# PAPER I

# SIMULTANEOUS ITERATIVE TRANSFORM TOMOGRAPHY
## (Generalized String Inversion)

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## ABSTRACT

I present the theoretical basis for a new method of traveltime inversion based on a semi-continuous representation of the slowness field. The inversion can be called a pseudo-transform method of reconstruction. I use an iterative solver to invert the pseudo-transform, thus the name iterative transform tomography. When discretized and numerically implemented, iterative transform tomography resembles a finely gridded series-expansion method of reconstruction such as ART or SIRT. Iterative transform tomography captures many of the benefits of both transform inversion methods and finite series-expansion methods. It is demonstrated to be a fast, robust, and flexible method for traveltime tomography. This paper presents the reconstruction theory and the description of a special case known as string tomography.

## INTRODUCTION

In a recent review paper (Lewitt, 1983) on reconstruction algorithms, transform reconstruction methods were characterized as having three main steps:

1. formulate a mathematical model in which the known and unknown are functions whose arguments come from a continuum of real numbers;
2. derive an inversion formula for the unknown continuous function;
3. adapt the inversion formula to discrete and noisy data.

In contrast, in series-expansion algorithms (Censor, 1983) the mathematical model is discretized at the beginning with a finite set of unknowns (basis functions) representing the image. Moreover, the most popular choice of basis function is the orthogonal pixel of constant slowness; therefore, the unknowns are the actual image values in homogeneous regions, thus inseparably mixing inversion and display. This choice of parameterization of the image provides a clear mathematical and physical understanding of the inversion model but can yield unsightly sharp boundaries in the reconstruction. Furthermore, the approximation of smoothly varying images is expensive in the sense that a large number of these pixels may be necessary to obtain an adequate representation. A "high resolution" result demands many model parameters, thus burdening tomography with inversion of a large and sparse projection matrix. In practice, resolution (the number of pixels N) is traded off against the expense of inversion.

It's readily seen that transform methods clearly separate inversion (step 2) from image discretization (step 3). Although the series-expansion approach is not restricted to orthogonal pixels, when used as often is the case, such orthogonal pixels combine two conflicting objectives - parameterization for inversion purposes and discretization for display purposes. Transform methods are very popular in the medical world due primarily to their speed, accuracy of reconstruction, and the conformance of the data to the requirements of the computer algorithms. Series-expansion methods are far more popular in geophysics, especially for crosswell tomography, primarily because of their flexibility in handling curved ray paths and their adaptability to noisy data taken from limited apertures. Despite these obvious advantages of series-expansion or algebraic methods in geophysics, transform (or pseudo-transform) methods may have powerful complementary benefits.

In this paper, I describe an inversion method which exploits the merits of both transform and series-expansion methods. The inversion algorithm, called iterative transform tomography, is performed by a pseudo-transform method which is adapted to curved ray paths and irregularly sampled acquired from limited-view geophysical apertures. The method is developed specifically for crosswell seismic traveltime tomography applications but is applicable to other geophysical inversion problems as well. The paper is divided into two major sections. Mathematical models for reconstruction are formulated in the first section; this includes a sub-section on the discrete model and series expansion reconstruction and a sub-section on the semi-continuous models and the new pseudo-transform reconstruction. In the second section, the theory is developed into a geophysical algorithm with two dimensional ray tracing for applications to crosswell seismic tomography.

## THEORY AND FORMULATION

Let S(x, z) represent the 2D continuous slowness field (inverse velocity). The problem is to estimate S(x, z) from a finite set of observed traveltimes $\{t_i\}$, $1 \le i \le M$. Two models are required to pose the inversion problem: (1) a mathematical model which captures albeit approximately the physical process of generating the observed traveltimes; and (2) a measurement model which describes the physical process of seismic wave propagation in heterogeneous media, i.e., the slowness field, which produced the observed traveltimes.

### Mathematical model

A projection through S(x, z) about a line connecting the source and receiver points is defined to be the traveltime function $\tau(r, s)$. The traveltime is a continuous function of the pair-variables (r, s), where "r" and "s" denote the receiver and source position, respectively. Mathematically, the values of $\tau(r, s)$ are assumed to be given by a two dimensional surface integral through the slowness field, where the support of this integral is the beam path denoted by B(S; r, s; x, z). The beam path depends on the slowness field and represents the area of the slowness field influencing the traveltime observed by the source-receiver pair (Michelena and Harris, 1991). Mathematically, the continuous beam equation is

$$\tau(r, s) = \int_\Omega B(S; r, s; x, z) S(x, z)\, dxdz. \tag{1}$$

An equivalent model for discrete traveltimes is

$$\tau_i = \int_\Omega B_i(S; x, z)\, S(x, z)\, dxdz, \qquad 1 \le i \le M \tag{2}$$

where subscript "i" denotes the ith source-receiver pair. For convenience, I will sometimes use the notation $\tau_i = B_i S$ instead of Eqn. (2) to describe a traveltime value calculated for the ith source-receiver pair. For mathematical purposes that will be discussed in detail in the following sections, it is necessary to assume that the slowness field has finite support, i.e., S(x, z) = 0, if (x,z) is outside the region $\Omega$, and square integrable, i.e., $\int_\Omega [S(x, z)]^2\, dxdz$ exists. If S(x, z) has these properties then S(x, z) will be called an image. Eqn. (1) and Eqn. (2) define the "mathematical model." Clearly, it is not the only choice possible and as will be discussed below this beam equation is often replaced with the "ray" equation. A test of validity is the ability of the mathematical model to adequately describe the observations.

Measurement model

The measurement system which produced the observed traveltime "$t_i$" is completely unaware of our mathematical model. For purposes of description only, I assume that a finite set of functionals, $F_1, F_2, ... F_M$, exists; each assigns to any real world slowness image a real number $t_i = F_iS$ equaling the observed traveltime $t_i$ for $1 \leq i \leq M$. Just as Eqn. (1) is called the mathematical model, the finite sequence $F_1, F_2, ... F_M$ is called the "measurement model." If the calculated traveltime $\tau_i$ approximately equals the measured traveltime $t_i$, then the mathematical model adequately describes the physical process for purposes of inversion. In practice, however, due to measurement error and physical effects not captured by Eqn. (1), the calculated traveltimes will only approximately equal the measured traveltimes. Further, for sake of the development of the inversion theory, I must explicitly assume that the functional $F_i$ describing the measurement is linear and continuous. That is, $F_i(S_1 + S_2) = F_iS_1 + F_iS_2$ and if $S_1(x, z) \approx S_2(x, z)$ for all (x, z), then $F_iS_1 \approx F_iS_2$. The importance of these assumptions is discussed in the following section.

In the next sub-section, I will introduce two alternative reconstruction models. The *discrete model* describes the slowness field with a finite set of numbers, e.g., pixel values, spline coefficients, etc. The discrete model is common to series-expansion reconstructive algorithms. The *semi-continuous model* describes the slowness field as a continuous function of position estimated from discrete data, thus the name. The semi-continuous model is the basis for the subject of this paper - iterative transform reconstructive algorithms.

Discrete Series-Expansion Reconstruction

Suppose S(x, z) is an image parameterized by the vector $a_1, a_2, ... a_N$, and observed traveltimes $\{t_i\}$are known for $1 \leq i \leq M$. Find the constants $\{a_i\}$. This is a *discrete series-expansion reconstruction problem.* Parameterizing the slowness field by a discrete set of values first requires definition of a set of basis functions $\{\varphi_j(x, z)\}$, $1 \leq j \leq N$ whose linear combination give an adequate approximation, e.g., $\hat{S}(x, z) \approx S(x, z)$, to the true slowness. I am free to choose the basis functions to suit convenience or other criteria. In effect, each basis function $\varphi_j(x, z)$ is itself an image according the definition given above, i.e, compact support and square integrable. For example, a common choice is the orthogonal set of constant pixel functions: $\varphi_j(x, z) = 1$ inside a small rectangular region called a pixel and $\varphi_j(x, z) = 0$ outside. Thus for any image constructed with this model, there exist real numbers $a_1, a_2, \ldots a_N$ such that

$$S(x, z) \approx \widehat{S}(x, z) \equiv \sum_{j=1}^{N} a_j \varphi_j(x, z) \tag{3}$$

The $a_j$'s in Eqn. (3) form a finite set which parameterize the slowness, i.e., a series-expansion. Linearity of the measurement model leads to an equation relating the measured traveltimes for this approximate image to the constants describing the slowness, that is

$$F_i\widehat{S}(x, z) = \sum_{j=1}^{N} a_j F_i \varphi_j \qquad 1 \le i \le M. \tag{4}$$

Hence, for purposes of the series-expansion reconstruction, the finite set of numbers $F_i\varphi_j$ over which the sum in Eqn. (4) is taken may describe the measurement model sufficiently. In practice, however, because the exact nature of $F_i$ is not known, one is forced to replace $F_i\varphi_j$ with an estimate $W_{ij}$ computed say from the mathematical model Eqn. (1):

$$F_i\varphi_j \approx W_{ij} = \int_\Omega B_i(S; x, z)\, \varphi_j(x, z)\, dxdz, \tag{5}$$

where $B_i$ is the beam operator defined by Eqn. (2). The value $W_{ij}$ is "calculated" for i<u>th</u> source-receiver pair and the j<u>th</u> basis image $\varphi_j(x, z)$, whereas $F_i\varphi_j$ is "measured" through $\varphi_j(x, z)$. When $\varphi_j(x, z)$ is the orthogonal pixel of constant unit slowness, $W_{ij}$ is simply the area of the i<u>th</u> beam intersecting the j<u>th</u> pixel. Substituting (5) into (4) gives

$$\hat{t}_i = \sum_{j=1}^{N} W_{ij}\, a_j \qquad 1 \le i \le M. \tag{6}$$

where $\hat{t}_i$ is the measured traveltime in the approximate image $\widehat{S}(x, z)$. Eqn. (6) nearly achieves the intermediate goal of discrete inversion, that is, to find a set of relationships between the observed traveltimes $\{t_i\}$ to the coefficients $\{a_j\}$. Unfortunately the left side of (6) is not exactly the known traveltimes but can be approximated by such, i.e., $\hat{t}_i \approx t_i$.

Combining all our approximations to this point yields the matrix system to be solved for the unknown constants:

$$\mathbf{t} = \mathbf{W}\mathbf{a} + \mathbf{e}, \tag{7}$$

where **t** is the (Mx1) column vector of observed traveltimes, **W** is the (MxN) projection matrix whose elements are defined in Eqn. (5), **a** is the (Nx1) column vector of unknown coefficients which parameterize the slowness field, and **e** is the vector of errors due to the many approximations such as the inaccuracies in measurements $\hat{t}_i \approx t_i$, the approximate nature of the series expansion $\hat{S} \approx S$, and the approximation for the projection matrix $W_{ij} \approx F_i\varphi_j$. The derivation leading to Eqn. (7) clearly identifies these approximations.

A rich literature exists on the subject of solving Eqn. (7) for the constants $\{a_j\}$. See Censor (183). While I will not address the general aspect of the problem here, I present here the solution based on well-known algebraic reconstruction techniques:

$$a_j^{(i+1)} = a_j^{(i)} + \frac{t_i - \tau_i}{\sum_{k=1}^{K} W_{ik}^2} W_{ij} \qquad \text{D-ART} \tag{8}$$

$$a_j^{(k+1)} = a_j^{(k)} + \frac{1}{M} \sum_{i=1}^{M} \frac{t_i - \tau_i}{\sum_{k=1}^{K} W_{ik}^2} W_{ij} \qquad \text{D-SIRT} \tag{9}$$

Eqns. (8) and (9) iteratively yield the coefficients discretely parameterizing the slowness image. In ART, the coefficients are updated sequentially with each projections. In SIRT, the coefficients are updated once for a set of projections whose number is denoted here by L. I refer to these as discrete (D) algebraic techniques. i.e., D-ART and D-SIRT. For more details, see Haykin (1985). In geophysics, especially for cross-well tomography, the most popular choice of basis function is the orthogonal pixel of constant slowness, i.e., $\varphi_j(x, z) = 1$ if (x, z) is inside the jth pixel and $\varphi_j(x, z) = 0$ otherwise; therefore $a_j$ represents the actual slowness in that pixel. This choice of basis functions provides a clear mathematical and physical understanding of the inversion model but can yield unsightly sharp boundaries in the reconstructed image. Furthermore, the approximation of smoothly varying images is expensive in the sense that a large number of these pixels may be necessary to obtain an adequate representation. A "high resolution" result demands many model parameters, thus burdening the inversion procedure with a large and sparse projection matrix **W**. In practice, resolution (the number of pixels N) is traded off against the expense of inversion. Although the series-expansion approach is not restricted to just orthogonal pixels, when used as often is the case, such orthogonal pixels combine two conflicting objectives - parameterization for inversion purposes and discretization for display purposes. In the next sub-section, I will introduce a method for avoiding this conflict.

Semi-Continuous Reconstruction

First, a statement of the *"continuous"* reconstruction problem: Suppose S(x, z) is an image and traveltimes t(r, s) are known for continuous values of source and receiver locations, find an estimate of S(x, z). This is a statement of the *continuous reconstruction problem*, that is, S(x, z) is not pixelated into a fixed number of unknowns as in the discrete reconstruction described above. Instead, a continuous estimate of S(x, z) is sought. More often than not, transform methods are used for reconstruction problems of this type. Though well-posed in general, this continuous formulation is unrealistic for geophysical applications. First, it assumes that the traveltimes are known for continuous source and receiver positions. And, second, it ignores the limited aperture effects which severely restrict the number and location of sources and receivers. Whereas the problem of limited aperture is not the subject of this paper and will not be addressed, the handling of discrete data is essential, thus demanding the semi-continuous model described below.

A more realistic reconstruction model uses discrete data but still seeks to form a continuous estimate of S(x, z), thus defining the the *"semi-continuous"* problem: Suppose S(x, z) is an image and discrete traveltimes $t_i$ are known for $1 \leq i \leq M$, find an estimate of the continuous image S(x, z) satisfying all observed traveltimes. Combining discrete observations $\{t_i\}$ with a continuous model for S(x, z) leads to the semi-continuous model. In contrast , the discrete model given by Eqn. (7) combines both discrete data with a discrete slowness representation and the continuous model combines continuous data with a continuous slowness representation.

A solution method known for the semi-continuous reconstruction problem is described in Appendix A, where an iterative solution similar to conventional ART or SIRT is presented. The solution can be summarized in two equations, from (A-3) and (A-5), which iteratively give an estimate for the continuous slowness:

$$\hat{S}^{(i+1)}(x, z) = \hat{S}^{(i)}(x, z) + \frac{t_i - \tau_i}{A_i} B_i(x, z) \qquad \text{SC-ART} \qquad (10)$$

$$\hat{S}^{(k+1)}(x, z) = \hat{S}^{(k)}(x, z) + \frac{1}{M_B} \sum_{i=1}^{M} \frac{t_i - \tau_i}{A_i} B_i(x, z) \qquad \text{SC-SIRT} \qquad (11)$$

where in Eqn. (11), $M_B$ is the number of beams touching the image point (x,z). Eqns. (10) and (11) describe iterative semi-continuous (SC) ART and SIRT. Unlike their counterparts (Eqns. (8) and (9)), however, these versions define solutions for the continuous slowness field itself not the coefficients of a discrete paramerization model, e.g, $\{a_j\}$. The constant $Ai = \int_\Omega B_i(x, z)\, dxdz$, where $B_i(x, z)$ functionally describes the beam path itself, i.e., $B_i$ = 1 on the ith beam path, $B_i = 0$ otherwise. Either of these equations resemble transform reconstruction methods more than series expansion methods: they estimate the slowness as a continuous function of space. Indeed, solutions of this type have been called iterative transform reconstructions (Hermann and Lent, 1974). I refer to Eqn. (11) as Simultaneous Iterative Transform Tomography.

The differences between the semi-continuous reconstructions and the discrete reconstruction become blurred when the SC-model is discretized for numerical solution or when the pixel size in the D-model is shrunk to zero. At that point, both become finely gridded iterative solutions. A numerical algorithm implementing the SC-SIRT for cross-well tomography is presented in the next section.

## CROSSWELL SEISMIC TOMOGRAPHY

The formulations summarized above are generally useful in their stated form for linear inversion problems, where the beam path is known and inversion is performed to obtain the slowness field. Seismic applications are often highly nonlinear in that the beam path depends on the slowness and both are unknown. That is, the variations in slowness are large enough that refraction is significant thus rendering the beam path as well as the slowness unknown. This dependence is easily seen in Eqns. (1) and (2) where the beam path is shown to depend explicitly on S. The usual and well known procedure of addressing this nonlinearity is to linearize Eqn. (1) by assuming that the slowness can be decomposed into a known component $S_o(x, z)$ and an unknown perturbation $\delta S(x, z)$, i.e.,

$$S(x, z) = S_o(x, z) + \delta S(x, z). \qquad (12)$$

Fermat's principle is then invoked in order to produce

$$\delta\tau(r, s) = \int_{\Omega} B(S_o; r, s; x, z)\delta S(x, z)\, dxdz\,, \tag{13}$$

where $\delta\tau = \tau - \tau_o$ is the difference between the total traveltime in the unknown slowness model S and the traveltime $\tau_o$ for the known slowness model $S_o$. The beam path for all traveltime calculations is found for the known (background) slowness field. For inversion purposes, Eqns. (12) and (13) define a linear mathematical model between the slowness perturbation and the traveltime perturbation. This model relies on Fermat's principle to argue explicitly that the traveltime computed in the slowness field S is not too different from the traveltime calculated in slowness field $S_o$ provided $\delta S$ is small. Inversion is actually performed for for the perturbation, usually within an iterative numerical scheme of assuming $S_o$, calculating residual traveltimes, solving for $\delta S$, then updating the slowness as $S_o$ for input to the next iteration. In most practical situations, convergence of the scheme is measured by the smallness of the traveltime perturbation $\delta\tau$ and/or the smallness of changes in the slowness.

String Tomography

A special case of the iterative transform method is the string algorithm, designed here for seismic tomography. The string algorithm uses conventional ray tracing rather than more general beam tracing. For geophysical applications, SIRT methods appear to be more robust than ART with respect to noise such as picking errors and limited view. For this reason, only the SC-SIRT algorithm of Eqn. (16) has been implemented to date; that is, model updates are performed for all projections (traveltimes) at once. The general string algorithm, schematically illustrated in Figure 1, features unlinked ray tracing and interpolation of the measured traveltime data.

The ray equation replaces the beam equation Eqn. (1):

$$\tau_i = \int_{L_i} S_o(x, z)\, dl, \qquad 1 \leq i \leq M \tag{15a}$$

Or,

$$\tau_i = \int_{\Omega} S_o(x, z)\, \phi_i(x, z)\, dxdz, \qquad 1 \leq i \leq M \tag{15b}$$

where $\phi_i(x, z)$ defines the ray path in two dimensions, i.e., $\phi_i = 1$ on the ray path and $\phi_i = 0$ otherwise. A simple numerical scheme solving the eikonal equation is used to find $\phi_i(x, z)$.

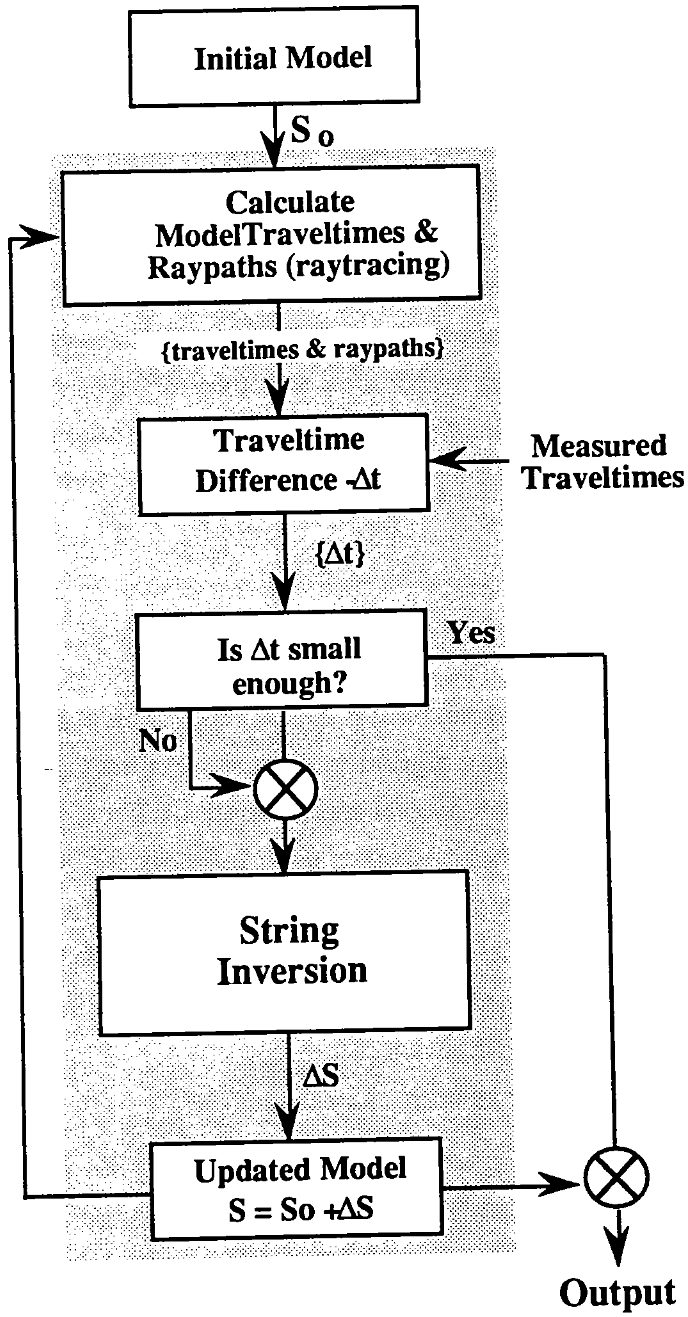


Figure 1. Schematic outline of the string inversion algorithm

The numerical scheme is a standard Runge-Kutta integration (McMechan, et, al 1987). but is specially implemented to run on a scalar workstation. Rays are not linked between source and receiver; instead, a common-source gather of traveltimes is computed by launching a sequence of rays from the source toward the receiver borehole. At the receiver well, a set of pseudo-receivers intercept these rays thus generating synthetic traveltimes at numerous points along the receiver wellbore. These synthetic traveltimes are then compared to the measured traveltimes to produce the SC-SIRT correction $\delta\widehat{S}$ to the background model:

$$\delta\widehat{S}^{(k+1)}(x, z) = \frac{1}{M_\phi} \sum_{i=1}^{M} \frac{t_i - \tau_i}{R_i} \phi_i(x, z), \qquad (16)$$

where $R_i$ , $\tau_i$ ,and $\phi_i$ , are the effective length, traveltime, and path function, respectively, of the i<u>th</u> ray in the k<u>th</u> model and $M_\phi$ is the number of rays touching the image point at (x,z). While Eqn. (16) expresses only one correction per ray trace iteration, it is possible to perform several iterations thus refining the estimate for $\delta\widehat{S}$ between ray tracings.

Recall that rays are not linked between the sources and receivers. Therefore, in order to compare the synthetic with the measured traveltimes, the measured values must be estimated at the pseudo-receiver locations. To accomplish this, a numerical wavefront is fit to the actual measured traveltimes as a function of depth and a "measured" traveltime is estimated for each of the synthetic pseudo-receivers. This procedure relies on the fact that the measured traveltimes are adequately sampled to reconstruct the wavefront; otherwise, important structure of S(x, z) manifested in the traveltime wavefront would be missed. In general the number of rays launched in the simulated gather is about 2 times greater than the actual number of measured traveltimes and the traveltime data used in Eqn. (16) are replaced by data generated from the interpolation equation

$$t_m = \sum_{i=m-2}^{m+2} \mathbf{I}_{i-m}\, t_i \qquad (17)$$

where $\mathbf{I}_{i-m}$ is an interpolation operator applied to the measured data $t_i$, and $t_m$ are the interpolated traveltimes. Aside from simplifying the ray tracing algorithm, the pseudo-receivers play a very important role in interpolating the image in a manner consistent with the model $S_o$. For all practical purposes, this method of traveltime interpolation creates a semi-continuous dataset thus satisfying the requirement of the semi-continuous reconstruction theory.

The next step is to add the background model $S_o$ to the estimate $\delta\hat{S}(x, z)$ and discretize the result:

$$\hat{S}(i\Delta r_x, j\Delta r_y) = S_o(i\Delta r_x, j\Delta r_y) + \frac{1}{2M_\phi} \sum_{m=1}^{2M} \frac{t_m - \tau_m}{R_m} \phi_m(i\Delta r_x, j\Delta r_y) \tag{18}$$

In effect, Eqn. (18) is the implementation of Lewitt's step (3) described in the Introduction. However, the image produced by Eqn. (18) is irregularly sampled on a grid determined by the ray coordinates. In practice, $\phi_i$ is regularly sampled along the ray path $(i\Delta r_x, j\Delta r_z)$ not a Cartesian grid. So, the final step in the overall tomography process is to regularize Eqn. (18) for display on an equi-spaced Cartesian grid:

$$\hat{S}(i\Delta x, j\Delta z) = \sum_{m=i-L}^{i+L} \sum_{n=j-L}^{j+L} B(m-i,n-j)\, \hat{S}(m\Delta r_x, n\Delta r_z) \tag{19}$$

The sample intervals $(\Delta x, \Delta z)$ and $(\Delta r_x, \Delta r_z)$ should be kept as small as practical in order to maintain consistent approximations for the semi-continuous model. Eqns. (18) and (19) complete the theoretical development of the iterative transform tomography algorithm.

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## Appendix A - Iterative Transform Inversion

Following Herman and Lent [], I will first derive a inversion formula which can be used to reconstruct S(x, z) from discrete traveltime data. This inversion is similar to algebraic reconstruction techniques, so I'll proceed with a similar derivation.

Iterative reconstruction algorithms such as ART and SIRT are well established. However, unlike conventional ART and SIRT which are based on discrete representation of the slowness as represented by (3), I seek here a continuous representation for the slowness image. There are many excellent treatments of the ART and SIRT algorithms based Kaczmarz's "method of projections", so I will not repeat those details here but will describe the geometrical interpretation of my reconstruction which differs for the semi-continuous model. The derivations rely on a Hilbert space representation as follows: The set of all continuous images $\{S_i\}$, each being defined in the sub-section on mathematical models, forms a vector space in which the inner product of any two images say $S_1(x, z)$ and $S_2(x, z)$ can be defined by

$$<S_1 \cdot S_2> = \iint_P S_1(x, z)\, S_2(x, z)\, dxdz \qquad \text{(A-1)}$$

where "P" is the entire two-dimensional Cartesian x-z plane. It is well known that with this inner product the set of all equivalent classes of images forms a Hilbert space, which I'll denote by **H**. The aim of the continuous reconstruction is to find an image, say one image $\hat{S}(x, z)$, which satisfies all M traveltimes $\{t_i\}$, $1 \le i \le M$.

Now, if one assumes that $\hat{S}(x, z)$ is a member of **H**, then the set of all images which satisfy the single traveltime $t_i$ also forms a closed subspace of **H** that I'll denote by $\mathbf{N_i}$, $1 \le i \le M$. If the image which satisfies all M traveltimes is denoted by $\mathbf{N_0}$, then clearly $\mathbf{N_0}$ is the intersection of the M closed subspaces. See Figure A-1. Evidently, $\mathbf{N_0}$ itself is a subspace of **H** and may have more than element (i.e., image). In this case, it is important to establish some additional optimally criteria. Herman and Lent [1976] point out that the minimum-norm solution to which the method of projections converges is the image in $\mathbf{N_0}$ which minimizes

$$<\hat{S} \cdot \hat{S}> = \iint_P [\hat{S}(x, z)]^2\, dxdz. \qquad \text{(A-2)}$$

Finally, the notion of orthogonal projections on a closed linear variety is required to complete the development. Solomon[] has shown that the orthogonal projection denoted by $S^{(k+1)}(x, z)$ of the image $S^{(k)}(x,z)$ onto the closed linear variety $\mathbf{N}_i$ is given by

$$\widehat{S}^{(i+1)}(x, z) = \widehat{S}^{(i)}(x, z) + \frac{t_i - \tau_i}{A_i} B_i(x, z) \tag{A-3}$$

where $t_i$ is the observed traveltime, $\tau_i$ is the calculated traveltime in the image model $\widehat{S}^{(k)}(x, z)$, $B_i(x, z)$ is the ray path, and $A_i = \int_\Omega B_i(x, z)\, dxdz$ is the characteristic of the beam path, i.e, the path integral given by Eqn. (2) with a unity slowness field everywhere. The new image $\widehat{S}^{(k+1)}(x, z)$ is continuously defined for the variables (x, z) at every point within the support of the i<u>th</u> beam path. Moreover, the sequence of images $\widehat{S}^{(0)}$, $\widehat{S}^{(1)}$, $\widehat{S}^{(2)}$, ... converges to the minimum norm element of $\mathbf{N}_0$. This says that if S(x, z) is the unique image which among all images to satisfy the traveltimes which minimizes (A-2), then

$$\lim_{k \to \infty} \iint_P [\widehat{S}^{(k)}(x,z) - S(x, z)]^2\, dxdz \to 0 , \tag{A-4}$$

See Figure A-1 for a geometric interpretation of this convergence for M = 2.

The similarity of Eqn. (A-3) to conventional discrete ART is of little surprise. This iterative solution might be called semi-continuous or SC-ART. Unlike conventional ART, the iterations for SC-ART produce the slowness image directly not the model coefficients parameterizing the slowness. In this sense, Eqn. (A-3) is a transform method. The method avoids discretization of the slowness field at the beginning as conventional algebraic methods do. For practical use, Eqn. (A-3) must be discretized just as any other transform inversion. Inversion according to Eqn. (A-3) is referred to as iterative transform tomography. Algorithms implementing iterative transform tomography deal with one equation at a time and their convergence is assured (Herman and Lent, 1976). It is possible, however, to alter the algorithms for a small subset of the equations or all equations simultaneously, e.g., a semi-continuous SC-SIRT method which is discussed in a later section.

$$\widehat{S}^{(k+1)}(x, z) = \widehat{S}^{(k)}(x, z) + \frac{1}{M_B} \sum_{i=1}^{M} \frac{t_i - \tau_i}{A_i} B_i(x, z), \tag{A-5}$$

where $M_B$ is the number of beams touching the image point at (x,z).

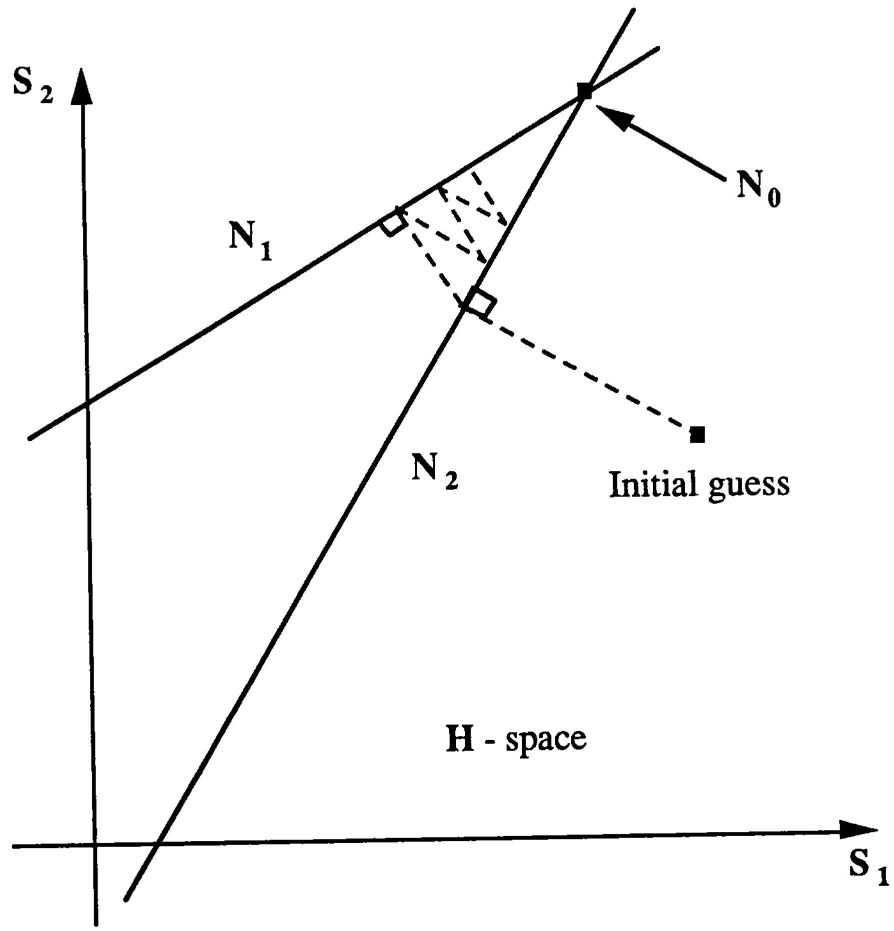


Figure A-1. Geometric interpretation of Hilbert space convergence of SC-ART algorithm for simple case of M = 2. $N_1$ and $N_2$ are closed linear varieties in the Hilbert space of possible images satisfying traveltimes $t_1$ and $t_2$, respectively. $N_0$ is the closed linear variety of images satisfying both $t_1$ and $t_2$.