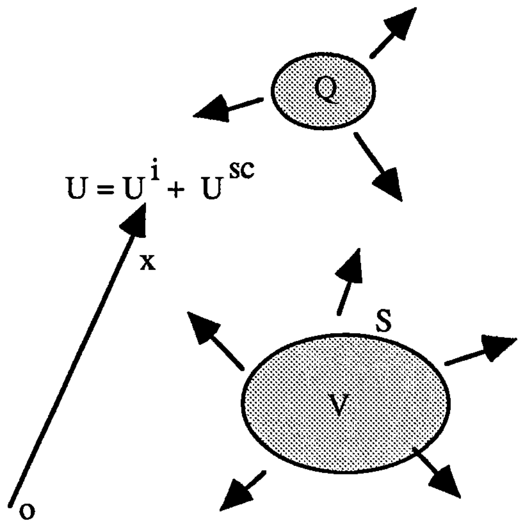

Fig. 1. Definition of a scattering model

With an appropriate Green's function, the scattering field can be written as an integral equation

$$u^{sc}(x,\omega) = \int_{-\infty}^{\infty} q_c(x',\omega) G(x-x',\omega) d^3x' \tag{3}$$

where $q_c(x',\omega) = f(x)[u^i + u^{sc}]$. $f(x) = (k^2(x) - k_0^2)$. is called the acoustic potential.

Under the distorted Born approximation, the scattering field is approximated as

$$u^{sc}(s,g) = \int_v f(x) \tilde{G}(s,x) \tilde{G}(x,g) d^3x \tag{4}$$

where s, g represent source and receiver locations respectively; $\tilde{G}(s,x)$, $\tilde{G}(x,g)$ are Green's functions for variable background.

WKBJ GREEN'S FUNCTION AND THE IMPEDANCE

The Green's function of variable background is very complicated and generally can be, at the best, decomposed into space-variant "plane waves", that have little use in terms of utilizing Fourier transform techniques which are essential part of diffraction reconstruction algorithms. By introducing a reference field of a uniform host medium, we rearrange the kennel of the equation (4), such that it is set as two components: one i.e. the impedance resulting from the ratio between the background and reference fields is account for the non-uniform effects of the medium together with the acoustic potential function; the other which is the Green's function of the reference background is used to describe the propagation effects in the reference background. The scattering field becomes

$$u^{sc}(s,g) = \int_v f(x) \frac{\tilde{G}(s,x)\tilde{G}(x,g)}{G(s,x)G(x,g)} G(s,x)G(x,g) d^3x \qquad (5)$$

Where $G(s,x)$, $G(x,g)$ are Green's functions for a uniform reference background. $\frac{\tilde{G}(s,x)\tilde{G}(x,g)}{G(s,x)G(x,g)}$ is defined as an impedance. Assume WKBJ solutions to the Green's functions, i.e.

$$\tilde{G}(s,x) = A(s,x)e^{-i\omega\tilde{\phi}(s,x)}$$

$$\tilde{G}(x,g) = A(x,g)e^{-i\omega\tilde{\phi}(x,g)}$$

$$G(s,x) = \frac{1}{|s-x|} e^{-i\omega\phi(s,x)}$$

$$G(x,g) = \frac{1}{|x-g|} e^{-i\omega\phi(x,g)}$$

Assume that the amplitude is only the function of the length of the ray path, i.e., only the geometrical spreading is considered. Then in the situation of relative weak variation of the background, the amplitudes of the background field and reference field are approximately the same. That is

$$\frac{A(s,x)}{|s-x|} \approx 1$$

$$\frac{A(x,g)}{|x-g|} \approx 1$$

as indicated in figure 2.

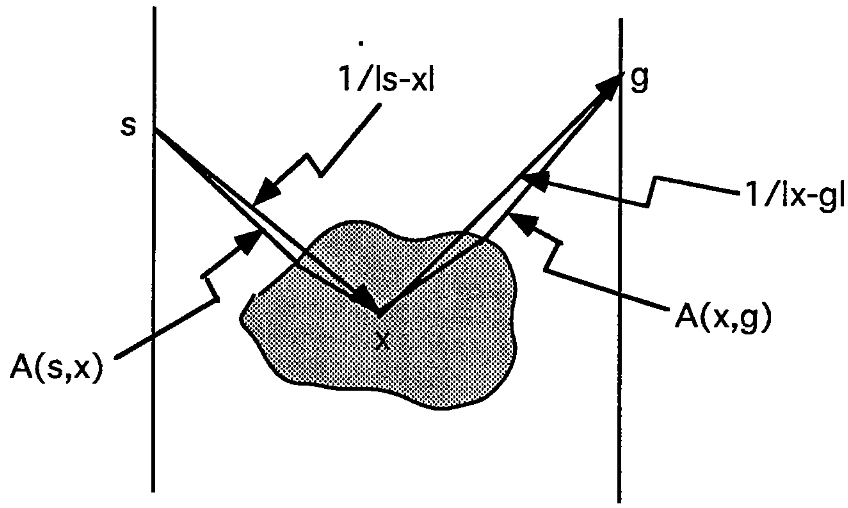

Fig. 2 Geometrical spreading along ray paths

Consequently, the impedance

$$\frac{\tilde{G}(s,x)\tilde{G}(x,g)}{G(s,x)G(x,g)} = e^{-i\omega[\tilde{\phi}(s,x)+\tilde{\phi}(x,g)-\phi(s,x)-\phi(x,g)]} = e^{-i\omega\hat{\phi}(s,g,x)} \tag{6}$$

where $\hat{\phi}(s,g,x)$ is the composite eikonal. The equation (6) means that the phase is more distinctive than the amplitude in relative smoothly varied background medium. It is true that in the case of coherence field superposition the phase play a very important role (Goodman, 1968). The implication of equation (6) is to relax Born approximation condition such that the larger contrast between inhomgeneity and background is allowed.

Note that the Green's function here is for 3-D. For a 2-D case the above arrangement is still applicable, since the geometrical spreading is eliminated in the ratio of (6). Substitute (6) into (5) we have

$$u^{sc}(s,g) = \int_v f(x)e^{-i\omega\hat{\phi}(s,g,x)}G(s,x)G(x,g)d^3x \tag{7}$$

where $f(x)e^{-i\omega\hat{\phi}(s,g,x)}$ is called the composite potential. The equation (7) is the modified inverse scattering model. we propose a Fourier holographic reconstruction method base on this modified inverse scattering model.

FOURIER-HOLOGRAPHIC RECONSTRUCTION

In equation (7) the composite eikonal $\hat{\phi}(s,g,x)$ is generally, a function both of the source/receiver locations and the image reconstruction positions. In another word, the eikonal is the ray path dependent. This is troublesome because the path is difficult to calculate. The objective here is to related the impedance to a holographic field so that the differential form of the total eikonal is the function only of the reconstruction position.

The measured scattering field in equation (7) is the interference of the scattering and background fields at the receiver locations. It includes the amplitude and phase as does a hologram. The information about the object function is encoded in the recorded field or the interference pattern. It can be read out by using a replica of the background field according to the principle of diffraction. The background field is defined as

$$u^{bk}(s,g) = \int_V e^{-i\omega\hat{\phi}(s,g,x)} G(s,x)G(x,g)d^3x \tag{8}$$

which has the same form as that the scattering field except the potential *f(x)* is unit here.

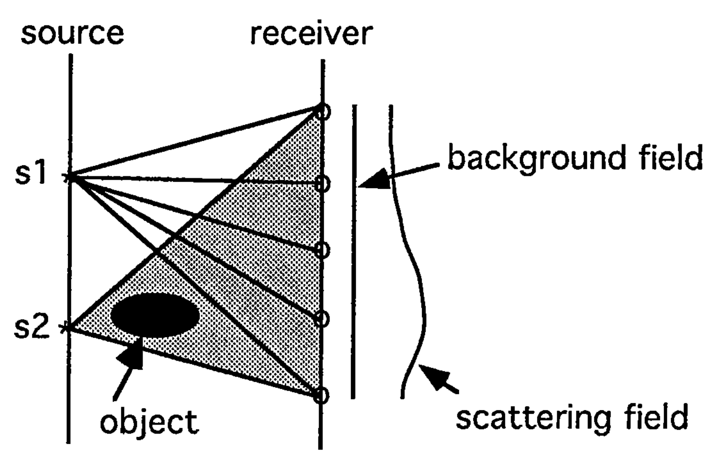

Fig. 3. Interference between the background field and the scattering field

A holographic field is constructed by multiplying the conjugate background field to the scattering field, i.e.

$$u^{sc}(s,g)u^{bk^*}(s,g) = \int_V d^3y \int_V d^3x f(x) e^{-i\omega[\hat{\phi}(s,g,x)-\hat{\phi}(s,g,y)]} G(s,x)G(x,g)G^*(s,y)G^*(y,g) \tag{9}$$

where $u^{bk^*}(s,g)$ is the conjugate background field. As showing in figure 4, the background field is used to "illuminate" the hologram and the information encoded in the hologram is readout according to the principle of diffraction (Goodman, 1968).

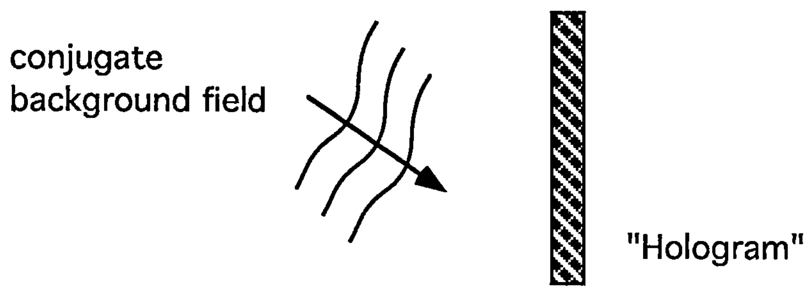

Fig. 4 Holographic reconstruction with conjugate background field

Now consider the reconstruction with the holographic field. Assuming that the object function *f(x)* is a localized function (Miller, et. al, 1987), therefore, at an arbitrary reconstruction point *y*, *f(x+y)* is not zero only for small *x*.

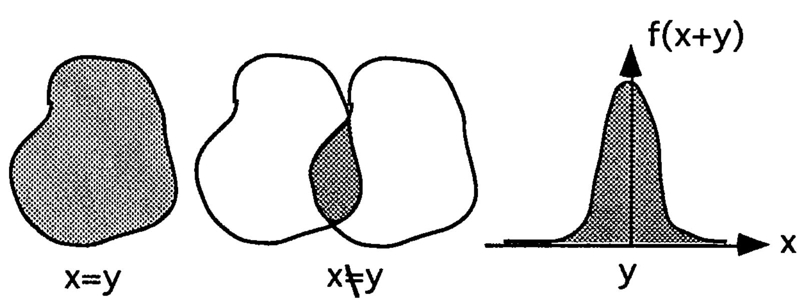

Fig. 5. The localized property of the object function

In other words, $f(x+y)$ is non zero only in the neighborhood of y. Consequently, we can take Taylor expansion of the total eikonal $\hat{\phi}(s,g,x)$ around the reconstruction point y, i.e.

$$\hat{\phi}(s,g,x) = \hat{\phi}(s,g,y) + \nabla\hat{\phi}(s,g,y)\cdot(x-y)+....$$

Take only first order

$$\hat{\phi}(s,g,x) - \hat{\phi}(s,g,y) = \nabla\hat{\phi}(s,g,y)\cdot(x-y)$$

and using the eikonal equation $|\nabla\hat{\phi}(s,g,y)| = \frac{1}{v(y)}$ we have

$$\hat{\phi}(s,g,x) - \hat{\phi}(s,g,y) = \frac{\hat{\alpha}\cdot(x-y)}{v(y)} \tag{10}$$

where $v(y)$ is composite velocity, i.e. $\frac{1}{v(y)} = \frac{1}{c(r)} - \frac{1}{c_0}$. The $\hat{\alpha}$ is the unit vector of the composite wave vector. It is interesting to see that the result obtained with the approach here is very similar to that of obtained by Beylkin, 1990. Notice that the direction of the composite eikonal gradient bisects the angle between the incident and the scatted rays as indicated in Figure 6. Even through the background and the reference fields are different, their propagation direction can be treat approximately as the same locally in the neighborhood of the reconstruction point.

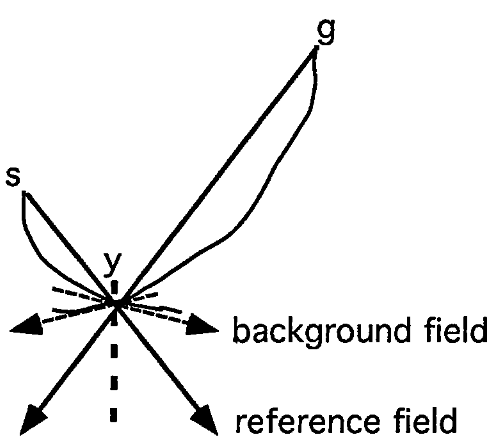

Fig. 6. The direction geometry of the total eikonal gradient.

Substitute (10) into (9), the holographic field is then

$$u^{sc}(s,g)u^{ref^*}(s,g) = \int_v d^3y \int_v d^3x f(x) e^{-i\frac{\omega}{v(y)}\hat{\alpha}\cdot(x-y)} G(s,x)G(x,g)G^*(s,y)G^*(y,g) \tag{11}$$

Let x-y=ξ. Since $f(\cdot)$ is localized and can be treated as a constant around the neighborhood of y, i.e. $f(\xi + y) \approx f(y)$, equation (11) can be rewritten as

$$u^{sc}(s,g)u^{ref^*}(s,g) = \int_v d^3y f(y) \int_v d^3\xi e^{-i\frac{\omega}{v(y)}\xi} G(s,\xi+y)G(\xi+y,g)G^*(s,y)G^*(y,g) \tag{12}$$

Next step is to decompose the Green's function in equation (12) into spatial invariant plane waves so that we can apply conventional Fourier diffraction tomographic techniques. Take Fourier transform of holographic field along source line s and receiver line g respectively, we get

$$F\{u^{sc}(s,g)u^{ref^*}(s,g)\}(k_s,k_g) = \int_v d^3y f(y) \int_v d^3\xi e^{-i\frac{\omega}{v(y)}\hat{\alpha}\cdot\xi} F\{G(s,\xi+y)G(\xi+y,g)G^*(s,y)G^*(y,g)\}(k_s,k_g) \tag{13}$$

Up to this point, the Green's function are general one. For the 3-Dimensional case, the frequency diversity has to be applied in order to decompose the Green's function into plane waves because of the lack of the data in the third spatial dimension. Now, we assume 2-D line source Green's function, in Fourier domain

$$F\{G(s,\xi+y)G(\xi+y,g)\}(k_s,k_g) = \frac{e^{-ik\cdot(\xi+y)}}{-4\gamma_s\gamma_g} e^{i\gamma_s d_s + i\gamma_g d_g} \tag{14}$$

$$F\{G^*(s,y)G^*(y,g)\}(k_s,k_g) = \frac{e^{i(k_x y_x - k_z y_z)}}{-4\gamma_s\gamma_g} e^{-i(\gamma_s d_s + \gamma_g d_g)} \tag{15}$$

Equation (13) can be written as

$$\int_v d^2\xi e^{-i\frac{\omega}{v(y)}\hat{\alpha}\cdot\xi} F\{G(s,\xi+y)G(\xi+y,g)\}(k_s,k_g) F\{G^*(s,\xi+y)G^*(\xi+y,g)\}(k_s,k_g)$$
$$=[\frac{e^{i(\gamma_s d_s+\gamma_g d_g)}}{-4\gamma_s\gamma_g} e^{-ik\cdot y}\int d^2\xi e^{-i\frac{\omega}{v(y)}\hat{\alpha}\cdot\xi} e^{-ik\cdot\xi}]*[\frac{e^{i(\gamma_s d_s+\gamma_g d_g)}}{-4\gamma_s\gamma_g} e^{-ik_x y_x-k_z y_z}] \quad (16)$$

here we inter-changed the order of the integration and the convolution. Notice that $\frac{\omega}{v(y)}\hat{\alpha}\cdot\xi+k\cdot\xi\approx\frac{\omega}{c(y)}\frac{k}{|k|}\cdot\xi=\frac{c_0}{c(y)}k\cdot\xi$. After the variable changing $\eta=(c_0/c(y))\xi$, the Fourier integral of right side of (16) can be evaluated in the isochron space as

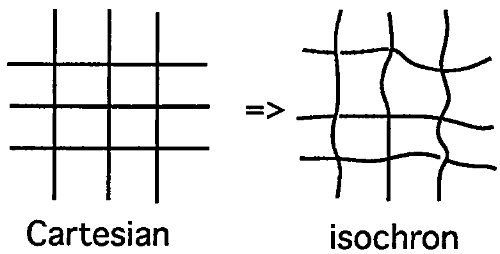

Fig. 7. Coordinate transformation from the Cartesian to isochron

i.e.

$$\int d^2\xi e^{-i(\frac{\omega}{v(y)}\hat{\alpha}\cdot\xi+k\cdot\xi)} = \delta(k)(c(y)/c_0)^2$$

Consequently, equation (16) becomes

$$\int_v d^3\xi e^{-i\frac{\omega}{v(y)}\hat{\alpha}\cdot\xi} F\{G(s,\xi+y)G(\xi+y,g)G^*(s,y)G^*(y,g)\}(k_s,k_g)$$
$$=[\frac{\delta(k)}{(c_0/c(y))^2}\frac{e^{-ik\cdot y}}{4\gamma_s\gamma_g}e^{i(\gamma_s d_s+\gamma_g d_g)}]*[\frac{e^{-i(k_x y_x-k_z y_z)}}{4\gamma_s\gamma_g}e^{-(i\gamma_s d_s+\gamma_g d_g)}] \quad (17)$$

where $*$ represent the convolution operations regarding to the measurement array wave numbers k_s and k_g. The resultant wave vectors are

$$k=(k_x,k_z), \qquad k_z=k_s+k_g,$$

$$k_x=\gamma_s-\gamma_g$$

$$\gamma_s=\sqrt{k^2-k_s^2} \qquad \gamma_s=\sqrt{k^2-k_g^2}$$

$$\delta(k_x,k_z)=\delta(\gamma_s-\gamma_g,k_s+k_g)$$

According to distribution analysis

$$\delta(k_x,k_z)=\left|\frac{(\gamma_s\gamma_g)^3}{k^2(\gamma_s{}^3+\gamma_g{}^3)}\right|\delta(k_s-k_g,k_s+k_g)$$

The coordinate transformation of the δ(.) function is indicated in the figure 8. It is obviously that the coverage of the (K_X,K_Z) space is poorer in the horizontal direction. Notice that the coverage here is only for arguments of the δ function.

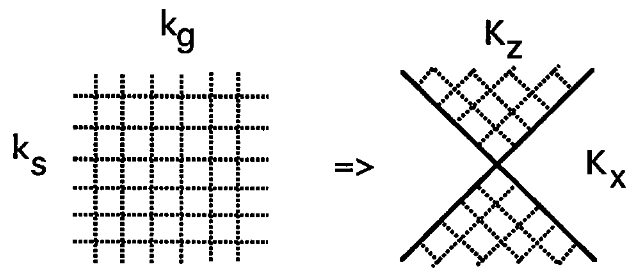

Fig. 8.. The coverage from (k_S,k_g) space to (K_X,K_Z) space

It is convenient to perform integration of (17) in the polar coordinates. Let $r^2=(k_s-k_g)^2+(k_s+k_g)^2$, $k_s=\rho\cos\theta$ and $k_g=\rho\sin\theta$, this leads to $r^2=2\rho^2$. Therefor,

$$\delta(k_s-k_g,k_s+k_g)=\frac{1}{2\pi\rho}\delta(\rho) \qquad (18)$$

substitute (18) into (17)

$$[\frac{\delta(k)}{(c_0/c(y))^2}\frac{e^{-ik\cdot y}}{4\gamma_s\gamma_g}e^{i(\gamma_s d_s+\gamma_g d_g)}]*[\frac{e^{-i(k_x y_x-k_z y_z)}}{4\gamma_s\gamma_g}e^{-(i\gamma_s d_s+\gamma_g d_g)}]$$
$$=\frac{1}{16(c_0/c(y))^2}\iint\rho'\,d\rho'\,d\theta'\frac{(\gamma'_s\gamma'_g)^3}{k'^2(\gamma'^3_s+\gamma'^3_g)}\frac{\delta(\rho')}{2\pi\rho'}\times \tag{19}$$
$$\frac{e^{i(\gamma'_s d_s+\gamma'_g d_g)}}{\gamma'_s\gamma'_g}e^{-ik'\cdot y}\frac{e^{-i(\gamma_s^- d_s+\gamma_g^- d_g)}}{\gamma_s^-\gamma_g^-}e^{i(k_x^- y_x-k_z^- y_z)}$$

Notice that when $\rho'=0$, $\gamma'_s=\gamma'_g=k$, $\gamma'_s=\gamma'_g=k$, and $\gamma_s^-=\gamma_s$, $\gamma_g^-=\gamma_g$. As a result

$$[\frac{\delta(k)}{(c_0/c(y))^2}\frac{e^{-ik\cdot y}}{4}e^{i(\gamma_s d_s+\gamma_g d_g)}]*[\frac{e^{-i(k_x y_x-k_z y_z)}}{4}e^{-(i\gamma_s d_s+\gamma_g d_g)}]$$
$$=\frac{k}{32(c_0/c(y))^2}\frac{e^{ik(d_s+d_g)}e^{-(i\gamma_s d_s+\gamma_g d_g)}}{\gamma_s\gamma_g}e^{i(k_x y_x-k_z y_z)} \tag{20}$$

Let the filtered data be denoted by

$$D(k_s,k_g)=32\frac{\gamma_s\gamma_g}{k}e^{-ik\,(d_s+d_g)}e^{+i(\gamma_s d_s+\gamma_g d_g)}F\{u^{sc}(s,g)u^{ref*}(s,g)\}(k_s,k_g)$$

The spectrum of the composite object function is then

$$O(k_x,-k_z)=\int dxdz\frac{f(y)}{(v_0/v(y))^2}e^{i(k_x x-k_z z)}=D(k_s,k_g) \tag{21}$$

Take inverse Fourier transform of (21), we get the reconstruction

$$O(x,z))=\int D(k_s,k_g)e^{-i(k_x x-k_z z)}|J|dk_s dk_g \tag{22}$$

The object function is therefore

$$f(x,z)=O(x,z)(c_0/c(x,z))^2 \tag{23}$$

where J is Jacobean transformation.

CONCLUSIONS

Assume that the amplitude is only the function of the length of the ray path, in the situation of relative weak variation of the background, an inversion method can be established based on a modified scattering model. The algorithm is the same as that of Fourier diffraction tomography in a constant host medium. The approach however, differs from the conventional diffraction tomography in that instead of the scattering field itself, the image is reconstructed from a holographic field which is the multiplication of the scattering and its conjugate background fields. The reconstruction is easy to be implemented and the computation is efficient.

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