

DISCRIMINATION BETWEEN RESERVOIR MODELS
IN WELL TEST ANALYSIS

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DOCTOR OF PHILOSOPHY

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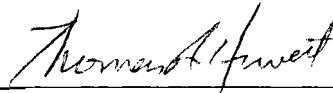
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Abstract

Uncertainty involved in estimating reservoir parameters from a well test interpretation originates from the fact that different reservoir models may appear to match the pressure data equally well. A successful well test analysis requires the selection of the most appropriate model to represent the reservoir behavior. This step is now performed by graphical analysis using the pressure derivative plot and confidence intervals. The selection by graphical analysis is influenced by human bias and, as a result, the result may vary according to the interpreter. Confidence intervals can provide a quantitative evaluation of the adequacy of a single model but is less useful to discriminate between models.

This study describes a new quantitative method, the sequential predictive probability method, to discriminate between candidate reservoir models. This method was originally proposed in the field of applied statistics to construct an effective experimental design and is modified in this study for effective use in model discrimination in well test analysis. This method is based on Bayesian inference, in which all information about the reservoir model and, subsequently, the reservoir parameters deduced from well test data are expressed in terms of probability.

The sequential predictive probability method provides a unified measure of model discrimination regardless of the number of the parameters in reservoir models and can compare any number of reservoir models simultaneously.

Eight fundamental reservoir models, which are the infinite acting model, the sealing fault model, the no flow outer boundary model, the constant pressure outer boundary model, the double porosity model, the double porosity and sealing fault model, the double porosity and no flow outer boundary model, and the double porosity and

constant pressure outer boundary model, were employed in this study and the utility of the sequential predictive probability method for simulated and actual field well test data was investigated.

The sequential predictive probability method was found to successfully discriminate between these models, even in cases where neither graphical analysis nor confidence intervals would work.

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This dissertation is dedicated to the memory of Professor Henry J. Ramey, Jr.

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Chapter 1

Introduction

1.1 Introduction

One of the primary objectives of petroleum engineers is concerned with the optimization of ultimate recovery from oil and gas reservoirs. In order to develop and produce oil and gas reservoirs and forecast their future reservoir performance, it is important to attain accurate reservoir descriptions.

Information about reservoir properties can be obtained from different sources such as geological data, seismic data, well logging data, core measurement data, and well test data. Well test data include valuable information on the dynamic behavior of reservoirs.

It is essential to incorporate all sources of information for a successful description of a reservoir. However, it is a relatively difficult task to integrate all sources of information quantitatively, since these sources of information have different resolutions. For example, permeabilities estimated from core measurements represent local values where the cores are obtained, while the permeability deduced from well test data is an average value over a specific volume near the wellbore.

In recent years, Deutsch (1992) developed a new methodology to integrate geological data with well test data using a simulated annealing technique. The permeability estimated from well test data is used as a constraint for the possible spatial distribution of elementary grid block permeability values near the wellbore. One aspect

of this technique is that the average permeability estimated by well test analysis is assumed to be a true value without any uncertainty. Theoretically, it is possible to relax this limitation if uncertainty can be expressed in a quantitative manner.

In practice, the unknown reservoir parameters estimated from well test data inherently contain uncertainty. Therefore, uncertainty involved in the estimated parameters needs to be expressed quantitatively to be combined with other sources of information. In general, probability distributions can be used to represent uncertainty quantitatively.

Before evaluating the uncertainty involved in the estimated parameters, it is necessary to reduce uncertainty by performing a successful well test interpretation. Successful well test analysis requires the selection of the most appropriate model to represent the reservoir behavior.

Up to now, confidence intervals suggested by Dogru, Dixon and Edgar (1977) and Rosa and Horne (1983) have been useful tools to provide a quantitative evaluation of the uncertainty involved in the estimated parameters in well test analysis. However, confidence intervals have been derived in the framework of sampling theory inference and the uncertainty involved in the estimated parameters evaluated using confidence intervals cannot be integrated quantitatively with other sources of information, since the uncertainty is not expressed in terms of probability. Hence, it is necessary to find a method to express the uncertainty involved in the estimated parameters in well test analysis in terms of probability in order to incorporate with other sources of information for a successful description of the reservoir.

Confidence intervals are also used to determine quantitatively whether the model is acceptable or not. However, it should be mentioned that determining the model appropriateness is inherently different from selecting the most appropriate model. Although confidence intervals have been found to be useful in providing a quantitative evaluation of whether a specific model is acceptable or not, they cannot be applied directly to discriminate between different models. Up to now, there is no standard procedure available for model discrimination in well test analysis.

Therefore, the main objective of this work is to find a method to express uncertainty in well test analysis in terms of probability and to develop a new quantitative

method to discriminate between possible reservoir models.

1.2 Previous Work

The problem of selecting the most appropriate model has been studied extensively in many fields in engineering, applied science, and economics. However, there is no unique statistical procedure available for selecting the most appropriate model. The main reason is that the following two conflicting qualitative criteria are involved:

1. The model should include as many parameters as possible to express the current data sufficiently, since the representation of the current data can generally be improved by adding more parameters.
2. The model should include as few parameters as possible to predict the future responses accurately, since in general the variances of the predictions increase with the number of parameters.

Hence, a suitable compromise between these two extremes is necessary for selecting the best model. The compromise should be decided quantitatively depending on the purpose of the study and the nature of the models compared. The nature of the models and the related considerations are categorized from the following points of view:

1. Is the model used to express the current data or to predict the future response?
In cases where the model is used to explain the current data as well as possible, all of the parameters which may have any contributions to the matches of the model to the data should be included in the model. On the other hand, in cases where the model is used to predict the future response as accurately as possible, any of the parameters which may degenerate the accuracy of the future response moderately should be excluded from the model. The parameters estimated from well test data are used to predict the future reservoir performance, and in well test analysis the model should be selected by taking account of the future prediction as well as the current data.

2. Is the model linear or nonlinear with respect to the parameters? Statistical inference techniques have been investigated extensively in a linear model framework. In cases where the model is nonlinear, an adequate linear approximation should be employed to apply the wealth of results from the linear model framework to the nonlinear model. In well test analysis, the reservoir models are generally nonlinear.
3. Are the parameters independent of one another or not? In cases where two parameters are completely correlated with each other, which means that the two parameters are not independent, it is sufficient to include just one of the parameters in the model, since the additional contribution of the other parameter to the model is negligible. In well test analysis, there is no physical reason to believe that the reservoir parameters such as permeability, skin, distance to the boundary and so on are correlated.
4. Is one model a subset of another or not? In cases where one model is a special case of another, it is possible to compare two models directly using an F test. The F test provides a conclusion as to whether additional parameters are necessary in the model or not. In cases where two models are not nested, these two models cannot be compared by the F test. In well test analysis, some reservoir models are nested and the others not.
5. Is the total number of data fixed or not? In cases where the total number of data is not fixed and it is possible to select data points to facilitate model selection, it is necessary to construct an effective experimental design. This problem is known as the optimal design problem. Many techniques have been proposed for the optimal design problem. Box (1968) showed that if a sequence of n experiments is designed to estimate m parameters, then an optimal design is usually obtained when the m best experiments are each replicated n/m times. However, in general the total number of well test data is fixed and it is not feasible to replicate well testing due to the expense.

6. Is the form of the error distribution known or not? The error is the difference between the actual pressure response and the true pressure response. In cases where the form of error distribution is known, it is much easier to select the most appropriate model. This information greatly reduces the uncertainty involved in the parameter estimates. However, the form of the error distribution is generally unknown in well test analysis.
7. Which statistical inference technique is used, sampling theory inference or Bayesian inference? Both inferences have their own advantages and disadvantages. Sampling theory inference can handle nonlinear models without any linear approximations, while Bayesian inference requires the nonlinear model to be approximated with a linear form. Therefore, in order to obtain exact confidence regions of the parameter estimates of the nonlinear model sampling theory inference should be employed, since Bayesian inference produces only approximate results in the case of nonlinear models. However, due to theoretical reasons, Bayesian inference can express uncertainty involved in the estimated parameters in terms of probability, while sampling theory inference cannot. Hence, Bayesian inference can incorporate other sources of information with the information obtained from well test analysis.

The importance and the difficulty in selecting the most appropriate model in well test analysis have been recognized widely. This section reviews the history of the model verification problem in the context of well test analysis.

Padmanabhan and Woo (1976) and Padmanabhan (1979) demonstrated the use of the covariance matrix as a means of evaluating the quality of the matches of the model to the well test data. The idea of a sequential approach was proposed. This idea is expressed as follows: starting from some prior information, which represents the initial estimates of the reservoir parameters, the measurements are taken one at a time in chronological order and used to improve the estimates in accordance with a learning algorithm. The updated estimate then serves as the prior information when the next measurement is to be processed. As long as the model is correct, the sequence of updated estimates will eventually converge to the true values of the

parameters. Observing how the covariance matrix, which represents a measure of the uncertainty involved in the parameter estimates, changes during this sequential procedure, provides information on how accurate the current estimates are. If some parameters are insensitive, which means that during the updating procedure the variances corresponding to the parameters do not change, one may conclude that these parameters need not be added to the model.

This procedure is quite attractive and provides some ideas about model adequacy, but no quantitative criteria are available to decide whether some parameters are insensitive or not. Therefore, the results are subjective and may be different among interpreters. This procedure also provides a useful idea about the treatment of the data. The total number of data is originally fixed but the number of data may be regarded as flexible by the chronological selection. This idea will be used effectively in a new method for model discrimination developed in this work.

Dogru, Dixon and Edgar (1977) demonstrated the idea of confidence intervals on parameter estimates. Confidence intervals for the estimated parameters and calculated pressures were presented using nonlinear regression theory. The model employed in the study was assumed to be true in advance, since the objective of the study was not to find the most appropriate model but to do the sensitivity analysis, which determines which parameters are sensitive or not to the measurement errors involved in well test data. Dogru, Dixon and Edgar (1977) also presented confidence intervals on future prediction of pressures based on a fixed number of history matching data. Dogru, Dixon and Edgar (1977) studied how uncertainty involved in the parameter estimates affects the future predictions. The idea of confidence intervals on future prediction pressures will be also used in a method for model discrimination developed in this work.

Rosa and Horne (1983) showed that by numerical inversion from Laplace space of not only the pressure change but also its partial derivatives with respect to the reservoir parameters, it is possible to perform nonlinear regression. Rosa and Horne (1983) also showed that confidence intervals can be calculated using the same nonlinear regression technique and proposed the use of confidence intervals to determine how well the reservoir parameters are estimated.

Barua (1984) proposed a scaling technique which makes parameters with different magnitudes comparable. In practice, confidence intervals are scaled by dividing by the estimated values of the parameters. The scaled confidence intervals are called relative confidence intervals.

Abbaszadeh and Kamal (1988) reviewed a general technique for automated type curve matching of well test data, in which confidence intervals and correlation coefficients were included as statistical methods to provide information on the reliability of the results obtained by nonlinear regression techniques.

In order to make it convenient to use confidence intervals as criteria to decide whether a model is acceptable or not, Horne (1990) defined acceptable confidence intervals for common reservoir parameters. These criteria of acceptability were defined heuristically, based on actual experience with interpretation of real and synthetic well test data.

Ramey (1992) demonstrated the importance of confidence intervals as a quantitative measure of quality of the results. Ramey (1992) compared the results obtained from a Horner method with those from a nonlinear regression technique using confidence intervals, and showed how the confidence intervals could be used to reveal inadequacies in the Horner method.

Horne (1992) made a review of the practical applications of computer-aided well test interpretation with specific attention to confidence intervals.

As long as two possible reservoir models are nested, which means one model can be expressed as a subset of the other model, it is possible to compare two models directly using an F test, as was proposed by Watson *et al.* (1988). The usefulness of an F test was demonstrated using simulated and actual field well test data. A homogeneous reservoir model and a double porosity reservoir model were used, since the homogeneous model may be recognized as a subset of the double porosity model.

A limitation of the F test is that this method can compare only two models at a time. Furthermore, an F test cannot discriminate quantitatively between two models which are not nested. For instance, a no flow outer boundary model and a double porosity model sometimes show more or less similar pressure responses, but an F test cannot be used to compare these two models, since these two models are not nested,

Before closing this section, it should be pointed out that all of the methods discussed above have been investigated in the framework of sampling theory inference.

1.3 Problem Statement

Selection of the most appropriate model is a crucial step for a successful well test interpretation. Existing quantitative methods to discriminate between candidate models such as confidence intervals or the F test, have some limitations. Hence, evaluating the quality of match as well as discriminating between possible models is sometimes left to engineering judgement using graphical visualization. This can be dangerously misleading and the result is subjective. In addition, while new reservoir models are still being proposed, these models cannot always be used effectively in actual well test analysis due to the limitations of engineering judgement. In other words, model verification techniques have not kept up with the progress of constructing new models.

The objectives of this study are:

1. To express the quality of parameter estimates quantitatively in the framework of Bayesian inference (as opposed to sampling theory inference used by previous works).
2. To develop a new quantitative method to discriminate between possible reservoir models.
3. To investigate the utility of this method for simulated and actual field well test data.

The main objective is the development of a new quantitative method to discriminate directly between reservoir models. The method is expected to provide a unified measure of model discrimination in cases where several models are possible. The method should compare any number of models simultaneously, whether they are nested or not.

1.4 Dissertation Outline

Chapter 2 discusses the basic concepts of well test analysis. Well test analysis can be understood from two different aspects: a signal analysis problem and an inverse problem.

Chapter 3 discusses practical problems involved in well test analysis. Reservoir models employed in this study are illustrated. The limitations of graphical analysis are discussed. Several approaches using artificial intelligence are described. The basic procedures of nonlinear regression are presented. Bayesian inference is introduced. In Bayesian inference all information about the reservoir parameters is expressed in terms of probability, and uncertainty involved in the parameter estimates can be expressed quantitatively. Confidence intervals are derived in the framework of Bayesian inference, and the problems inherently involved in the application of confidence intervals for model discrimination are discussed. The F test is also examined.

Chapter 4 describes a new quantitative method for model discrimination, which is called the sequential predictive probability method. The idea was originally proposed by Box and Hill (1967) in the field of applied statistics to construct an effective experimental design and is implemented for use in model discrimination in well test analysis. This method is based on Bayesian inference. The method is a direct extension of the use of confidence intervals, yet overcomes the weak points of confidence intervals.

Chapter 5 demonstrates the utility of the sequential predictive probability method, for model discrimination in well test analysis. Various factors affecting the method are discussed. The advantages of the method over confidence interval analysis and graphical analysis are demonstrated. Application to simulated well test data and to actual field well test data is examined.

Chapter 6 concludes the principal contributions of this study and makes recommendations for future work.

Chapter 2

Well Test Analysis

This chapter discusses the basic concepts of well test analysis. The concepts described in this chapter provide the background for the methodology described in later chapters.

Section 2.1 discusses well test analysis as a signal analysis problem. The diffusive nature of the pressure response is discussed.

Section 2.2 discusses well test analysis as an inverse problem. Uncertainty is inherent in all inverse problems. The importance and difficulty of model discrimination are discussed.

2.1 Signal Analysis Problem

Well testing is performed to obtain information about unknown reservoir properties to predict the future reservoir performance. An input signal (an impulse) perturbs the reservoir and an output signal (a response) is monitored during a well test. This is a typical signal analysis problem (Gringarten, 1986).

The input signal is usually a step function change in the flow rate of a well, created either by opening it to production or closing it to shut-in, and the output signal is the corresponding change in pressure at the well.

The simplest and most frequently discussed form of the input signal is a constant rate production, which is a one-step function in the flow rate. This test is called a

drawdown test. One of the practical difficulties in a drawdown test is to maintain the flow at a fixed rate during the entire test period. Therefore, a buildup test where the well is shut-in after a constant rate production is more frequently used, since the constant flow rate condition (the flow rate is zero) is easily achieved. The input signal in a buildup test is a two-step function. In some cases, multirate flow tests where the input signals are multistep functions are employed. One example of a multirate flow test is a pulse test. In a pulse test, the input signals are sequences of production and shut-in periods.

Signal analysis suggests the use of different shapes of input signals, since different input signals generate different output signals, which could contain different information about the reservoir. For example, Rosa (1991) proposed the use of cyclic flow variations to characterize the permeability distribution in areally heterogeneous reservoirs.

Signal analysis concepts also highlight the significance of wellbore storage effects. A major change in the flow rate of a well is generally created at the surface, by opening or closing the master valve of the well. While wellbore storage effects dominate, there is little sand face flow occurring and, as a result, almost no input signal is being imposed on the reservoir. Therefore, wellbore storage effects need to be included in the specification of the actual input signal, even though wellbore storage effects are not reservoir properties.

Pressure propagation throughout a reservoir is an inherently diffusive process and the diffusive nature of the pressure response has several consequences:

1. The diffusive nature of the pressure response is governed largely by average conditions rather than small local heterogeneities (Horne, 1990). Therefore, the use of the pressure response for detecting heterogeneities has an inherent limitation. During a well test, only abrupt changes in physical properties such as mobility and storativity within the reservoir are likely to be detected.
2. Due to their diffusive nature pressure changes propagate throughout the reservoir at an infinite velocity. Once the input signal is applied to the reservoir, the pressure response involves all the information about the reservoir such as the

average permeability, skin, the boundary effect, the heterogeneity effect and so on. Therefore, it is theoretically possible to obtain all the information about the reservoir from the very beginning of a well test.

3. The farther a point in the reservoir from a well, the later the information involved in that point is significant to the pressure response at the well. In practice the boundary effect becomes significant to the pressure response only after a certain time, and the concepts of radius of investigation and stabilization time are frequently used. Several criteria have been proposed for defining both radius of investigation and stabilization time. The principle reason for the differences between these criteria results from the manner in which the time when the boundary effect becomes significant is defined. In other words, the differences come from the magnitudes of the tolerances used, since theoretically the pressure response at the well involves all the information about the reservoir from the very beginning of a well test.

Here it is important to understand the scale of the resolution of well test analysis. Hewett and Behrens (1990) showed four classes of the range of scales in a reservoir. These are the microscopic scale (the scale of a few pores within the porous medium), the macroscopic scale (the scale of core plugs and laboratory measurements of flow properties), the megascopic scale (the gridblock scale in full-field models), and the gigascopic scale (the reservoir scale).

Reservoir simulation models are based on mass conservative equations derived for the macroscopic scale, which is the scale of the representative elementary volume where the details of the macroscopic structure of the porous medium are replaced by a fictitious continuum of properties. In cases where the grid block size is the megascopic scale, several scaling-up techniques are employed.

The scale of the resolution achievable in well test analysis is generally involved in the gigascopic scale, since the pressure response tends to yield integrated properties of the reservoir without sufficient resolution for detecting small heterogeneities.

2.2 Inverse Problem

The objective of well test analysis is to identify the reservoir system and estimate the reservoir properties from the pressure response. This is achieved by building a mathematical model of the reservoir which generates the same output response as that of the actual reservoir system. This is an inverse problem that in general cannot be solved uniquely.

Strictly speaking, each reservoir behaves differently so it is necessary to have the same number of mathematical models as there are reservoirs. However, as mentioned above, the resolution attainable in well test analysis has limitations due to the diffusive nature of the pressure response. This makes it possible to study a finite number of mathematical models. This theoretical explanation has been confirmed by many years of successes of well test analysis in real field experiences.

The observed pressure data (the actual pressure response) cannot be identical to the pressure response calculated using a mathematical model for two reasons: measurement errors and the simplified nature of model (Watson *et al.*, 1988). Measurement errors can be greatly reduced by the use of accurate pressure measurement devices. However, even if a correct model is used, modeling error could still exist, since a simple mathematical model is employed to represent a complex reservoir behavior. Therefore, the discrepancy between the observed pressure data and the calculated pressure response is inherent in well test analysis. In other words, there is a limitation within any effort to reduce the differences. These errors introduce uncertainty into well test analysis.

Hence, the final solution of the inverse problem is to find the most appropriate model which generates the pressure response as close to the actual pressure response as possible.

What makes it more difficult to perform well test analysis is that several different models may show adequate matches to the observed data. One of the cases encountered commonly is the detection of boundaries. In practice the boundary effect becomes significant only after a certain time. This means that either an infinite acting model or a boundary model can provide more or less equivalent matches of the

2. Use nonlinear regression to estimate the parameter values.
3. Verify the results using confidence intervals as criteria to decide whether the model is acceptable or not.

Up to now, confidence intervals suggested by Dogru, Dixon and Edgar in 1977 and Rosa and Horne in 1983 have been useful tools for a quantitative evaluation of models. However, confidence intervals sometimes provide inappropriate results due to limitations involved inherently in obtaining them. This work investigates a new quantitative method to discriminate between reservoir models. This method successfully selects the most appropriate model and provides more consistent results than confidence intervals.

In the next chapter, confidence intervals and their related topics are described fully, since the new method is a direct extension of confidence intervals, yet reinforces the weak points.

Chapter 3

Confidence Intervals

This chapter discusses practical problems involved in the use of confidence intervals in computer-aided well test analysis.

Section 3.1 derives mathematical models employed in this study. The terms of “homogeneous” and “heterogeneous” in the context of well test analysis are discussed. Implications of the averaging process are considered. The characteristics of several heterogeneous models such as a composite model, a multilayered model, and a double porosity model are described.

Section 3.2 discusses graphical analysis using the pressure derivative plot. Artificial intelligence approaches to model identification are also discussed.

Section 3.3 discusses the nonlinear regression technique employed in this work. Nonlinear regression techniques significantly improve the quality of parameter estimation. The concepts of the least squares method, weighted least squares method and least absolute value method are unified.

Section 3.4 discusses some basic principles of Bayesian inference. Bayesian inference is required to develop the sequential predictive probability method.

Section 3.5 discusses the use of confidence intervals for model verification. The statistical aspects of nonlinear regression enable us to calculate confidence intervals. Confidence intervals are derived in the framework of Bayesian inference. The applications of confidence intervals for model verification are demonstrated through simulated data. The problems inherently involved in the application of confidence

intervals for model discrimination are discussed. The difference between approximate (linearized) confidence intervals and exact confidence intervals is shown. The F test approach is also examined.

3.1 Mathematical Model

In developing the fundamental diffusivity equation, the following simplifying assumptions are made:

- Darcy's law applies;
- flow is radial through the porous medium with negligible gravitational forces;
- flow is single phase and isothermal;
- the porous medium is homogeneous and isotropic with uniform formation thickness;
- the well is completed across the entire formation thickness;
- the fluid is slightly compressible with constant viscosity;
- the total system compressibility is small and constant;
- pressure gradients are small everywhere; and,
- no chemical reactions occur between fluid and rock.

With these assumptions, the fluid flow in the reservoir is governed by the diffusivity equation:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r} \frac{\partial p}{\partial r} = \frac{\phi \mu c_t}{k} \frac{\partial p}{\partial t} \quad (3.1)$$

where p is pressure, r is the radial distance from the wellbore, t is time, ϕ is the porosity, μ is the viscosity, c_t is the total system compressibility, and k is the absolute permeability.

Mathematical models can be constructed by solving the diffusivity equation for different boundary conditions. Mathematical models are then defined by three different components which describe the basic behavior of the reservoir, the well and the surroundings (the inner boundary conditions), and the outer boundaries of the reservoir (the outer boundary conditions). In general, the early time pressure response is dominated by the inner boundary conditions, the intermediate time pressure response is characterized by the basic behavior of the reservoir, and the late time pressure response is influenced by the outer boundary conditions.

The basic assumption in building a mathematical model is that the reservoir properties are uniform throughout the various regions of the reservoir. In the context of well test analysis, the terms “homogeneous” and “heterogeneous” are related to reservoir behavior, not to reservoir geology (Gringarten, 1984). The term “homogeneous” means that only one medium is involved in the flow process. On the other hand, the term “heterogeneous” indicates changes in mobility and storativity.

One of the most important reservoir parameters determined from well test analysis is the effective absolute permeability of the reservoir. The effective absolute permeability is a function of the location and is therefore heterogeneous at the macroscopic scale. Considerable efforts have been devoted to understanding the influence of heterogeneity on the effective absolute permeability estimated from well test analysis and to deriving methods for averaging permeabilities in heterogeneous distributions. An important issue that must be addressed is the volume and type of averaging.

Warren and Price (1961) studied the performance characteristics of heterogeneous reservoirs at the megascopic scale and investigated the effect of permeability variation on both the steady state and the transient flow of a single phase fluid. Based on simulated experiments, the following important conclusions were obtained:

1. The most probable behavior of a heterogeneous system approaches that of a homogeneous system with a permeability equal to the geometric mean of the individual permeabilities.
2. The permeability determined from a pressure buildup curve for a heterogeneous reservoir gives a reasonable value for the effective permeability of the drainage

area.

3. A qualitative measure of the degree of heterogeneity and its spatial configuration are obtained from a comparative study of core analysis and pressure buildup data.

Dagan (1979) presented upper and lower bounds on the effective absolute permeability. The lower bound equals the harmonic mean: which corresponds to the effective absolute permeability of a layered formation to flow perpendicular to the layering direction. The upper bound equals the arithmetic mean, which corresponds to the effective permeability to flow parallel to the layering direction.

Alabert (1989) proposed a power average of the block permeabilities within a specific averaging volume to model the full nonlinear averaging of block permeabilities as measured by a well test. The assumption is that the elementary block permeability values average linearly after a nonlinear power transformation.

Although these several averaging techniques have been proposed, the principal point is that in the context of well test analysis a reservoir with small variations in permeability in space can often be represented by a homogeneous model due to the limited resolution of the pressure transient response.

A reservoir model which shows pressure response characteristics due to abrupt changes in mobility and storativity is regarded as a heterogeneous model. Well known heterogeneous reservoir models include the composite model, the multilayered model, and the double porosity model. These models have been studied extensively by many authors. In this work, the basic features of these models are shown and some of the important works relating to well test analysis are described.

A composite reservoir model is made up of two or more radial regions centered at the wellbore. Each region has its own reservoir properties which are uniform within the region. A composite reservoir system may be created artificially. Enhanced oil recovery projects, like water flooding, gas injection, CO_2 miscible flooding, in-situ combustion, steam drive, and so on artificially create conditions wherein the reservoir can be viewed as consisting of two regions with different rock and/or fluid properties.

The following four parameters are used generally to characterize a two-region composite reservoir model:

1. Mobility ratio (M)

$$M = \frac{(k/\mu)_1}{(k/\mu)_2} \quad (3.2)$$

2. Storativity ratio (F_s)

$$F_s = \frac{(\phi c_t)_1}{(\phi c_t)_2} \quad (3.3)$$

3. Discontinuity radius for a two-region reservoir (R)

4. Skin effect at the discontinuity (S_f)

Ambastha (1988) presented the pressure derivative behavior of a well in a two-region radial composite reservoir model at a constant flow rate. Ambastha (1988) also presented the pressure derivative behavior of a well in a three-region radial composite reservoir model at a constant flow rate. Extension from a composite reservoir model with two regions to that with more than two regions was straightforward by adding the corresponding parameters.

In water-injection and falloff tests, the injected water usually has a lower temperature than the initial reservoir temperature. In addition, because of the differences in oil and water properties, a saturation gradient is established in the reservoir. Hence, well test analysis of injection and falloff tests should take into account the following two important effects: the saturation gradient and the temperature effect.

Abbaszadeh and Kamal (1989) presented procedures to analyze falloff data from water-injection wells. Abbaszadeh and Kamal (1989) included the effect of the saturation gradient in the invaded region without considering the temperature effect. Bratvold and Horne (1990) presented the generalized procedures to interpret pressure injection and falloff data following cold-water injection into a hot-oil reservoir by accounting for both temperature and saturation effects.

A multilayered reservoir model is composed of more than one layer. Each layer has its own reservoir properties which are uniform within each layer. Geologically, it is confirmed that many reservoirs have strongly heterogeneous characteristics with respect to the vertical direction due to the sedimentation process.

Two different multilayered reservoir models have been proposed, depending on the presence or absence of interlayer crossflow. A multilayered reservoir is called a crossflow system if fluid can move between layers, and is called a commingled system if layers communicate only through the wellbore. A commingled system can be regarded as a limiting case of a crossflow system where the vertical permeabilities of all layers are assumed to be zero. In particular, a two-layer reservoir model without formation crossflow is often called a double permeability model.

Park (1989) presented the computer-aided well test analysis of multilayered reservoirs with formation crossflow. The work by Bidaux et al. (1992) showed the comprehensive characteristics of pressure transient behavior in multilayered reservoir models.

A double porosity model is used to represent naturally fractured reservoir behavior. Naturally fractured reservoirs may be considered as initially homogeneous systems whose physical properties have been deformed or altered during their deposition (Da Prat, 1990). A double porosity model considers two interconnected media of different porosity, that is the interconnected fractures of low storage capacity and high permeability and the low permeable formation matrix. Da Prat (1981) presented the characteristics of the pressure transient behavior of such a system. Gringarten (1984) demonstrated practical applications of a double porosity model to real field well test data.

The two important parameters used to characterize the double porosity behavior are the storativity ratio (w) and the transmissivity ratio (λ) defined by Warren and Root (1963).

1. Storativity ratio (w) is defined as the ratio of the storage capacity of the fractures to the total storage capacity:

$$\omega = \frac{(\phi c_t)_f}{(\phi c_t)_f + (\phi c_t)_m} \quad (3.4)$$

2. Transmissivity ratio (λ) is the parameter used to describe the interporosity flow, sometimes called as interporosity flow coefficient:

$$\lambda = \alpha \frac{k_m}{k_f} r_w^2 \quad (3.5)$$

where α is the interporosity flow shape factor which depends on the geometry of the interporosity flow between the matrix and the fracture.

In addition to the three heterogeneous models discussed above, large scale heterogeneity problems have also been studied. Sageev and Horne (1983) studied pressure transient analysis for a drawdown test in a well near an internal circular boundary, such as may be found in a gas cap or a large shale lens. The main objective of the work by Sageev and Horne (1983) was to estimate the size and the distance to the internal circular discontinuity from well test data. Type curves were calculated with different sizes of the internal circular discontinuity. From the visual inspection of these type curves, the effect of a no flow boundary hole with a relative size of 0.3 or less appears to be insignificant, where the relative size is the ratio of the diameter of the hole to the distance from the well to the center of the hole. This result is an important demonstration of the insensitivity of the diffusive pressure response to local heterogeneities and one of the examples of nonuniqueness in inverse problems.

Grader and Horne (1988) extended the work by Sageev and Horne (1983) for interference well testing.

All of these heterogeneous models treat their heterogeneities as a combination of different zones within which reservoir properties are assumed to be uniform. New approaches to define reservoir heterogeneity as continuously variable have been proposed, in which the reservoir properties may be described as some form of a statistically stationary random field with small variance.

Oliver (1990) proposed a method to use well test data to estimate the effective absolute permeability for concentric regions centered around the wellbore. Oliver (1990) studied the averaging process, including identification of the region of the reservoir that influences permeability estimates, and a specification of the relative contribution

of the permeability of various regions to the estimate of average permeability. Oliver (1990) showed that permeability estimates obtained from the slope of the plot of pressure versus the logarithm of the elapsed time are weighted averages of the permeabilities within an inner and outer radius of investigation. The significant limitation of this method is that it relies on very accurate pressure measurement devices and also on small permeability deviations from a constant overall mean.

Rosa (1991) extended the idea of Oliver (1990) and proposed the use of cyclic flow variations to characterize the permeability distribution in areally heterogeneous reservoirs from well test data.

Sato (1992) proposed the use of the perturbation boundary element method for well testing problems in heterogeneous reservoirs. Sato (1992) reported that heterogeneity has little effect on pressure responses if the average property value within a drainage area is not much different from the near-well property value.

However, to date little has been done to analyze well test data in a reservoir characterized by continuously variable permeability and studies in this area are currently ongoing.

One form of reservoir heterogeneity that affects pressure responses more significantly is the reservoir boundary. Four main types of outer boundary conditions are generally employed:

1. Infinite acting outer boundary
2. Sealing fault outer boundary
3. No flow outer boundary (closed outer boundary)
4. Constant pressure outer boundary

For infinite acting outer boundary conditions, no boundary effects have been detected.

For finite outer boundary conditions, typical boundary effects appear after a certain time, depending on the nature of the outer boundary.

For sealing fault outer boundary conditions, the well is not completely closed in on all sides but responds to only one impermeable boundary. The boundary effect is calculated by superposition and the pressure response at late time is that of two identical wells, which are the actual well and the image well. The semilog straight line has a doubling of slope on a semilog plot.

A reservoir approaches pseudosteady state behavior at late time for a no flow boundary. Pseudosteady state behavior is characterized by a unit slope straight line on a dimensionless derivative plot.

A reservoir approaches steady state behavior at late time for a constant pressure outer boundary. On a pressure derivative plot, steady state behavior is characterized by pressure derivatives of zero.

In this work, the following eight fundamental models were considered according to the basic behavior of the reservoir and the outer boundary conditions, since these models are regarded as basic and are commonly employed in actual well test analysis:

- infinite acting model (three parameters: k , S , and C)
- sealing fault model (four parameters: k , S , C , and r_e)
- no flow outer boundary model (four parameters: k , S , C , and r_e)
- constant pressure outer boundary model (four parameters: k , S , C , and r_e)
- double porosity model with pseudosteady state interporosity flow (five parameters: k , S , C , ω , and A)
- double porosity with pseudosteady state interporosity flow and sealing fault model (six parameters: k , S , C , w , λ , and r_e)
- double porosity with pseudosteady state interporosity flow and no flow outer boundary model (six parameters: k , S , C , w , A , and r_e)
- double porosity with pseudosteady state interporosity flow and constant pressure outer boundary model (six parameters: k , S , C , ω , A , and r_e)

As will be mentioned in later chapters, the proposed analysis method can discriminate between candidate reservoir models as long as the reservoir models can be expressed approximately as a linear form with respect to the reservoir parameters. Therefore, it is straightforward to extend the utility of the proposed method to other reservoir models than the eight fundamental models listed above.

3.2 Graphical Analysis

Solving the inverse problem consists of three steps. The first step is model recognition (model identification), the second step is parameter estimation, and the third step is model verification.

3.2.1 Model Recognition

The primary step is the recognition of the reservoir model, since without defining the model, the corresponding reservoir parameters cannot be estimated.

Graphical analysis using the pressure derivative plot proposed by Bourdet *et al.* (1983a) has become a standard procedure for model recognition. The procedure is based on the visual inspection of the pressure derivative plot. The pressure derivative plot provides a simultaneous presentation of the following two sets of plots.

o $\log(\Delta p)$ versus $\log(\Delta t)$

o $\log(\Delta p')$ versus $\log(\Delta t)$

where

$$\Delta p' = \frac{\partial \Delta p}{\partial \log(\Delta t)} = \Delta t \frac{\partial \Delta p}{\partial \Delta t} \quad (3.6)$$

The advantage of using the log-log plot is that it is able to display the whole data and show many distinct characteristics in a single graph.

The pressure derivative plot with the pressure plot has two main advantages over the pressure plot alone from two different aspects: one is model recognition and the

other is parameter estimation. First, from the aspect of model recognition, the pressure derivative plot reveals more characteristics of the response than the pressure plot. The pressure derivative plot improves the resolution of the data from a visual point of view. In addition, heterogeneous reservoir behavior such as that of a double porosity model can exhibit distinct characteristics on the pressure derivative plot. Hence, it is easier to recognize possible reservoir models. Second, in cases where parameter estimation is performed by manual type curve matching, matching is achieved for both the pressure data and the pressure derivative data simultaneously. This greatly enhances the reliability of the match.

However, it should be mentioned that graphical analysis is useful only as long as the flow condition is simple and the data are not strongly affected by errors. In cases where the flow conditions are no longer simple and/or the data involve large errors, interpretation requires expert skills to recognize the characteristics of the reservoir behavior. Model recognition is influenced by human bias and, as a result, the conclusions may vary according to the interpreter. Hence, Artificial Intelligence (AI) methods have been proposed for model recognition (Allain and Horne, 1990, Al-Kaabi and Lee, 1990, Allain and Houzé, 1992).

Allain and Horne (1990) showed an AI approach for model recognition using a rule-based expert system that is based on recognition of distinct features of the pressure derivative curve such as maxima, minima, stabilization, and upward and downward trends. For instance, an infinite acting model with wellbore storage and skin can be expressed as a combination of upward trend, maximum, downward trend, and stabilization.

Horne (1992) summarized the purpose of model recognition by an AI approach as follows:

1. An AI model recognition program is capable of detecting all reservoir models that are consistent with the data, which a human interpreter may not find.
2. Association of specific data ranges with specific flow regimes can reveal inconsistencies involved in the data.

Horne (1992) also speculated that a subject of research in AI may be the development of a multitasking AI program to incorporate multiple forms of information and qualitative data such as geological description or drilling records.

Although several AI methods have been proposed for model recognition, no method has yet become a standard procedure. Therefore, the conventional graphical procedure using the pressure derivative plot for model recognition was employed in this work.

Calculating the pressure derivative may encounter practical problems, since differentiation exaggerates noise and the pressure derivative tends to be noisier than the pressure data itself. To smooth the noisy pressure derivative, the use of a 0.2 log cycle differentiation interval is proposed (Bourdet *et al.*, 1989). Using data points that are separated by at least 0.2 of a log cycle can smooth the noise, but cannot be applied within the last 0.2 log cycle of the data. In cases where the boundary effect appears at very late time in the data, this differentiation process may disguise the boundary effect.

Although the pressure derivative plot suggested by Bourdet *et al.* (1983a) has been used widely, several other pressure derivative plots have been proposed by other authors.

Onur and Reynolds (1989) proposed a combined plot of

- $\log(\Delta p)$ versus $\log(\Delta t)$
- $\log\left(\frac{\Delta p}{2\Delta p'}\right)$ versus $\log(\Delta t)$

The main advantage of this formulation is that type curve matching becomes a one-dimensional movement of the data on type curves. Therefore, compared to the pressure derivative plot by Bourdet *et al.* (1983a), the degrees of freedom are reduced when a manual type curve matching is attempted, and the quality of match could be improved.

Duong (1989) proposed a combined plot of pressure and pressure-derivative ratio:

- $\log(2\Delta p)$ versus $\log(\Delta t)$
- $\log\left(\frac{\Delta p}{2\Delta p'}\right)$ versus $\log(\Delta t)$

The main advantage of this plot is the same as that of Onur and Reynolds (1989), namely that type curve matching becomes a one-dimensional movement of the data on type curves.

The underlying motivation in constructing new pressure derivative plots is to make it easier to perform manual type curve matching procedure through data transformation. However, these new pressure derivative plots also suffer from the same problem that differentiation exaggerates noise. Furthermore, as long as parameter estimation is performed by nonlinear regression using the pressure data itself, the results are the same regardless of the type of plot.

The pressure derivative plot aids the model recognition and parameter estimation process by emphasizing the characteristics of the pressure response. It should be pointed out that the pressure derivative plot does not add any extra information about the reservoir. The pressure derivative plot should be understood as a magnifying glass revealing the identifiable characteristic response of a reservoir, otherwise hidden in the pressure response (Stanislav and Kabir, 1990). In other words, even if the data are transformed into other forms using some operation such as differentiation, the information involved in the transformed data still remains the same as the original data.

On the basis of the limited data, several different reservoir models may appear to satisfy the available information about the reservoir and seem to provide more or less equivalent matches of the data. Whether graphical analysis using the pressure derivative plot or an AI method is employed, the model recognition procedure itself cannot select the most appropriate model. Hence a model verification procedure is required to select the most appropriate model among the possible reservoir models.

3.2.2 Parameter Estimation

After a reservoir model has been chosen, the next step is the estimation of the unknown reservoir parameters. Parameter estimation is performed by either one of the following procedures:

1. Manual curve matching.
2. Automated model fitting using nonlinear regression.

In a manual curve matching procedure, the data are laid over the type curves, and moved horizontally and vertically (two-dimensional movement) until a match is achieved from a visual point of view within a limited number of type curves. At the point of matching, correspondence between p_D and Δp and between t_D and At has been achieved and the reservoir parameters can be estimated.

The type curves of $\log(p_D)$ versus $\log(t_D/C_D)$ for various wellbore storage and skin values, C_De^{2S} are used commonly for an infinite acting reservoir model with wellbore storage and skin (Gringarten et al., 1978).

The drawbacks of manual curve matching are as follows:

1. Although type curves have been constructed for many different reservoir models! the number of published type curves is limited and they do not cover all possible reservoir models.
2. Most published type curves are valid only under the condition of a constant rate production drawdown test.
3. Even though the use of the derivative plot together with the pressure plot reduces the risk of incorrect matches, the procedure is inherently subjective.
4. Type curve matching does not provide any quantitative information about the validity of the estimated parameter values.

Rosa and Horne (1983) showed the utility of nonlinear regression algorithms to estimate the reservoir parameters from well test analysis. The advantages of automated model fitting using nonlinear regression can be expressed by comparison with the drawbacks of manual type curve matching as follows:

1. Nonlinear regression can be performed for any possible reservoir models by generating the corresponding pressure transient solution.

2. Nonlinear regression can handle multirate or variable rate flow tests. The strategy is to compute the pressure response for a constant rate production drawdown test based on the reservoir model. From this solution, the pressure response for an arbitrary flow rate history may be computed by applying superposition.
3. The results are free from human bias.
4. Nonlinear regression can provide quantitative information about the quality of the estimated parameter values in conjunction with statistical inference.

One of the objectives in this work is to express the quality of parameter estimates quantitatively, and nonlinear regression is employed for parameter estimation in this work.

3.2.3 Model Verification

Once parameter estimation has been performed, the final step is to determine how well the reservoir parameters are estimated and to verify the model adequacy.

In graphical analysis, the model verification problem is left to engineering judgement. Graphical visualization, in which the actual pressure data and the calculated pressure response based on the estimated values of the parameters are compared, is most often used as a guide for evaluating the quality of the estimation. Therefore, model verification is subjective.

On the other hand, confidence intervals obtained from nonlinear regression are a powerful tool that provides quantitative information about model verification that is not available in graphical analysis.

In cases where several reservoir models are possible from the model recognition procedure, model discrimination should be accomplished to select the most appropriate reservoir model.

Whether graphical visualization or confidence interval analysis is employed for model verification, a common procedure for model discrimination is selecting a simple model first. If the result is not satisfactory, then the next model is employed in order of complexity and model verification is applied to this model. This procedure is

repeated until the result is acceptable (Gringarten, 1986, Watson *et al.*, 1988, Ramey, 1992).

The underlying idea in this procedure is based on a belief that a model that has too many parameters might result in parameter estimates that have larger uncertainty associated with them. Therefore, a simple model is selected first. This idea, is frequently used in general inverse problems without any verification, but as will be shown in Chapter 5 this is not always true in well test analysis. No reliable technique is available to discriminate between possible reservoir models quantitatively. This is the motivation of this study, and the main objective is to develop a better quantitative method for model discrimination.

It should be mentioned that model verification should consider the geological and petrophysical information about the reservoir if available. As discussed in Section 3.2.1, the development of a multitasking AI program to incorporate multiple forms of information is a current subject of research in AI approaches although it is beyond the scope of this study to incorporate other information quantitatively with well test analysis. In this study model discrimination is performed using well test data only.

3.3 Nonlinear Regression

In this section, the nonlinear regression technique is briefly reviewed, since nonlinear regression is closely related to both confidence intervals and the sequential predictive probability method.

The performances of different nonlinear regression algorithms in well test analysis have been studied by several authors (Rosa and Horne, 1983, Abbaszadeh and Kamal, 1988, Barua *et al.*, 1988, Rosa, 1991). Horne (1992) made a recent review of nonlinear regression techniques available in well test analysis.

Rosa and Horne (1983) showed that by numerical inversion from Laplace space of not only the pressure change but also its partial derivatives with respect to the reservoir parameters into real space, it is possible to perform nonlinear regression. This approach has made it possible to apply nonlinear regression to a wide variety of well test models whose solution is known only in Laplace space.

One of the theorems of the Laplace transform theory is employed to calculate the partial derivatives in real space from those in the Laplace space:

$$\frac{\partial F(\boldsymbol{\theta}, t)}{\partial \theta_i} = L^{-1} \left\{ \frac{\partial f(\boldsymbol{\theta}, z)}{\partial \theta_i} \right\} \quad (3.7)$$

where

L^{-1} = inverse Laplace operator

f = function F in the Laplace space

t = time

z = argument of a function in the Laplace space

Rosa and Horne (1983) used the Gauss-Marquardt method with penalty function and interpolation and extrapolation technique, and reported it to be a reliable nonlinear regression algorithm to determine the parameter estimates in typical well test applications.

In this work, it is the purpose to make statistical inferences at the stage where the optimal parameter estimates have already been obtained by nonlinear regression. Although several alternative methods have been proposed to overcome the few drawbacks of the Gauss-Marquardt method, this method performs well for the reservoir models employed in this study. Hence, in this study, the Gauss-Marquardt method with penalty function and interpolation and extrapolation technique was employed as the nonlinear regression algorithm.

3.3.1 Nonlinear Regression Algorithm

In nonlinear regression using the least squares method, the objective is to minimize the sum of the squares of the differences between the observed pressure data and the calculated pressure responses based on the reservoir model:

$$E = \sum_{i=1}^n [y_i - F(\boldsymbol{\theta}, x_i)]^2 \quad (3.8)$$

where

- E = objective function
- F = reservoir model function
- θ = unknown reservoir parameters
- x_i = dependent variable (time)
- y_i = independent variable (pressure)
- n = number of data

The reservoir model function F is generally a nonlinear function of the unknown reservoir parameters, so too is the objective function E (Eq. 3.8). Due to the non-linearity, the unknown parameters need to be modified iteratively until the objective function cannot be made any smaller.

The Gauss-Marquardt method with penalty function and interpolation and extrapolation technique is a modification of Newton's method. First of all, Newton's method needs to be described to explain the Gauss-Marquardt method. The general procedure of Newton's method is described as follows.

Newton's method has its mathematical basis in Taylor's theorem. Taylor's theorem states that if a function and its derivatives are known at a certain point, then approximations to the function can be made at all points in the sufficiently close neighborhood of the point.

In Newton's method, the objective function is approximated with a quadratic model by truncating the Taylor series around an initial guess for a set of unknown parameters (θ^0):

$$\begin{aligned}
 E^* = E|_{\theta^0} &+ \sum_{j=1}^m (\theta_j - \theta_j^0) \left. \frac{\partial E}{\partial \theta_j} \right|_{\theta^0} \\
 &+ \sum_{j=1}^m \sum_{k=1}^m (\theta_j - \theta_j^0) (\theta_k - \theta_k^0) \left. \frac{\partial^2 E}{\partial \theta_j \partial \theta_k} \right|_{\theta^0}
 \end{aligned} \tag{3.9}$$

where

$$\left. \frac{\partial E}{\partial \theta_j} \right|_{\theta^0} = -2 \sum_{i=1}^n [y_i - F(\theta^0, x_i)] \left. \frac{\partial F}{\partial \theta_j} \right|_{\theta^0} \tag{3.10}$$

$$\left. \frac{\partial^2 E}{\partial \theta_j \partial \theta_k} \right|_{\theta^0} = -2 \sum_{i=1}^n \left[-\frac{\partial F}{\partial \theta_j} \frac{\partial F}{\partial \theta_k} + [y_i - F(\theta^0, x_i)] \left. \frac{\partial^2 F}{\partial \theta_j \partial \theta_k} \right|_{\theta^0} \right] \quad (3.11)$$

Using Eq. 3.10 and Eq. 3.11, the gradient of the objective function (**g**) and the Hessian matrix (**H**) are defined by:

$$\mathbf{g} \equiv \{g_j\} = \left\{ \left. \frac{\partial E}{\partial \theta_j} \right|_{\theta^0} \right\} \quad (3.12)$$

$$\mathbf{H} \equiv \{h_{jk}\} = \left\{ \left. \frac{\partial^2 E}{\partial \theta_j \partial \theta_k} \right|_{\theta^0} \right\} \quad (3.13)$$

Hence, Eq. 3.9 can be expressed in a matrix form as:

$$E^* = E|_{\theta^0} + (\delta\theta)^T \mathbf{g} + \frac{1}{2} (\delta\theta)^T \mathbf{H} (\mathbf{S0}) \quad (3.14)$$

where **S0** is set to $\theta - \theta^0$.

Here E^* is the Newton approximation to the value of E around θ^0 . The minimization of E^* calls for its derivatives with respect to **S0** to be zero at a stationary point:

$$\frac{\partial E^*}{\partial (\delta\theta)} = 0 \quad (3.15)$$

Substituting Eq. 3.14 into Eq. 3.15 reduces to:

$$\delta\theta = -\mathbf{H}^{-1} \mathbf{g} \quad (3.16)$$

S0 obtained from Eq. 3.16 is merely an extremum solution for E^* . Therefore, iterations are required to obtain the solution for E .

A new solution is obtained from:

$$e = \theta^0 + \delta\theta \quad (3.17)$$

The iterative process is repeated until convergence is achieved. Note that when convergence is achieved $\delta\theta$ is negligible and the stationary value of E^* is exactly that of E and, as a result, the objective function is at an extremum. This technique is called Newton's method.

Newton's method may encounter two difficulties, depending on the nature of the model function. Firstly, due to the second derivatives terms involved in the diagonal elements of the Hessian matrix, there is no guarantee that the Hessian matrix is positive definite and consequently no guarantee that new solutions always approach the minimum point during iteration. Secondly, the iteration process may converge slowly or even diverge in cases where the Hessian matrix is ill-conditioned. Either strong correlations between some parameters or insensitivity of the model function to some parameters makes the Hessian matrix ill-conditioned. Therefore, several modifications are required to overcome these difficulties and to achieve a fast convergence to a minimum point.

In the Gauss method, the second derivatives — are regarded as if they were zero. This is usually a good approximation at a minimum point, since the gradient of the objective function is zero. This modification makes the Hessian matrix positive definite and guarantees the convergence to a minimum point.

The Marquardt method is useful when the Gauss-modified Hessian matrix is poorly conditioned. Adding a constant to the diagonal elements of the Hessian matrix improves the condition of the Hessian matrix and prevents the Hessian matrix from being numerically singular.

The interpolation and extrapolation technique modifies the step length to speed up the rate of convergence. This technique is sometimes also known as a line search. Penalty functions may be used to improve the rate of convergence, by limiting the search to a feasible region.

Finally, the objective function is expressed at a minimum point as follows:

$$E|_{\theta} = E|_{\hat{\theta}} + \sum_{j=1}^m \sum_{k=1}^m (\theta_j - \hat{\theta}_j) (\theta_k - \hat{\theta}_k) \left. \frac{\partial^2 E}{\partial \theta_j \partial \theta_k} \right|_{\hat{\theta}} \quad (3.18)$$

In addition, this form is equivalent to the following form:

$$F(\theta) = F(\hat{\theta}) + \sum_{j=1}^m \left(\frac{\partial F}{\partial \theta_j} \right)_{\theta=\hat{\theta}} \cdot (\theta_j - \hat{\theta}_j) \quad (3.19)$$

It should be noted that even if the reservoir model is incorrect, the nonlinear regression technique forces the model to fit the data and will still provide results. This

implies that a blind use of nonlinear regression technique may lead to dangerous misleading results. This is one of the main reasons why model verification is necessary.

3.3.2 Statistical Inference

From the viewpoint of computation, the objective in nonlinear regression is to minimize the objective function defined by Eq. 3.8. The reason why the least squares estimator is usually used is explained in the framework of statistical inference.

From a statistical point of view, the model function is implicitly assumed to have the following form:

$$y_i = F(\mathbf{e}, x_i) + \epsilon_i \quad (3.20)$$

where ϵ_i is a random error characterized by a probability density function.

Suppose that the model is linear with respect to the parameters. In cases where the errors ϵ_i are independent and uncorrelated to each other, and each follows the same normal distribution with zero mean and variance of σ^2 , the least squares estimator ($\hat{\theta}$) is the unbiased estimator of θ with the minimum variance (Seber, 1977). Hence, $\hat{\theta}$ is best used as a point estimator.

The least squares estimator is used commonly in inverse problems, since it is the most unbiased assumption due to the Central Limit Theorem, which states that the sum of independent random variables with finite variances tends to be normally distributed (Casella and Berger, 1990).

If the model is nonlinear, the above discussion is no longer valid. However, it is assumed that for a region in the parameter space sufficiently close to the parameter estimates, the nonlinear model function can be expressed approximately in a linear form using Taylor series expansion up to the first order. This assumption holds asymptotically (Seber and Wild, 1989), and as long as a large number of data is used this is a reasonable approximation to general reservoir model functions.

Therefore, for large n , even in nonlinear regression, the least squares estimator ($\hat{\theta}$) could be the unbiased estimator of θ with the minimum variance and $\hat{\theta}$ could be best used as a point estimator.

3.3.3 Least Absolute Value Method

One of the drawbacks of the least squares method is that it is significantly affected by outliers, which are data points that can be considered bad observations caused by malfunctioning of the pressure measurement devices or other test-related operations.

In cases where outliers exist, the least absolute value method can be employed as a robust method. The robustness lies in the fact that the techniques can handle cases where the errors do not follow a commonly assumed normal distribution but a double exponential (Laplace) distribution which has larger tails. Typical forms of the normal distribution and the double exponential distribution are given in Fig. 3.1. The normal distribution has zero mean and a variance of 1.0 psi^2 and the double exponential distribution has zero mean and a variance of 2.0 psi^2 . Fig. 3.1 shows that the double exponential distribution has longer tails than the normal distribution.

The least absolute value method can provide a smooth transition between full acceptance and total rejection of a given observation, providing a systematic way of rejecting outliers by automatically assigning them less weight in the objective function. In other words, the least absolute value method does not require a subjective decision from the interpreter whether or not to reject the extreme observations. A weak point of the least absolute method is that it is expensive in computation compared to the least squares method.

Rosa (1991) and Vieira and Rosa (1993) presented several algorithms for the least absolute value method and showed that the least absolute value method performed better than the least squares method in cases where outliers exist. Rosa (1991) and Vieira and Rosa (1993) proposed the modified least absolute value method and the combination of the modified least absolute value method and the least absolute value method. The modified least absolute value method is robust in cases where poor initial estimates are used. On the other hand, the least absolute value method is less expensive in computation than the modified least absolute value method. Hence, the combination of the methods uses the advantages of both methods and consists of starting with the modified least absolute value method and switching to the least absolute value method when the estimated parameters are sufficiently close to the optimal values.

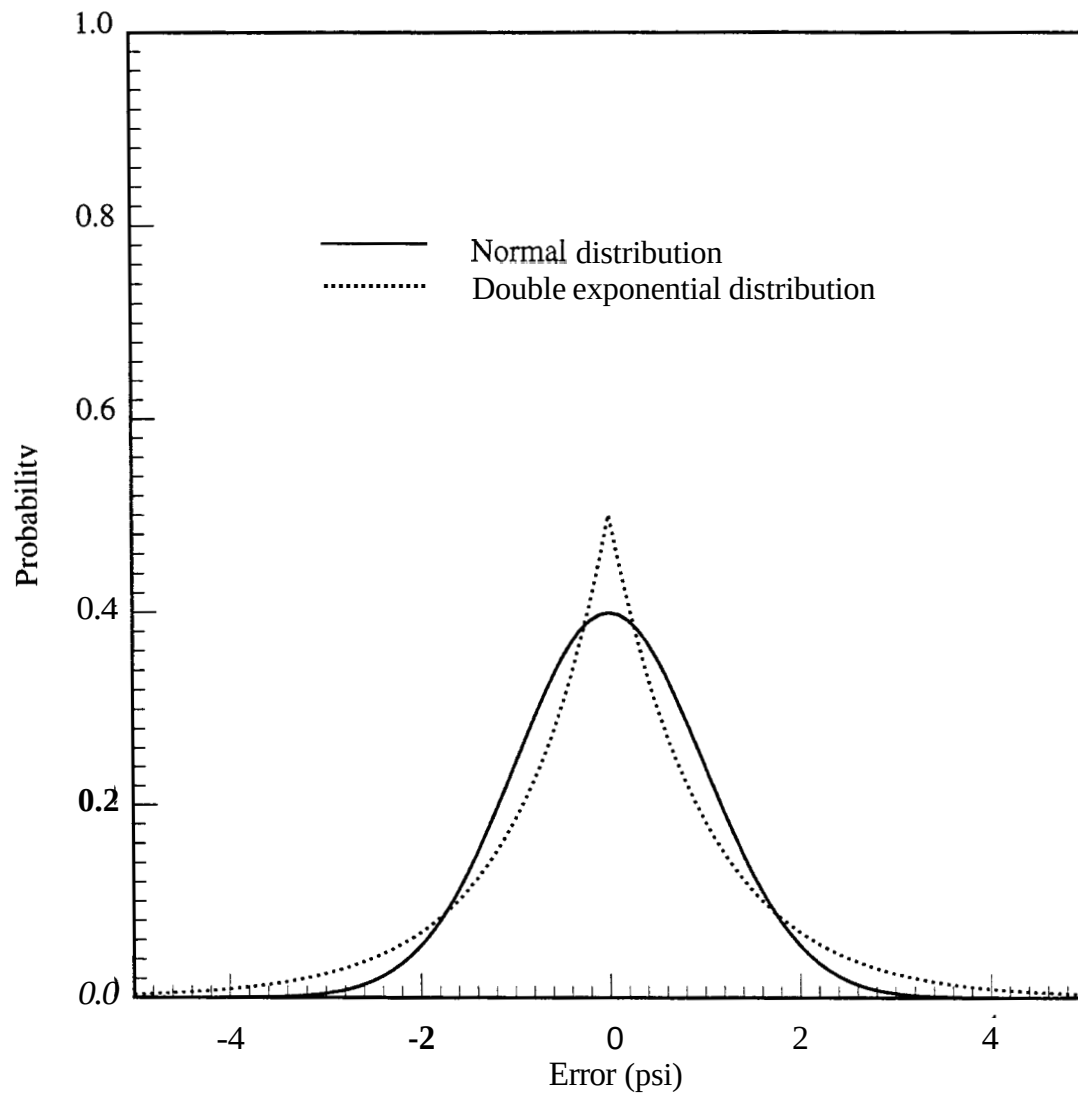


Figure 3.1: Typical forms of the normal distribution and the double exponential (Laplace) distribution

Another method to handle the outliers is the weighted least squares method. This method assigns smaller weight factors to the outliers and is not expensive in computation, since computation is done in the framework of the least squares method. A zero weight factor is equivalent to eliminating the data points. The fundamental drawback is that the outliers need to be identified prior to parameter estimation, which requires a subjective decision from the interpreter.

One of the most difficult problems in deciding which data points are outliers is that the outliers may be caused by either bad observations or the use of an incorrect model. In other words, it is possible to decide whether there exist outliers or not only after the correct model has been selected. In this work, the objective is the selection of the correct model and the least squares method was employed.

The least squares method, the weighted least squares method and the least absolute value method are categorized from a statistical point of view as follows:

1. Least squares method:

The errors have normal distributions with zero means and variances of the same magnitude.

$$Prob(\epsilon_i) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left(-\frac{\epsilon_i^2}{2\sigma^2}\right) \quad (3.21)$$

2. Weighted least squares method:

The errors have normal distributions with zero means and variances of the different magnitudes.

$$Prob(\epsilon_i) = \frac{1}{\sqrt{2\pi}\sigma_i} \cdot \exp\left(-\frac{\epsilon_i^2}{2\sigma_i^2}\right) \quad (3.22)$$

3. Least absolute value method:

The errors have double exponential distributions with zero means and variances of the same magnitude.

$$Prob(\epsilon_i) = \frac{1}{\sqrt{2}\sigma} \cdot \exp\left(-\sqrt{2}\left|\frac{\epsilon_i}{\sigma}\right|\right) \quad (3.23)$$

The normal distribution and the double exponential distribution can be classified under exponential power distributions (Box and Tiao, 1973). The general form of the exponential power distribution is given by:

$$Prob(\epsilon) = \frac{\omega(\beta)}{\sigma} \cdot \exp \left[-c(\beta) \left| \frac{\epsilon}{\sigma} \right|^{2/(1+\beta)} \right] \quad (3.24)$$

where

$$c(\beta) = \left[\frac{\Gamma \left[\frac{3}{2}(1 + \beta) \right]}{\Gamma \left[\frac{1}{2}(1 + \beta) \right]} \right]^{1/(1+\beta)} \quad (3.25)$$

and

$$\omega(\beta) = \frac{\left[\Gamma \left[\frac{3}{2}(1 + \beta) \right] \right]^{1/2}}{(1 + \beta) \left[\Gamma \left[\frac{1}{2}(1 + \beta) \right] \right]^{3/2}} \quad (3.26)$$

where $\sigma > 0$ and $-1 < \beta \leq 1$.

When $\beta = 0$, the distribution reduces to the normal distribution. When $\beta = 1$, the distribution becomes the double exponential distribution. When β tends to -1 , the distribution tends to the uniform (rectangular) distribution.

Hence, depending on the form of the error distribution, it is possible to construct the objective function such as:

$$E = \sum_{i=1}^n [y_i - F(\theta, x_i)]^{2/(1+\beta)} \quad (3.27)$$

However, the form of the error distribution is generally unknown in inverse problems. As discussed in Section 3.3.2, the normal distribution assumption is the most unbiased assumption due to the Central Limit Theorem. This is the statistical reason why the least squares method is generally employed.

3.4 Bayesian Inference

In this section, some basic principles of Bayesian inference are described.

An essential element of Bayesian inference is Bayes theorem. The concept of likelihood function is introduced to represent the information about the parameters deduced from the data. The purpose of introducing Bayes theorem is that Bayes theorem expresses the process of learning from data quantitatively.

3.4.1 Bayes Theorem

From the definition of the conditional probability:

$$Prob(A \cap B) = Prob(A) \cdot Prob(B|A) \quad (3.28)$$

where $Prob(A \cap B)$ is the joint probability of A and B both occurring.

Since the event $A \cap B$ is identical to the event $B \cap A$, Eq. 3.28 can be rewritten as:

$$Prob(B \cap A) = Prob(B) \cdot Prob(A|B) \quad (3.29)$$

Combining Eq. 3.28 and Eq. 3.29 reduces to:

$$Prob(A|B) = \frac{Prob(B|A) \cdot Prob(A)}{Prob(B)} \quad (3.30)$$

This relation is called Bayes theorem and is valid for any events A and B .

In the case of continuous random variables X and Y , Eq. 3.30 is expressed in terms of probability density function as:

$$f(x|y) = \frac{f(y|x) \cdot f_X(x)}{f_Y(y)} \quad (3.31)$$

where

$f_X(x)$ = the marginal probability density function of the random variable X

$f_Y(y)$ = the marginal probability density function of the random variable Y

$f(x|y)$ = the conditional probability density function of the random variable X given $Y = y$

$f(y|x)$ = the conditional probability density function of the random variable Y given $X = x$

3.4.2 Likelihood Function

Suppose that $f(\mathbf{y}|\boldsymbol{\theta})$ denotes the joint probability density function of the random variables $\mathbf{Y} = (Y_1, \dots, Y_n)$ for a fixed value of $\boldsymbol{\theta}$. Given that $\mathbf{Y} = \mathbf{y}$ ($Y_1 = y_1, \dots, Y_n = y_n$) is observed, $f(\mathbf{y}|\boldsymbol{\theta})$ may be regarded as a function not of $\mathbf{y}$ but of $\boldsymbol{\theta}$.

The function of $\boldsymbol{\theta}$ defined by:

$$L(\boldsymbol{\theta}|\mathbf{y}) = f(\mathbf{y}|\boldsymbol{\theta}) \quad (3.32)$$

is called the likelihood function.

It should be mentioned that in the probability density function $f(\mathbf{y}|\boldsymbol{\theta})$ $\mathbf{y}$ and $\boldsymbol{\theta}$ are considered as random and as fixed variables, respectively, while in the likelihood function $L(\boldsymbol{\theta}|\mathbf{y})$ $\mathbf{y}$ is considered to be observed values and $\boldsymbol{\theta}$ is considered to be varying over all possible parameter values. Even if $\boldsymbol{\theta}$ can vary over all possible parameters in $L(\boldsymbol{\theta}|\mathbf{y})$, $\boldsymbol{\theta}$ is not regarded as random.

3.4.3 Bayesian Inference

Suppose that $\mathbf{y} = (y_1, \dots, y_n)$ is a vector of n observations whose probability distribution $Prob(\mathbf{y}|\boldsymbol{\theta})$ depends on the values of m parameters $\boldsymbol{\theta} = (\theta_1, \dots, \theta_m)$.

Suppose also that $\boldsymbol{\theta}$ itself has a probability distribution $Prob(\boldsymbol{\theta})$.

From Bayes theorem (Eq. 3.31), given the observed data $\mathbf{y}$, the conditional probability distribution of $\boldsymbol{\theta}$ is:

$$Prob(\boldsymbol{\theta}|\mathbf{y}) = \frac{Prob(\mathbf{y}|\boldsymbol{\theta}) \cdot Prob(\boldsymbol{\theta})}{Prob(\mathbf{y})} \quad (3.33)$$

where

$$Prob(\mathbf{y}) = \int_m Prob(\mathbf{y}|\boldsymbol{\theta}) \cdot Prob(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (3.34)$$

where the integral is taken over all possible range of $\boldsymbol{\theta}$ (m dimensions), and where $Prob(\mathbf{y})$ is regarded as the expectation value of $Prob(\mathbf{y}|\boldsymbol{\theta})$ with respect to the probability distribution $Prob(\boldsymbol{\theta})$. The quantity $Prob(\mathbf{y})$ may be regarded as merely a

normalizing constant necessary to ensure that the posterior probability distribution $Prob(\theta|y)$ integrates to one.

$Prob(\theta)$, which expresses what is known about θ without knowledge of the data, is called the prior probability distribution of θ . $Prob(\theta|y)$, which expresses what is known about $Prob(\theta)$ given knowledge of the data, is called the posterior probability distribution of θ given y .

Substituting Eq. 3.32 into Eq. 3.33 leads to:

$$Prob(\theta|y) = \frac{L(\theta|y) \cdot Prob(\theta)}{Prob(y)} \quad (3.35)$$

where

$$Prob(y) = \int_m L(\theta|y) \cdot Prob(\theta) d\theta \quad (3.36)$$

Eq. 3.35, which is also called Bayes theorem, indicates that the probability distribution for θ posterior to the data y is proportional to the product of the probability distribution for θ prior to the data and the likelihood function for θ given y :

$$\begin{aligned} \text{posterior distribution} &\propto \\ \text{likelihood function} \times \text{prior distribution} &\quad (3.37) \end{aligned}$$

Note that the posterior probability distribution $Prob(\theta|y)$ has all the prior and data information incorporated in it. The prior information about θ enters the posterior probability distribution through the prior probability distribution, while the information about θ coming from the data enters through the likelihood function. In other words, **Eq. 3.35** provides a mathematical formulation of how previous knowledge may be combined with new knowledge.

The posterior probability distribution $Prob(\theta|y)$ is employed in Bayesian inference to make statistical inference about the parameters.

3.4.4 Important Probability Distributions

In Bayesian inference, information about the parameters is expressed in terms of probability. There are two extreme states: one is a state of complete uncertainty and the other is a state of no uncertainty.

In cases where no information is available about the parameters, which is a state of complete uncertainty, uniform probability distributions are employed. These are common situations before any data are collected. Therefore, in cases where nothing is known about the parameters before the experiments, uniform probability distributions are used as the prior probability distributions for the parameters.

A uniform probability density function $f(\theta)$ is defined by:

$$f(\theta) = c \quad c > 0 \quad (3.38)$$

$$\begin{aligned} \int_m f(\theta) d\theta &= c \int_m d\theta \\ &= \infty \end{aligned} \quad (3.39)$$

However, a basic property of a probability density function $f(\theta)$ is that it integrates over its admissible range to one:

$$\int_m f(\theta) d\theta = 1 \quad (3.40)$$

Eq. 3.39 implies that no matter how small c is, the integral becomes infinite. Probability density functions of this kind are sometimes called improper probability distributions. Hence, in order to make the uniform probability density function feasible as the proper prior probability distribution, it is generally assumed that the probability density function is locally uniform in the region where the likelihood function is appreciable, but not over its entire admissible range. Therefore, the non-informative prior probability distribution is generally called a locally uniform prior probability distribution.

Eq. 3.35 implies that in cases where the locally uniform prior probability distribution is employed?the constant c in the numerator and that in the denominator cancel each other out. Hence, the locally uniform prior probability distribution has no effect on the posterior probability distribution.

In cases where there is no uncertainty involved in the parameters?these probability distributions form the Dirac delta function (distribution):

$$\delta(\theta - \tilde{\theta}) = 0 \quad \theta \neq \tilde{\theta} \quad (3.41)$$

$$\int_m \delta(\theta - \tilde{\theta}) d\theta = 1 \quad (3.42)$$

where $\tilde{\theta}$ is a true and fixed value.

Now the multinormal distribution is defined by:

$$Prob(\theta) = \frac{1}{(2\pi)^{m/2} |V|^{1/2}} \cdot \exp \left[-\frac{1}{2} (\theta - \tilde{\theta})^T V^{-1} (\theta - \tilde{\theta}) \right] \quad (3.43)$$

where $\tilde{\theta}$ is the mean vector and V is the covariance matrix.

As all components of the covariance matrix go to zero, the multinormal distribution approaches the Dirac delta function (distribution). As will be shown later, a typical form of the posterior probability distribution is the multinormal distribution. The situation where all components of the covariance matrix go to zero is equivalent to that where the number of data goes to infinity. Therefore, as the number of data increases?uncertainty involved in the parameters reduces and approaches zero.

On the other hand, as all components of the covariance matrix go to infinity?the multinormal distribution approaches the uniform probability distribution.

The relationship between the uniform distribution?the normal distribution, and the Dirac delta function (distribution) is illustrated schematically in Fig. 3.2. For simplicity?the number of the parameters is assumed to be one. The mean value $\tilde{\theta}$ is assumed to be a true value of the parameter θ . The variance of the uniform distribution is infinite. As information about the parameter is obtained and uncertainty

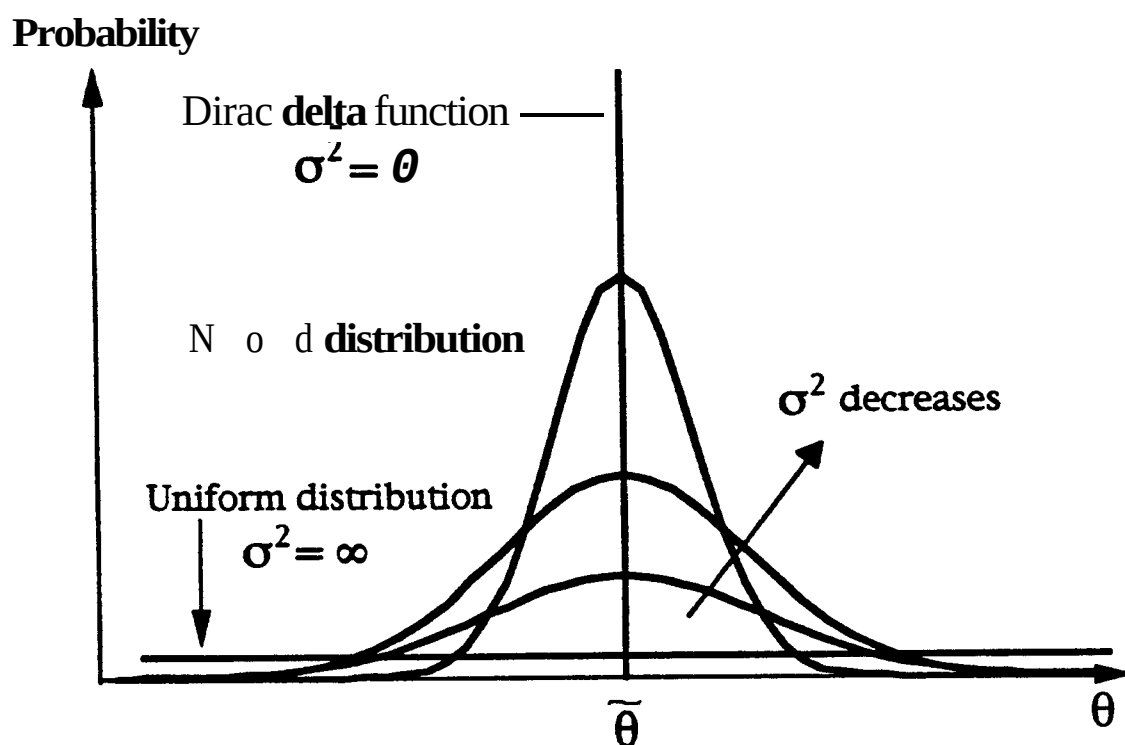


Figure 3.2: Schematic illustration of the relationship between the uniform distribution, the normal distribution and the Dirac delta function (distribution)

involved in the parameter is reduced, the variance of the normal distribution becomes smaller. Consequently, the spread of the normal distribution becomes narrower and the normal distribution has shorter tails. The normal distribution approaches the Dirac delta function (distribution) which has a zero variance.

3.5 Confidence Intervals

In this section, confidence intervals are derived and then the problems inherently involved in their application to model discrimination are discussed.

3.5.1 Confidence Intervals

The concept of confidence intervals can be explained by either sampling theory inference or Bayesian inference (Casella and Berger, 1990). Both inferences have completely different philosophies. In sampling theory inference the true value is assumed to be fixed but unknown, while in Bayesian inference the true value is a random variable with a specific probability distribution. Confidence intervals are calculated against the estimated values in sampling theory inference and against the true values in Bayesian inference. However, it is important to mention that both confidence intervals have numerically identical shapes as long as noninformative prior probability distributions are employed in Bayesian inference (Box and Tiao, 1973), and that it is possible to choose either inference. In this work, Bayesian inference is employed, since Bayesian inference makes it possible to develop the idea of the sequential predictive probability method used later. In Bayesian inference all information about the reservoir model and, consequently, the reservoir parameters should be expressed in terms of probability.

As discussed in Section 3.4.4, if there is no information available about the parameters before well testing, the information about these parameters is expressed as uniform probability distributions in the parameter space. Well test data include valuable information about these parameters and the purpose of well test analysis is to deduce this information and to use it to update the probability distributions in

the parameter space. Confidence intervals can provide a quantitative evaluation of the information obtained. This quantitative evaluation is not available in graphical analysis.

The direct application of confidence intervals to well test analysis requires two basic assumptions which are the same as discussed in Section 3.3.2. The first is that the errors, which are the differences between the actual pressure data and the true pressure responses, are independently normally distributed around the true pressure responses. This assumption forms a statistical basis of the least squares method. The other assumption is that for a region in the parameter space sufficiently close to the parameter estimates, the objective function can be expressed approximately in a linear form using a Taylor series expansion up to the first order.

The first assumption is used commonly in inverse problems, since it is the most unbiased assumption due to the Central Limit Theorem. The second assumption holds asymptotically, and as long as a large number of data is used this is a reasonable approximation to general reservoir model functions.

As long as the two assumptions mentioned above hold, the updated probability distributions of the unknown parameters form a multinormal distribution in parameter space. A feature of a multinormal distribution is that this distribution is fully characterized by only two parameters, its mean vectors and its covariance matrix. In nonlinear regression, these mean vectors are the parameter estimates and the covariance matrix is calculated as the inverse Hessian matrix of the objective function evaluated using the final values of the parameter estimates.

The mathematical derivation in the framework of Bayesian inference may be described as follows.

It is supposed that for a region in the parameter space sufficiently close to the parameter estimates, the model function can be expressed approximately in a linear form using Taylor series expansion up to the first order (recall Eq. 3.19):

$$F(\boldsymbol{\theta}) = F(\hat{\boldsymbol{\theta}}) + \sum_{j=1}^m \left(\frac{\partial F}{\partial \theta_j} \right)_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}} \cdot (\theta_j - \hat{\theta}_j) \quad (3.44)$$

It is assumed that the observed pressure data y_i is normally distributed about the

true response $F(\boldsymbol{\theta}, x_i)$ with a known variance σ^2

$$\begin{aligned} \text{Prob}(y_i|F(\boldsymbol{\theta}, x_i)) &= \text{Prob}(y_i|\boldsymbol{\theta}) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left[-\frac{1}{2\sigma^2}(y_i - F(\boldsymbol{\theta}, x_i))^2\right] \end{aligned} \quad (3.45)$$

After n independent observed pressure data, the likelihood function for the parameters is:

$$\begin{aligned} L(\boldsymbol{\theta}|y_1, \dots, y_n) &= \text{Prob}(y_1, \dots, y_n|\boldsymbol{\theta}) \\ &= \text{Prob}(y_1|\boldsymbol{\theta}) \cdots \text{Prob}(y_n|\boldsymbol{\theta}) \\ &= \prod_{i=1}^n \left[\frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left[-\frac{1}{2\sigma^2}(y_i - F(\boldsymbol{\theta}, x_i))^2\right] \right] \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^n} \cdot \exp\left[-\frac{1}{2\sigma^2}(\mathbf{R} - \mathbf{J} \cdot (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}))^T (\mathbf{R} - \mathbf{J} \cdot (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}))\right] \end{aligned} \quad (3.46)$$

where

$$\mathbf{R} = \begin{pmatrix} y_1 - F(\hat{\boldsymbol{\theta}}, x_1) \\ \vdots \\ y_n - F(\hat{\boldsymbol{\theta}}, x_n) \end{pmatrix} \quad (3.47)$$

and

$$\mathbf{J} = \begin{pmatrix} \left(\frac{\partial F}{\partial \theta_1}\right)_{x_1} & \cdots & \left(\frac{\partial F}{\partial \theta_m}\right)_{x_1} \\ \vdots & \ddots & \vdots \\ \left(\frac{\partial F}{\partial \theta_1}\right)_{x_n} & \cdots & \left(\frac{\partial F}{\partial \theta_m}\right)_{x_n} \end{pmatrix} \bigg|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}} \quad (3.48)$$

Since the least squares method is equivalent to the maximum likelihood function and the likelihood function is maximized if and only if

$$\mathbf{R}^T J = \mathbf{0} \quad (3.49)$$

then

$$L(\boldsymbol{\theta}|y_1, \dots, y_n) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \cdot \exp\left[-\frac{1}{2\sigma^2}(\mathbf{R}^T \mathbf{R} + (\mathbf{0} - \hat{\boldsymbol{\theta}})^T J^T J (\mathbf{0} - \hat{\boldsymbol{\theta}}))\right] \quad (3.50)$$

Now the Hessian matrix in the Gauss method, which is scaled by dividing by 2, is expressed as follows:

$$H = \begin{pmatrix} \sum_{i=1}^n \left(\frac{\partial F}{\partial \theta_1}\right) \left(\frac{\partial F}{\partial \theta_1}\right) & \cdots & \sum_{i=1}^n \left(\frac{\partial F}{\partial \theta_1}\right) \left(\frac{\partial F}{\partial \theta_m}\right) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n \left(\frac{\partial F}{\partial \theta_m}\right) \left(\frac{\partial F}{\partial \theta_1}\right) & \cdots & \sum_{i=1}^n \left(\frac{\partial F}{\partial \theta_m}\right) \left(\frac{\partial F}{\partial \theta_m}\right) \end{pmatrix} / \theta = \hat{\boldsymbol{\theta}} \quad (3.51)$$

Then

$$J^T J = H \quad (3.52)$$

If locally uniform prior probability distributions (noninformative prior probability distributions) are assumed for the parameters, then from Bayes theorem the posterior probability distribution of the parameters after n observations is:

$$\begin{aligned} Prob(\boldsymbol{\theta}|y_1, \dots, y_n) &= \frac{L(\boldsymbol{\theta}|y_1, \dots, y_n) \cdot Prob(\boldsymbol{\theta})}{\int_m L(\boldsymbol{\theta}|y_1, \dots, y_n) \cdot Prob(\boldsymbol{\theta}) d\boldsymbol{\theta}} \\ &= \frac{L(\boldsymbol{\theta}|y_1, \dots, y_n)}{\int_m L(\boldsymbol{\theta}|y_1, \dots, y_n) d\boldsymbol{\theta}} \end{aligned} \quad (3.53)$$

where $Prob(\mathbf{0})$ is the locally uniform prior probability distributions (noninformative prior probability distributions).

By definition of the multinormal distribution:

$$\int_m \frac{|H|^{1/2}}{(2\pi\sigma^2)^{m/2}} \cdot \exp\left[-\frac{1}{2\sigma^2}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T H (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})\right] d\boldsymbol{\theta} = 1 \quad (3.54)$$

Hence, the equation reduces to:

$$Prob(\boldsymbol{\theta}|y_1, \dots, y_n) = \frac{|H|^{1/2}}{(2\pi\sigma^2)^{m/2}} \cdot \exp\left[-\frac{1}{2\sigma^2}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T H (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})\right] \quad (3.55)$$

Therefore, the parameters $\boldsymbol{\theta}$ form a multinormal distribution about $\hat{\boldsymbol{\theta}}$ with the covariance matrix $\sigma^2 H^{-1}$. Eq. 3.55 expresses uncertainty involved in the estimated parameters quantitatively.

In cases where variance σ^2 is unknown, slight modifications are applied for the derivations above. σ^2 can be inferred from the error mean square (s^2). The error mean square is calculated from:

$$s^2 = \frac{SSR}{n - m} \quad (3.56)$$

where

$$SSR = \sum_{i=1}^n [y_i - F(\boldsymbol{\theta}, x_i)]^2 \quad (3.57)$$

Now s^2 is an unbiased estimator of σ^2 and σ^2 has an inverted gamma distribution about s^2 with $n - m$ degrees of freedom (Seber and Wild, 1989).

$$Prob(\sigma^2|s^2) = \frac{2}{\Gamma(\nu/2)} \left(\frac{\nu s^2}{2}\right)^{\frac{\nu}{2}} \frac{1}{\sigma^{\nu+1}} \cdot \exp\left[-\frac{\nu s^2}{2\sigma^2}\right] \quad (3.58)$$

where ν is $n - m$ degrees of freedom.

It should be mentioned that $\boldsymbol{\theta}$ and σ^2 are independently distributed (Box and Tiao, 1973). This indicates that $\hat{\boldsymbol{\theta}}$ is unchanged even in cases where σ^2 is replaced by s^2 .

Hence, the posterior probability distribution of $\boldsymbol{\theta}$ can be obtained by integrating out σ^2 from the joint posterior probability distribution of $\boldsymbol{\theta}$ and σ^2 .

$$\begin{aligned}
& Prob(\boldsymbol{\theta}|y_1, \dots, y_n) \\
&= \int_0^\infty Prob(\boldsymbol{\theta}, \sigma^2|y_1, \dots, y_n) d\sigma^2 \\
&= \int_0^\infty Prob(\boldsymbol{\theta}|y_1, \dots, y_n, \sigma^2) \cdot Prob(\sigma^2|y_1, \dots, y_n) d\sigma^2 \\
&= \int_0^\infty Prob(\boldsymbol{\theta}|y_1, \dots, y_n, \sigma^2) \cdot Prob(\sigma^2|s^2) d\sigma^2 \tag{3.59}
\end{aligned}$$

Substituting Eq. 3.54 and Eq. 3.58 into Eq. 3.59 leads to:

$$\begin{aligned}
& Prob(\boldsymbol{\theta}|y_1, \dots, y_n) = \\
& \frac{\Gamma(n/2)|H|^{1/2}s^{-m}}{[\Gamma(1/2)]^m\Gamma(\nu/2)(\sqrt{\nu})^m} \left[1 - \frac{(\mathbf{e} - \hat{\boldsymbol{\theta}})^T H (\mathbf{e} - \hat{\boldsymbol{\theta}})}{\nu s^2} \right]^{-(\nu+m)/2} \tag{3.60}
\end{aligned}$$

where ν is $n - m$ degrees of freedom.

Therefore, in cases where σ^2 is unknown, the parameters $\boldsymbol{\theta}$ form a multivariate student t distribution about $\hat{\boldsymbol{\theta}}$ with the covariance matrix $s^2 H^{-1}$ with the $n - m$ degrees of freedom.

Note that as n increases a multivariate student t distribution approaches a multinormal distribution and when $n - m$ is greater than 30 the multinormal distribution replaces the multivariate student t distribution sufficiently. Hence, in cases where $n - m$ is greater than 30, Eq. 3.55 can replace Eq. 3.60.

Fig. 3.3 shows a probability distribution of the parameters $\boldsymbol{\theta}$ ($= (\theta_1, \theta_2)^T$) and its corresponding marginal probability distribution of θ_1 and that of θ_2 . The marginal probability distribution of θ_1 is calculated by integrating out θ_2 over the probability distribution of $\boldsymbol{\theta}$ and the marginal probability distribution of θ_2 is calculated by integrating out θ_1 over the probability distribution of $\boldsymbol{\theta}$. In cases where there are more than two parameters, the marginal probability distribution of $\boldsymbol{\theta}$ is calculated by integrating out θ_i ($i \neq j, i = 1, \dots, m$) over the probability distribution of $\boldsymbol{\theta}$. The number of the marginal probability distributions is the same as the number of the parameters.

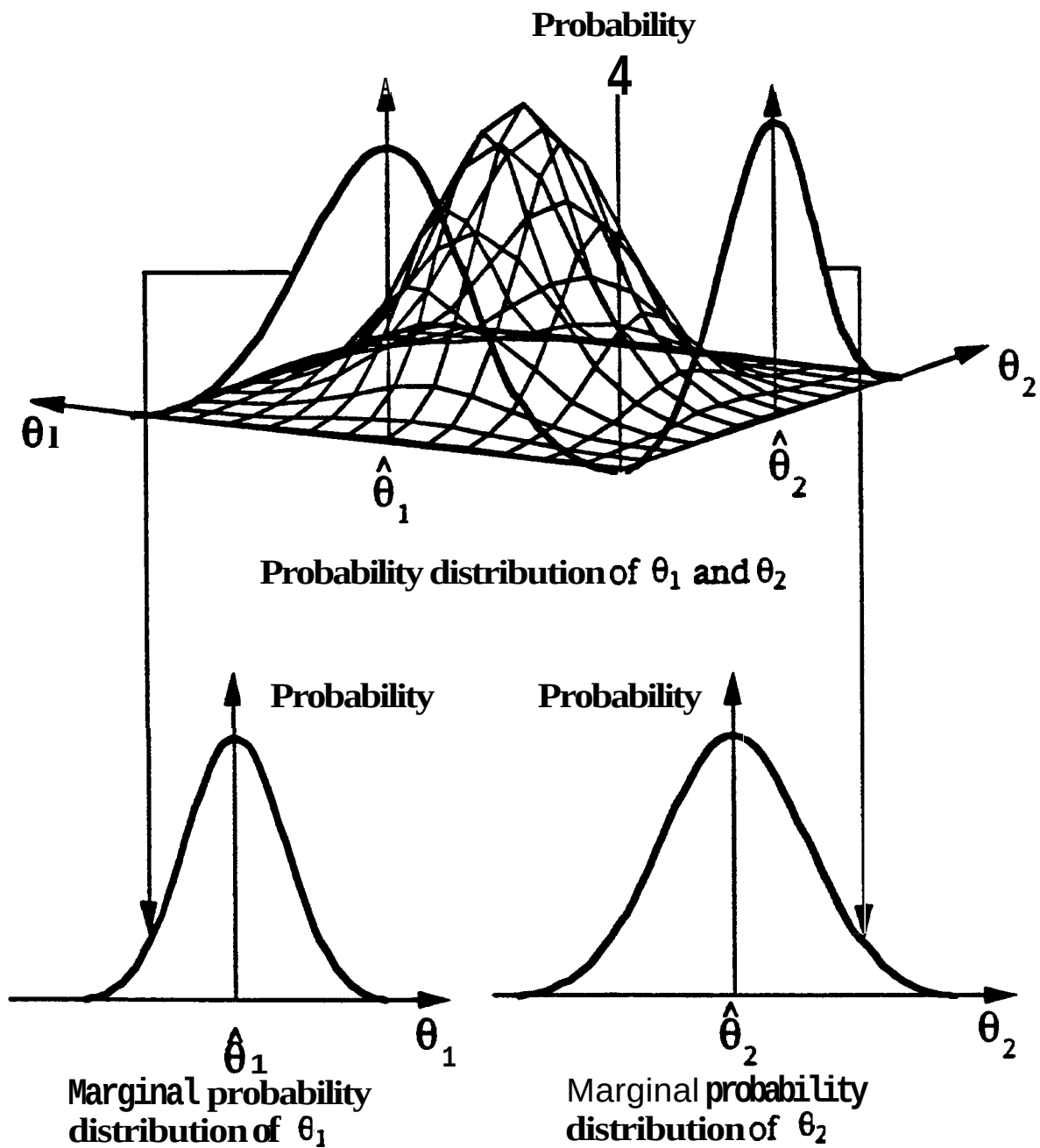


Figure 3.3: Probability distribution of θ_1 and θ_2 (upper) and its corresponding marginal probability distribution of θ_1 and that of θ_2 (lower)

The marginal probability distribution of θ_j ($j = 1, \dots, m$) for Eq. 3.55 is expressed by:

$$Prob(\theta_j|y_1, \dots, y_n) = \frac{1}{\sqrt{2\pi\sigma_{\theta_j}^2}} \cdot \exp\left[-\frac{1}{2\sigma_{\theta_j}^2}(\theta_j - \hat{\theta}_j)^2\right] \quad (3.61)$$

where $\hat{\theta}_j$ is the estimated value of a parameter θ_j . σ_{θ_j} is the standard deviation of each parameter, defined by:

$$\sigma_{\theta_j}^2 = \sigma^2 H_{jj}^{-1} \quad (3.62)$$

H_{jj}^{-1} is the j th diagonal element of the inverse Hessian matrix evaluated at $\theta = \hat{\theta}$.

Eq. 3.61 expresses uncertainty involved in the estimated parameter θ_j quantitatively.

As more information is obtained about the parameters from well test data, the spreads of the multinormal distribution become narrower and this distribution has shorter tails. Consequently, the spread of each marginal probability distribution becomes narrower and each marginal probability distribution has shorter tails. Confidence intervals are used to evaluate the spreads of the marginal probability distributions.

By definition, a 95% confidence interval covers 95% of the area under the probability distribution curve and is a range within which there is a 95% probability of the parameter values being true. As the probability distribution is a multinormal distribution, the corresponding marginal probability distribution of each parameter is symmetric about the parameter estimate and the 95% confidence interval is also symmetric about the parameter estimate.

Two types of confidence intervals are often used: one is the absolute value of the range and the other is the relative value of the range. The relative value of the range is a range scaled by dividing the absolute value of the range by the estimated value of the parameter so that it is free from the magnitude of the parameter estimate.

The basic equations for the definition of confidence intervals and correlation coefficients are derived as follows.

In cases where variance σ^2 is unknown, a $(1 - \alpha) \times 100\%$ absolute confidence interval on each parameter is computed from the following relationship:

$$\theta_j - \sigma_{\theta_j} \cdot t_{1-\frac{\alpha}{2}} \leq \theta_j \leq \theta_j + \sigma_{\theta_j} \cdot t_{1-\frac{\alpha}{2}} \quad (3.63)$$

where θ_j is the estimated value of a parameter θ_j . $t_{1-\frac{\alpha}{2}}$ is the tabulated value that cuts off $\frac{\alpha}{2} \times 100\%$ in the upper tail of the student t distribution with $n - m$ degrees of freedom.

σ_{θ_j} is the standard deviation of each parameter, defined by:

$$\sigma_{\theta_j}^2 = s^2 H_{jj}^{-1} \quad (3.64)$$

H_{jj}^{-1} is the j th diagonal element of the inverse Hessian matrix evaluated at $\theta = \hat{\theta}$.

As mentioned before, as n increases a student t distribution approaches a normal distribution and when $n - m$ is greater than 30 the normal distribution replaces the student t distribution sufficiently. Fig. 3.4 shows the relationship between the normal distribution and the student t distribution. Three cases of the student t distribution are shown according to the degrees of freedom. All distributions have zero mean and a variance of 1.0 psi^2 . In cases where $n - m$ is greater than 30, the value of $t_{1-\frac{\alpha}{2}}$ can be replaced by the corresponding value of the normal distribution. For $\alpha = 5$, the corresponding value of the normal distribution is 1.96.

Hence, in cases where $n - m$ is greater than 30, a 95% absolute confidence interval on each parameter can be computed by:

$$\hat{\theta}_j - 1.96 \cdot \sigma_{\theta_j} \leq \theta_j \leq \hat{\theta}_j + 1.96 \cdot \sigma_{\theta_j} \quad (3.65)$$

In cases where variance σ^2 is known in advance, Eq. 3.65 is used for a 95% absolute confidence interval on each parameter.

Fig. 3.5 shows a 95% absolute confidence interval on the parameter θ . The probability distribution of θ in Fig. 3.5 can correspond to one of the marginal probability distributions of the probability distribution of θ in Fig. 3.3.

In cases where variance σ^2 is unknown, a $(1 - \alpha) \times 100\%$ relative confidence interval on each parameter is computed from the following relationship:

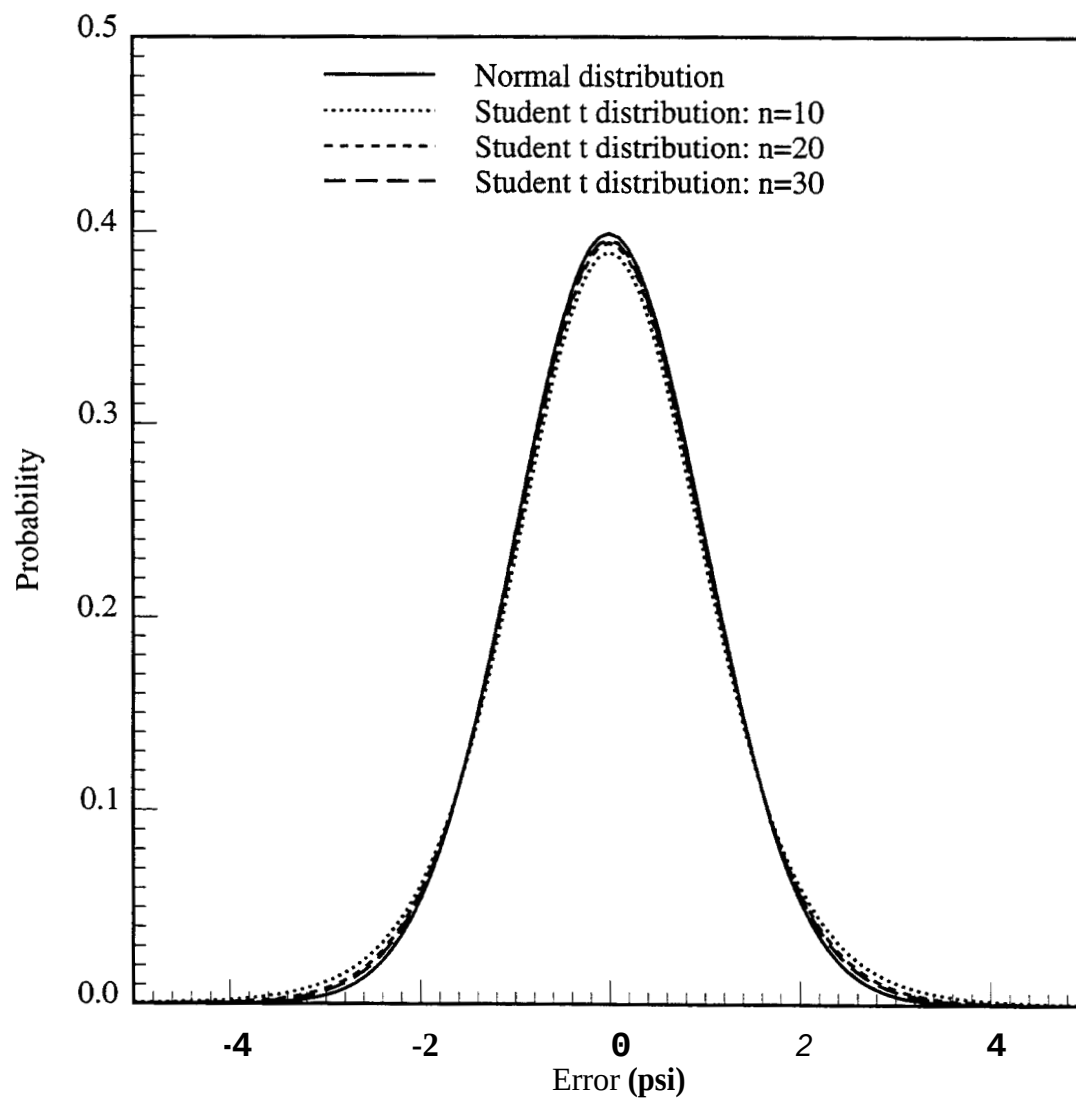


Figure 3.4: Relationship between the normal distribution and the student t distribution

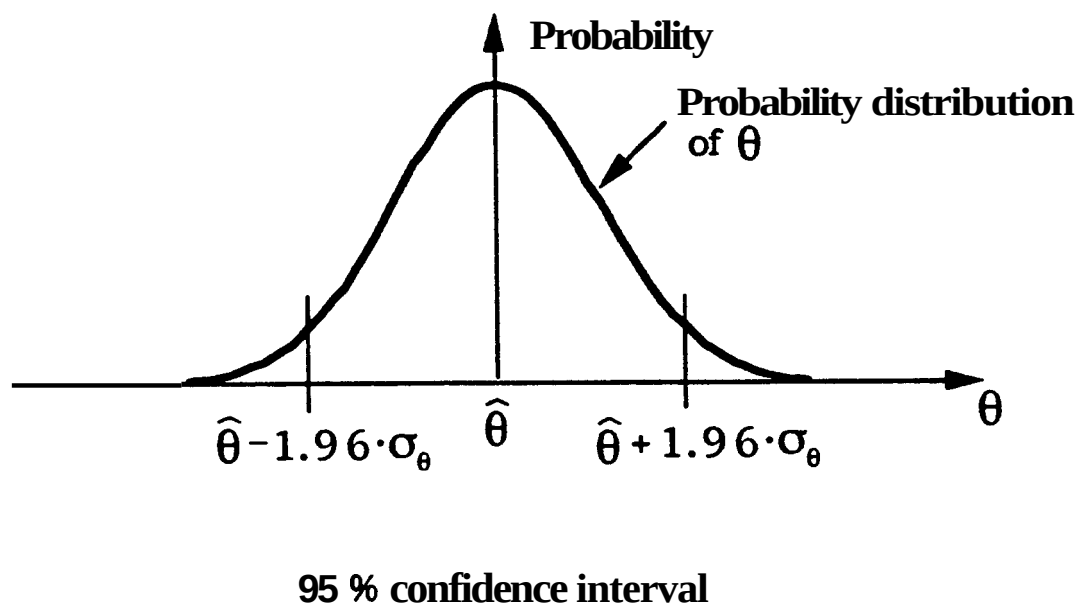


Figure 3.5: 95% absolute confidence interval

$$1 - \frac{\sigma_{\theta_j} \cdot t_{1-\frac{\alpha}{2}}}{\hat{\theta}_j} \leq \frac{\theta_j}{\hat{\theta}_j} \leq 1 + \frac{\sigma_{\theta_j} \cdot t_{1-\frac{\alpha}{2}}}{\hat{\theta}_j} \quad (3.66)$$

Similarly, in cases where $n - m$ is greater than 30 or variance σ^2 is known in advance, a 95% relative confidence interval on each parameter can be computed by:

$$1 - \frac{1.96 \cdot \sigma_{\theta_j}}{\hat{\theta}_j} \leq \frac{\theta_j}{\hat{\theta}_j} \leq 1 + \frac{1.96 \cdot \sigma_{\theta_j}}{\hat{\theta}_j} \quad (3.67)$$

Here confidence intervals are calculated for the corresponding parameters, and the number of confidence intervals is the same as that of the parameters.

The correlation coefficient between any two parameters is computed from the off-diagonal elements of the inverse Hessian matrix evaluated at $\theta = \hat{\theta}$:

$$\rho_{ij} = \frac{H_{ij}^{-1}}{\sqrt{H_{ii}^{-1} H_{jj}^{-1}}} \quad (3.68)$$

The correlation coefficient has a limit of $-1 < \rho_{ij} < 1$. A zero value of ρ_{ij} implies that there is no mathematical correlation between parameter θ_i and θ_j . When $\rho_{ij} = 1$, parameter θ_i and parameter θ_j have a perfect positive correlation.

As long as there are mathematical correlations among parameters, none of the parameters are uniquely determined.

To express both confidence intervals and correlations simultaneously, confidence regions can be constructed.

A $(1 - \alpha) \times 100\%$ confidence region for the parameters is given by:

$$(\theta - \hat{\theta})^T H (\theta - \hat{\theta}) \leq m s^2 F_{1-\alpha}(m, n - m) \quad (3.69)$$

where $F_{1-\alpha}(m, n - m)$ is a tabulated F value at $1 - \alpha$ confidence level with m and $n - m$ degrees of freedom.

Notice that the boundary of Eq. 3.69 forms an ellipsoid centered about $\hat{\theta}$ in the parameter space.

In order to make it convenient to apply these confidence intervals to model verification, attention is paid to the variances of the probability distributions of the

Table 3.1: Acceptable confidence intervals (from Horne (1990))

Parameters	Relative interval	Absolute interval
Permeability (k)	10	
Wellbore storage constant (C)	10	
Distance to the boundary (r_e)	10	
Fracture length (x_f)	10	
Storativity ratio (w)	20	
Transmissivity ratio (λ)	20	
Skin		10
Initial pressure (P_i)		1.0 (<i>psi</i>)

parameters rather than the correlations between the parameters. Limits on acceptable confidence intervals were suggested by Horne (1990) (Table 3.1). These criteria of acceptability were defined heuristically, based on actual experience with interpretation of real and synthetic well test data.

The underlying philosophy is that as long as the model selected is correct and there are sufficient data available to support this model, all parameters should be within these acceptable ranges. Thus, as long as all the parameter estimates are within the acceptable ranges, the model can be accepted. Otherwise, the model should be rejected, since confidence intervals do not support the model from a statistical point of view.

The variance of the probability distribution of each parameter is the product of the error mean square and the corresponding diagonal element of the inverse Hessian matrix (Eq. 3.64). It is worth mentioning that the error mean square is a function of the actual data but the inverse Hessian matrix is a function of the reservoir model and not of the data.

The error mean square is used to represent the variance of the errors, which is a finite value as long as a correct model is selected. If a correct model is selected, the error mean square represents the variance of the errors quite well, independent of the number of the data and the time region of the data. However, the use of an incorrect

Table 3.2: Reservoir and fluid data

Wellbore Radius (r_w)	<i>ft</i>	0.25
Reservoir Thickness (h)	<i>ft</i>	20.0
Formation Volume Factor (B_o)	<i>bbl/STB</i>	1.0
Viscosity (μ)	<i>cp</i>	1.0
Porosity (ϕ)		0.2
Initial Pressure (P_i)	<i>psia</i>	3000.0
Total Compressibility (c_t)	<i>psi⁻¹</i>	1.0×10^{-6}
Flow Rate (q)	<i>STB/D</i>	500.0

model makes the error mean square larger than the actual value of the variance of the errors.

The inverse Hessian matrix as well as the Hessian matrix are functions of the number of the reservoir parameters, which is equivalent to the selection of the model, the correlations between the parameters, the number of the data, and the time region of the data. In general, reservoir models are nonlinear functions of different parameters over different time ranges. It is a basic characteristic of the diagonal elements of the inverse Hessian matrix that they decrease monotonically as a function of the number of data. When a sufficient number of data is available, the diagonal elements of the inverse Hessian matrix become small.

Here it is demonstrated how confidence intervals are used to decide whether a model is acceptable or not. Simulated drawdown data were employed. The purpose in using simulated data is to show how confidence intervals work on a problem in which it is known in advance whether the reservoir model is correct or not. Two cases were considered: one is the case where a correct model is used and the other is the case where an incorrect model is used. In each case, four cases were considered according to the number of data points in order to demonstrate the basic characteristics of confidence intervals.

First, the case where the correct model was used is described. Drawdown pressure data were calculated using an infinite acting model and random errors were added. The reservoir and fluid data are given in Table 3.2. The true parameter values are

$k = 50md$, $S = 10$, and $C = 0.01STB/psi$. A random number generator was used to simulate random normal errors with zero mean and a variance of $1.0 psi^2$. According to the number of data points, the following four cases were considered: (a) 51 data points (the time period from 0.01 hour to 3.162 hours), (b) 61 data points (the time period from 0.01 hour to 10 hours), (c) 71 data points (the time period from 0.01 hour to 31.62 hours), and (d) 81 data points (the time period from 0.01 hour to 100 hours). The infinite acting model with three parameters (k , S , and C) was employed. In each case, the parameters were estimated using nonlinear regression for the corresponding data points. The matches of the model to the data and pressure data are shown in Fig. 3.6 and Fig. 3.7.

For simplicity, the estimation of permeability is illustrated. The marginal probability distributions of permeability are given in Fig. 3.8. The corresponding 95 % confidence intervals on permeability are given in Table 3.3. The starting time of the infinite acting behavior is calculated by:

$$t_D = C_D (60 + 3.5S) \quad (3.70)$$

where

$$t_D = \frac{0.000264kt}{\phi\mu c_i r_w^2} \quad (3.71)$$

$$C_D = \frac{5.615C}{2\pi\phi c_i h r_w^2} \quad (3.72)$$

From Eq. 3.70, the starting time of the infinite acting behavior is 3.22 hours, which is just after the 51st data point (3.162 hours). Hence, in cases (b), (c), and (d) the data contain valuable information about permeability. In Table 3.3, the estimated values of permeability are all close to the true value ($k = 50md$). Therefore, all probability distributions of permeability are distributed around the true value in Fig. 3.8. As the number of data points increases, the data contain more information about permeability and the corresponding variance about permeability (σ_θ) decreases in Table 3.3. In Fig. 3.8, the spread of the probability distribution becomes narrower and the normal distribution approaches the Dirac delta function and converges to

Table 3.3: 95% confidence intervals on permeability in the case where the correct model was used

	(a)	(b)	(c)	(d)
Data points	51	61	71	81
Parameter estimate	52.05	49.76	49.70	50.11
s^2	0.7966	0.9272	0.9904	0.9710
H^{-1}	2.0510	0.2204	0.0674	0.0305
$\sigma_\theta^2 = s^2 H^{-1}$	1.6338	0.2044	0.0668	0.0296
σ_θ	1.2782	0.4521	0.2584	0.1721
95% confidence interval (%)	4.81	1.78	1.02	0.67
Decision	acceptable	acceptable	acceptable	acceptable

Table 3.4: 95% confidence intervals on permeability in the case where the incorrect model was used

Case	(a)	(b)	(c)	(d)
Data points	51	61	71	81
Parameter estimate	52.02	49.57	39.94	24.00
s^2	0.7972	0.9830	32.05	1142
H^{-1}	2.0470	0.2171	0.0283	0.0016
$\sigma_\theta^2 = s^2 H^{-1}$	1.632	0.2134	0.9053	1.8650
σ_θ	1.2775	0.4620	0.9515	1.3657
95% confidence interval (%)	4.81	1.83	4.67	11.15
Decision	acceptable	acceptable	acceptable	unacceptable

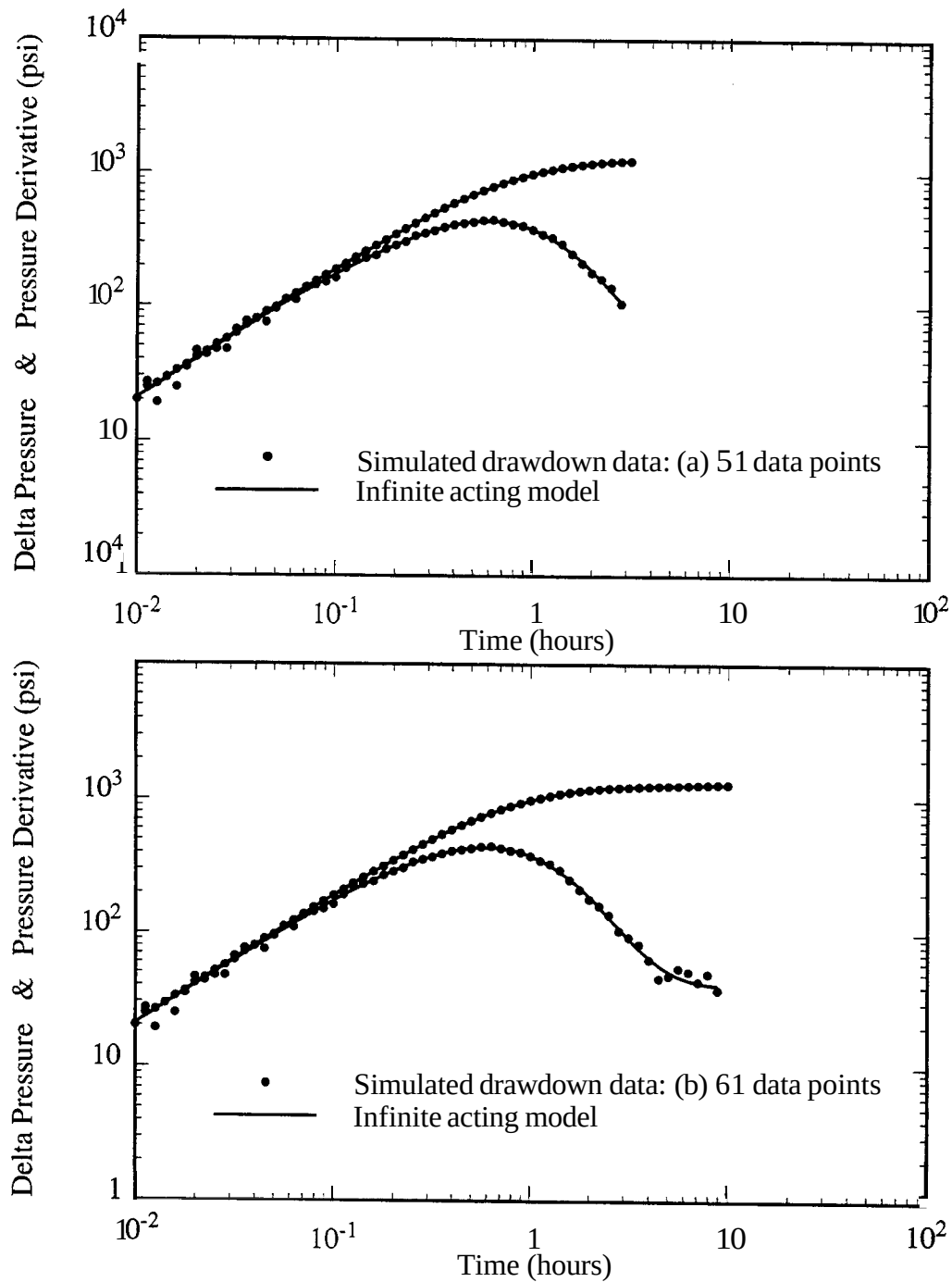


Figure 3.6: Simulated drawdown data and matches of the correct model to the data: (a) 51 data points (upper) and (b) 61 data points (lower)

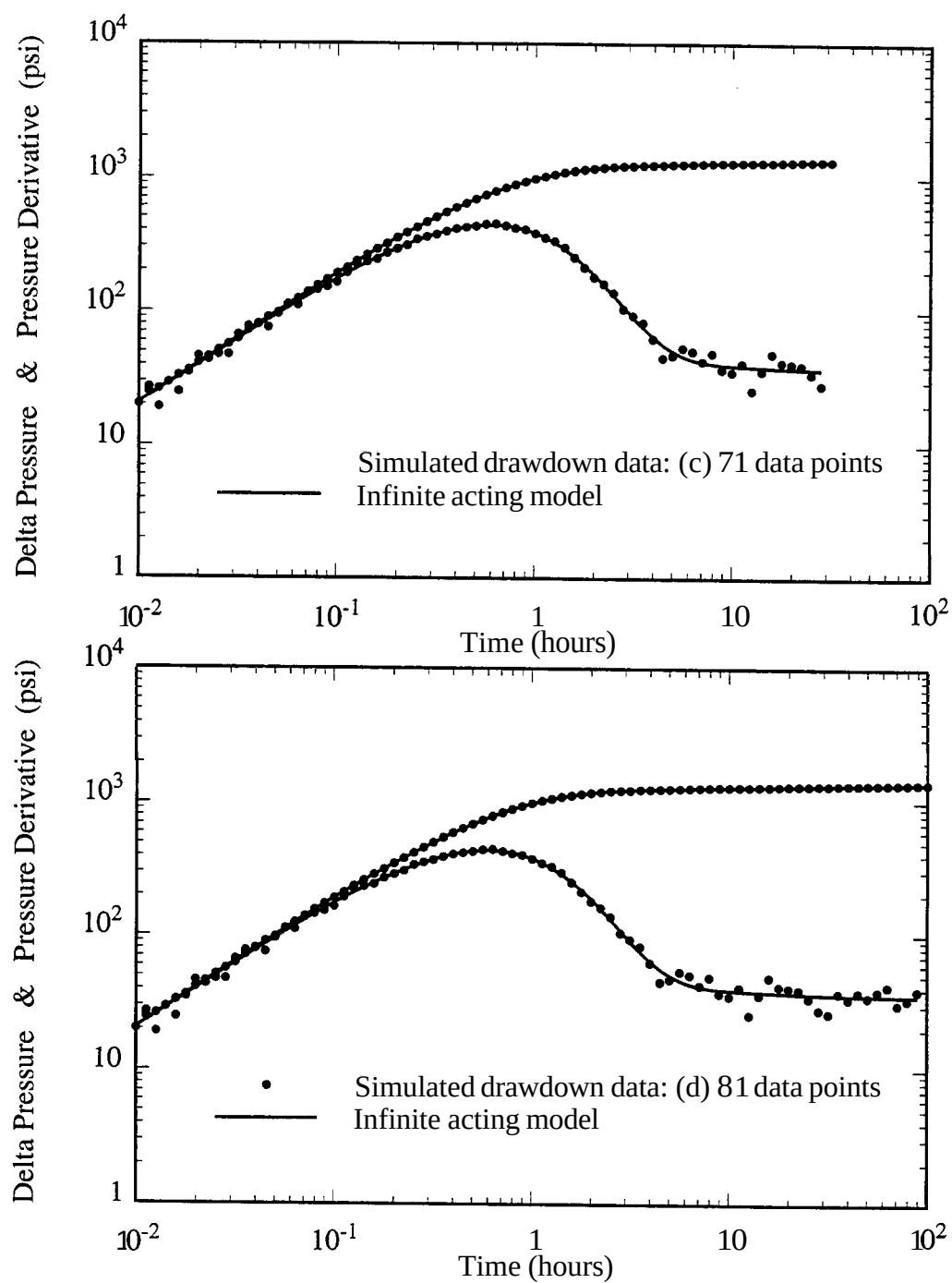


Figure 3.7: Simulated drawdown data and matches of the correct model to the data: (c) 71 data points (upper) and (d) 81 data points (lower)

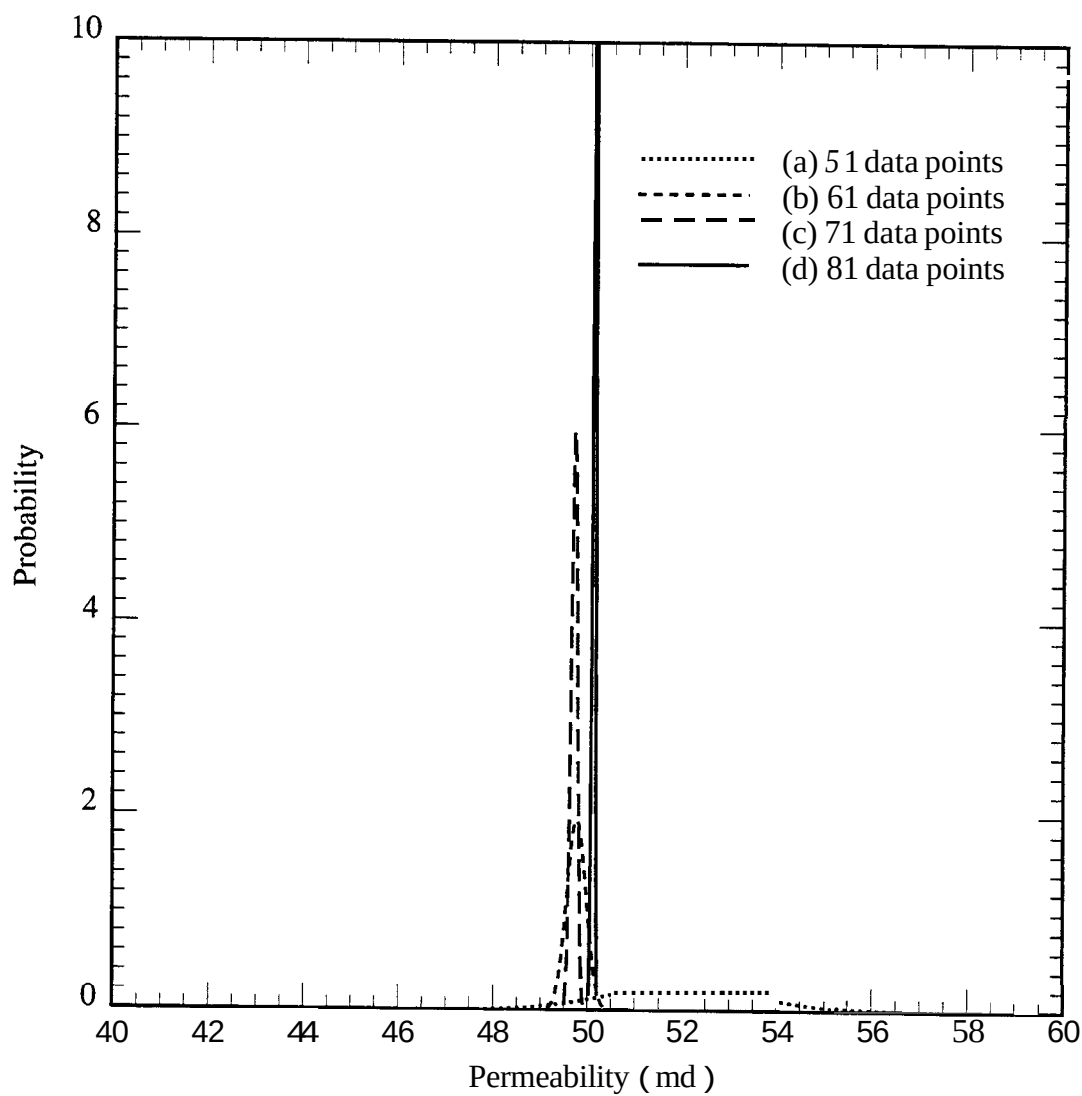


Figure 3.8: Probability distributions of permeability in the case where the correct model is used: (a) 51 data points, (b) 61 data points, (c) 71 data points, and (d) 81 data points

the true value. All data points in case (a) are taken prior to the starting time of the infinite acting behavior and the 95% relative confidence interval is relatively wide. In case (b), (c) and (d) the 95% confidence intervals are fairly narrow. From Table 3.1, the limit on the acceptable confidence interval for permeability is 10 %. According to this limit, in all cases the model is accepted as a correct model.

Next, the case where the incorrect model was used is described. Drawdown pressure data were calculated using a no flow outer boundary model and random errors were added. The same reservoir and fluid data were used as in the previous case (Table 3.2). The true parameter values are $k = 50md$, $S = 10$, $C = 0.01STB/psi$, and $r_e = 1500ft$. The random errors were identical to the previous case. The same four cases were considered: (a) 51 data points, (b) 61 data points, (c) 71 data points, and (d) 81 data points. The infinite acting model with three parameters (k , S , and C) was employed, which is an incorrect model. In each case, the parameters were estimated using the corresponding data points. The matches of the model to the data and pressure data are shown in Fig. 3.9 and Fig. 3.10.

The marginal probability distributions of permeability are given in Fig. 3.11. The corresponding 95 % confidence intervals on permeability are given in Table 3.4. The starting time of the pseudosteady state is calculated by:

$$t_{DA} = 0.1 \quad (3.73)$$

where

$$t_{DA} = \frac{0.000264kt}{\phi\mu c_t \cdot \pi r_e^2} \quad (3.74)$$

From **Eq. 3.73**, the starting time of the pseudosteady state is 10.71 hours, which is just after the 61st data point (10 hours). Hence, in cases (c) and (d) the infinite acting model is not expected to match the data adequately. In Table 3.4, the estimated values of permeability in cases (a) and (b) are all close to the true value ($k = 50md$). However, those in cases (c) and (d) are far from the true value. Therefore, the probability distributions of permeability in cases (a) and (b) are distributed around the true value but those in cases (c) and (d) are not (Fig. 3.11). As the number of

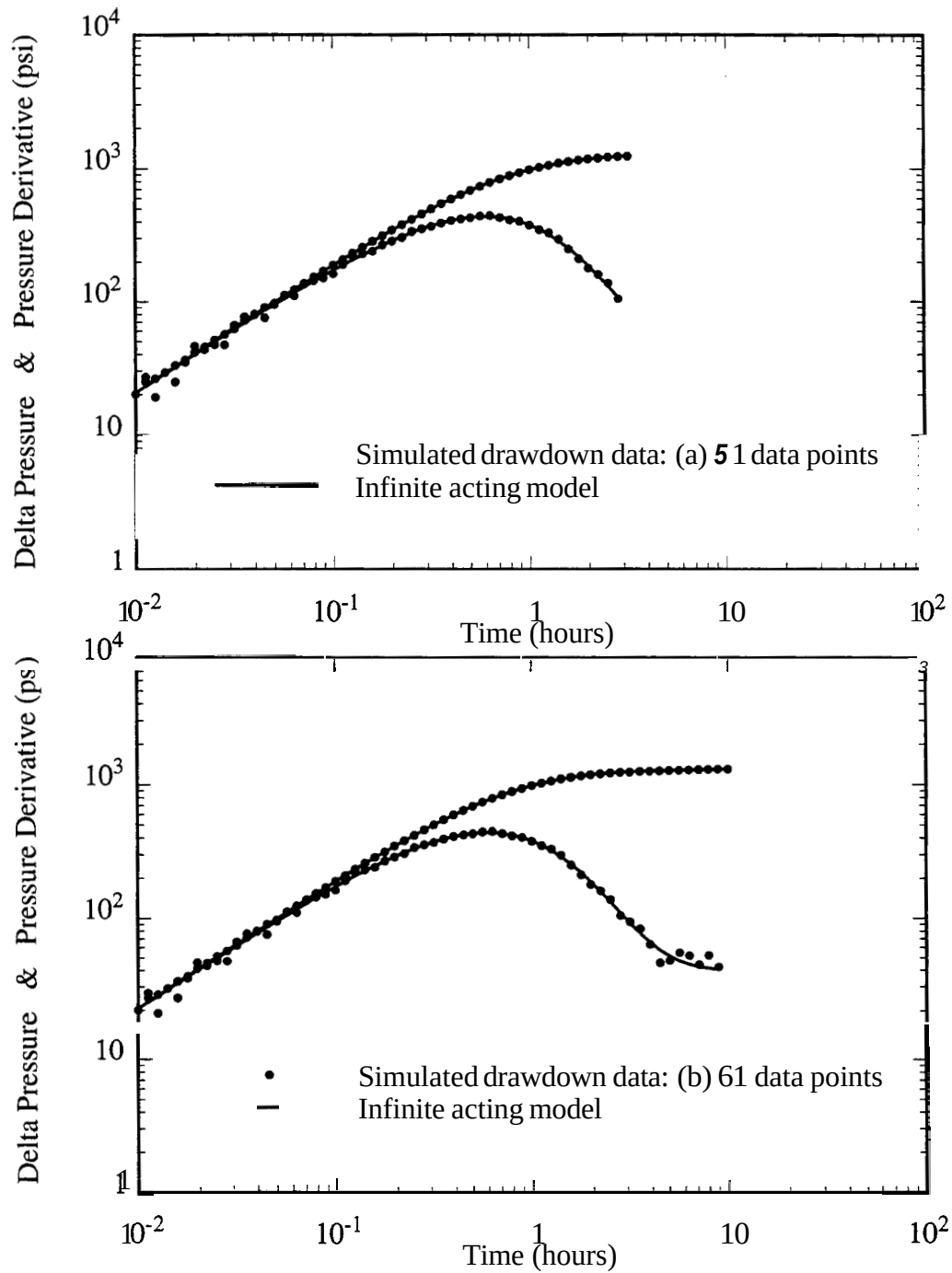


Figure 3.9: Simulated drawdown data and matches of the incorrect model to the data: (a) 51 data points (upper) and (b) 61 data points (lower)

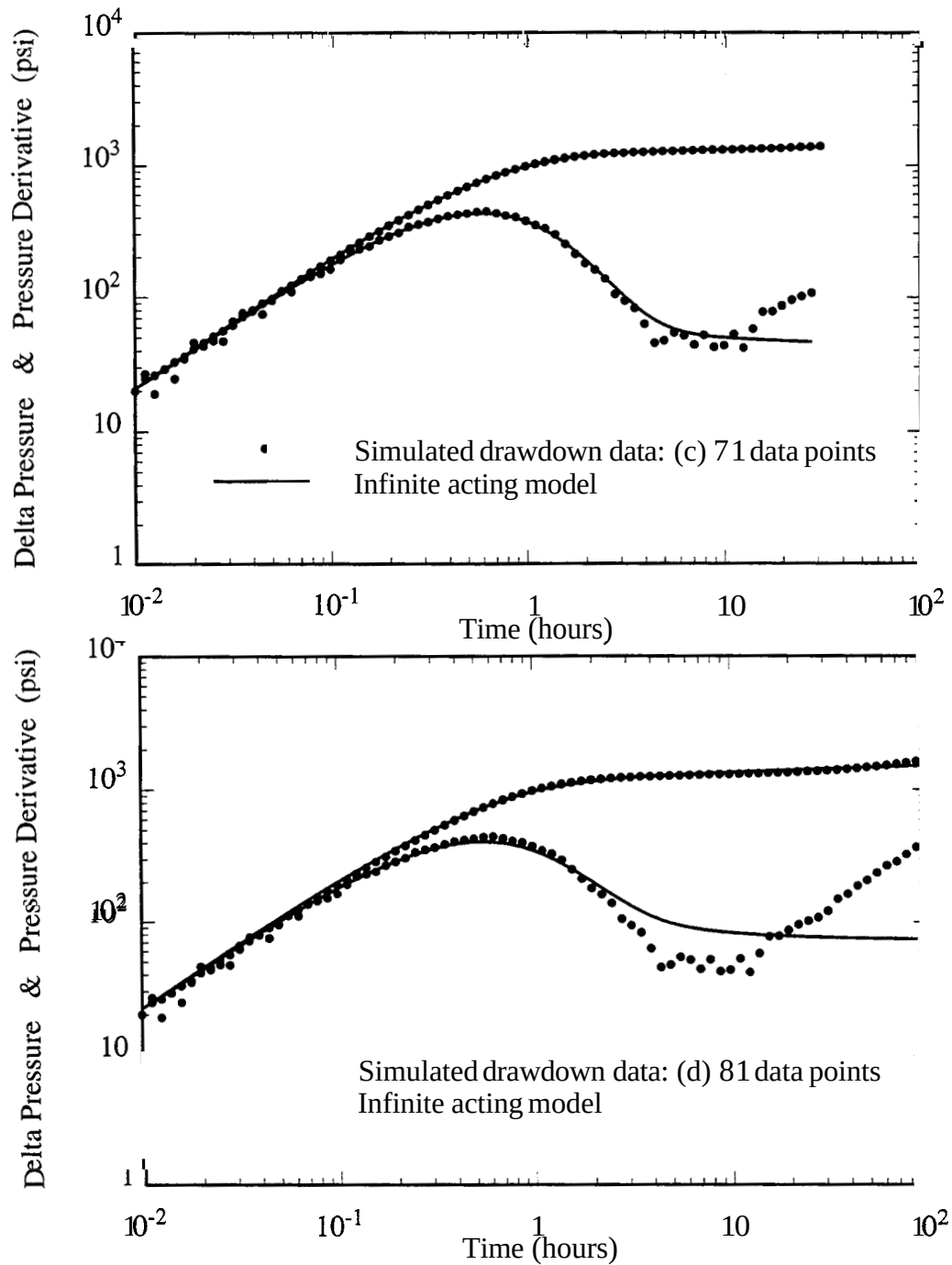


Figure 3.10: Simulated drawdown data and matches of the incorrect model to the data: (c) 71 data points (upper) and (d) 81 data points (lower)

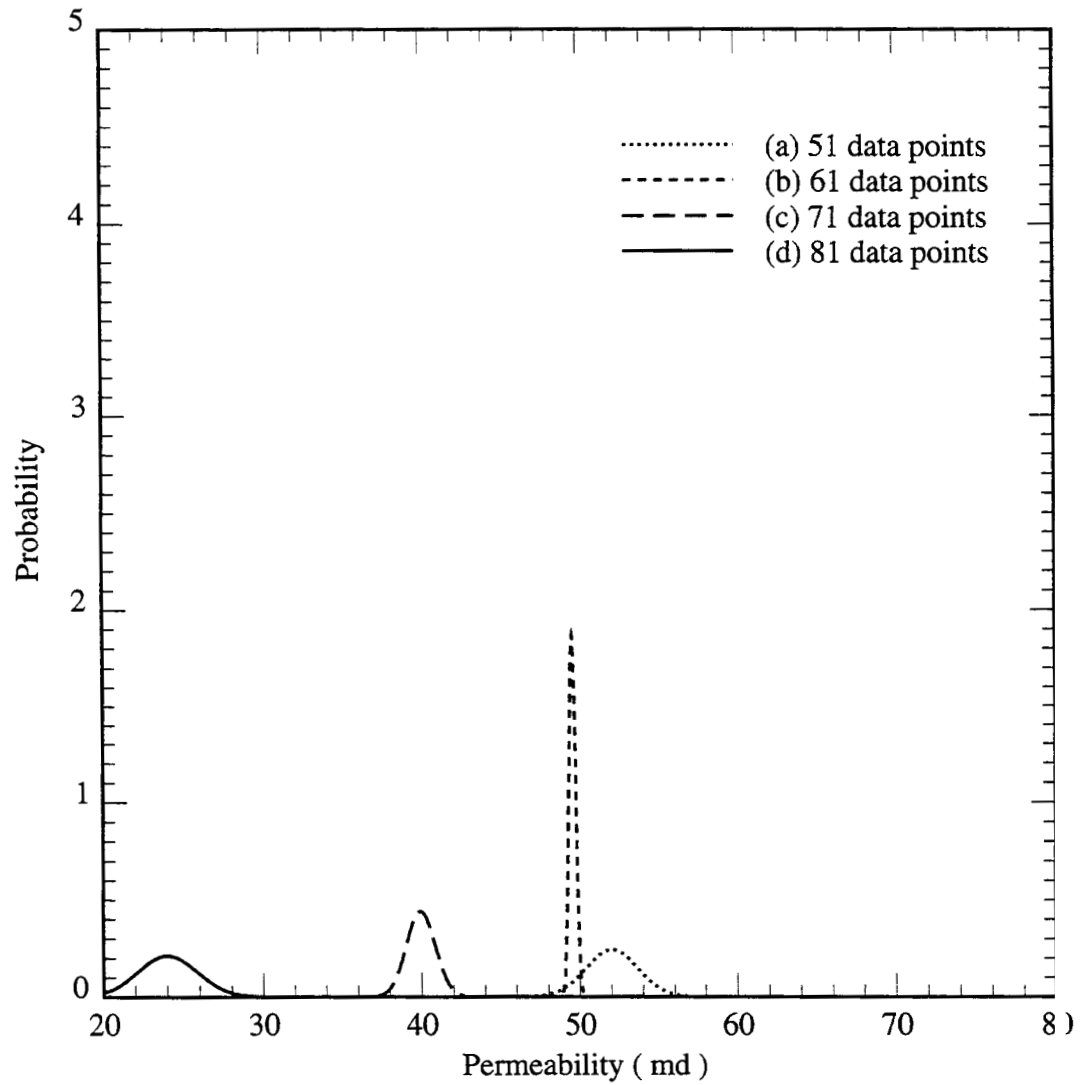


Figure 3.11: Probability distributions of permeability in the case where the incorrect model is used: (a) 51 data points, (b) 61 data points, (c) 71 data points, and (d) 81 data points

data points increases from case (a) to case (b), the data contain more information about permeability and the corresponding variance about permeability (σ_θ) decreases in Table 3.4. However, in cases (c) and (d) the model no longer represents the data adequately and the estimated variances (s^2) become large in Table 3.4. As a result, the corresponding variance about permeability is increasing from case (c) to case (d). In Fig. 3.11, the spread of the probability distribution becomes narrower from case (a) to case (b) but it becomes wider from case (c) to case (d). The normal distribution does not approach the Dirac delta function and does not converge to the true value. All data points in case (a) are taken prior to the starting time of the infinite acting behavior and the 95% relative confidence interval is relatively wide. In case (b), the 95% confidence interval is fairly narrow, since more data containing information about permeability are available. However, in cases (c) and (d) the 95% confidence intervals are fairly wide. According to the limit on the acceptable confidence interval for permeability of 10 %, in cases (a), (b), and (c) the model is accepted as a correct model but in case (d) the model is not accepted as a correct model.

As described above, confidence intervals are used to decide whether a model is acceptable or not. In both cases where either the correct model or the incorrect model is used, confidence intervals produce consistent results in the final case (d). Note that only the confidence interval for permeability is shown but in actual applications of confidence intervals to model verification, the confidence intervals for all parameters should be investigated. Confidence intervals are easy to calculate, since all components related to confidence intervals are evaluated during the nonlinear regression. In addition, confidence intervals are easy to use for model verification, as described above.

However, it should be mentioned that confidence interval analysis has two weak points from the point of view of model discrimination. One is practical and the other is theoretical.

Firstly, the practical weak point is discussed. Confidence intervals are directly proportional to the variances of the probability distributions of the parameters and several different cases could be possible as a combination of the error mean square

(estimated variance) (s^2) and the diagonal elements of the inverse Hessian matrix (H_{jj}^{-1}). For example, in case (c) in Table 3.4 the estimated variance is **32.05 psi²**, which is much larger than the true variance (1.0 psi²). However, the diagonal element of the inverse Hessian matrix is relatively small (**0.9053**) due to the large number of data points (71 data points) and the corresponding variance of probability distribution of permeability is merely **0.9053**. As a result, the **95%** confidence interval is 4.67 (%), which is relatively narrow and within the acceptable range even though the incorrect model is employed. In Fig. 3.10 the graphical analysis using the pressure derivative plot suggests the existence of no flow boundary, since a steep rising straight line is recognized on the pressure derivative plot. Hence, confidence intervals are sometimes within apparently acceptable ranges, even if an incorrect model is employed. This is the practical weak point.

Secondly, the theoretical weak point is discussed. Confidence interval analysis is useful for model verification but is not suitable for model discrimination. In other words, confidence interval analysis can decide whether a specific model is acceptable or not, but it says little about which model is better. In cases where several models are possible, confidence interval analysis cannot provide a unified measure of model discrimination. In order to make it convenient to use confidence interval analysis for model verification, attention is paid to the variances of the probability distributions of the parameters rather than the correlations between the parameters. However, in general reservoir parameters have correlations with each other in nonlinear regression and theoretically these correlations should be involved for model verification. Moreover, and more importantly, Eq. 3.55 indicates that the probability distribution of the parameters has the same dimension as the number of parameters. Hence, different models with different number of parameters have probability distributions with different dimensions. In order to compare probability distributions with different dimensions completely, direct comparison of the corresponding confidence interval alone is not sufficient.

These weak points of confidence interval analysis will be discussed further in Section 5.2 through an example.

3.5.2 Exact Confidence Intervals

Confidence intervals mentioned in Section 3.5.1 are valid only if the model is linear. Under the assumption that for a region in the parameter space sufficiently close to the parameter estimates the model function can be approximately expressed in the linear form using Taylor series expansion up to the first order, confidence intervals are approximately true.

Exact confidence regions for a nonlinear function can be constructed in the framework of sampling theory inference, since Bayesian inference holds only as long as a model is linear.

A $(1-\alpha) \times 100\%$ exact confidence region for the parameters in a nonlinear function can be obtained by:

$$E(\boldsymbol{\theta}) \leq E(\hat{\boldsymbol{\theta}}) \left[\frac{m}{n-m} F_{1-\alpha}(m, n-m) + 1 \right] \quad (3.75)$$

or

$$E(\boldsymbol{\theta}) - E(\hat{\boldsymbol{\theta}}) \leq m s^2 F_{1-\alpha}(m, n-m) \quad (3.76)$$

where

$$\begin{aligned} E(\boldsymbol{\theta}) &= \text{objective function evaluated at } \boldsymbol{\theta} \\ E(\hat{\boldsymbol{\theta}}) &= \text{objective function evaluated at } \hat{\boldsymbol{\theta}} \\ &= \text{SSR (Eq. 3.57)} \\ s^2 &= \frac{\text{SSR}}{n-m} \text{ (Eq. 3.56)} \\ F_{1-\alpha}(m, n-m) &= \text{a tabulated } F \text{ value at } 1-\alpha \text{ confidence level} \\ &\quad \text{with } m \text{ and } n-m \text{ degrees of freedom} \\ n &= \text{number of data} \\ m &= \text{number of parameters} \end{aligned}$$

The boundary of Eq. 3.75 always coincides with an actual contour of the objective function in the parameter space.

Comparison between the boundary defined by Eq. 3.69 and that by Eq. 3.75 reveals the validity of the linear approximation.

3.5.3 F test

Watson *et al.* (1988) presented a systematic and quantitative method to select from among possible reservoir models the most appropriate model for a given set of pressure data. The method has its basis in sampling theory inference. This method may be considered as a bridge between confidence intervals and the sequential predictive probability method, and thus needs to be documented in full.

First of all, a hierarchy of reservoir models is constructed according to the number of independent reservoir parameters involved in the reservoir models. The simplest model is defined as a homogeneous infinite acting reservoir model, which has the least number of parameters (two parameters: k and C). More complicated models have more parameters. In the hierarchy, simpler models can be derived from more complicated models on the same branch by setting certain parameters to known values. In other words, models should be subsets or supersets of each other. For example, a single porosity (homogeneous) model can be obtained from a double porosity model by setting the storativity ratio, ω , and the interporosity flow coefficient, λ , to 1 and 0, respectively. Hence, a single porosity model is a subset of a double porosity model and a double porosity model is a superset of a single porosity model.

The underlying philosophy is to select the simplest model that describes the data adequately within the hierarchy. A model that has too many parameters may result in parameter estimates that have larger uncertainty associated with them. The most accurate estimates of reservoir parameters should be given by the model with the fewest independent parameters that fully describes that data. On the other hand, any set of data that can be matched by a simple model can also be matched by a more complicated model on the same hierarchy branch, and the more complicated model is likely to show a better match to the data due to the decrease of the degrees of freedom. To accommodate these two aspects and make a choice between a simple model and a more complicated model, the concept of an F test is introduced.

Before an F test is applied, it is important to guarantee that at least one model within the hierarchy satisfactorily describes the data. Therefore, this method relates to the model verification problem, not the model recognition problem.

In general, a statistical test is a decision to accept or reject a statistical hypothesis,

In this case, the statistical hypothesis is a statement about the values of one or more parameters in a reservoir model and a test statistic is an F statistic. The decision to accept or reject the hypothesis is made quantitatively by comparing the test statistic calculated from the data with a tabulated critical value of the test statistic. It is important to mention that decisions based on the limited number of data cannot be made with complete certainty. Therefore, the concept of a confidence interval of $(1 - \alpha) \times 100\%$ is introduced to qualify the decisions.

By definition, an F statistic is a ratio of two sample variances associated with two normally distributed populations, each with identical variances. As long as errors are distributed normally, the error mean square, which represents the overall model fitting, is the most unbiased estimate of the sample variance.

Consider the set of unknown reservoir parameters, θ , with dimension m , involved in a reservoir model. Suppose θ can be partitioned into two subsets, θ_1 and θ_2 , with dimension $m_1 = m - m_2$ and m_2 , respectively. The minimum values of the objective function (Eq. 3.8) are evaluated for two different situations.

If the model is expressed by all elements of θ which can vary freely when the objective function (Eq. 3.8) is being minimized, the model is considered a complicated model. The minimum value of the objective function obtained using a complicated model is denoted as E_{comp} . On the other hand, in cases where some of the elements, θ_2 , are fixed with specific values and only the rest of the parameters, θ_1 , can be changed, the model reduces to a simple model. The minimum value of the objective function obtained using a simple model is denoted as E_{simp} . Then, the test statistic F is defined as:

$$F = \frac{(E_{simp} - E_{comp}) / m_2}{E_{comp} / (n - m)} \quad (3.77)$$

where n is the number of data.

A statistical decision is made by comparing the value of F obtained from Eq. 3.77 with a critical value, $F_{1-\alpha}(m_2, n - m)$, which can be found in standard statistical tables. If F exceeds $F_{1-\alpha}(m_2, n - m)$, the statistical hypothesis is rejected at the confidence interval of $(1 - \alpha) \times 100\%$. This means that the parameters, θ_2 , are worth adding to the reservoir model and the complicated model is warranted.

The value of F calculated by Eq. 3.77 is proportional to the reduction in the minimum value of the objection function using a complicated model. If this reduction is relatively large, which means the overall model fitting of the model to the data is moderately improved, the additional parameters are significant for representing the data. If the reduction is relatively small, which means that the data have been described adequately by a simple model, the additional parameters are not necessary.

Watson *et al.* (1988) demonstrated the utility of an F test as a model discrimination criterion for both simulated well test data and actual field well test data. However, due to the theoretical reasons discussed above, an F test cannot discriminate quantitatively between two models which are not nested.

Chapter 4

Sequential Predictive Probability Method

This chapter describes a new quantitative method for model discrimination) which is called the sequential predictive probability method. The derivation and the implementation of the method for use in model discrimination in well test analysis are the main objectives of this study.

Section 4.1 presents the mathematical derivation of the sequential predictive probability method. The method can be derived as a direct extension of the derivation of confidence intervals) based on Bayesian inference. The derivation is straightforward and the final form is simple) but the simple form contains significant advantages for model discrimination.

Section 4.2 discusses the theoretical features of the sequential predictive probability method. The most important feature of the method is that it can provide a unified measure of model discrimination in cases where several models are possible. This is not available in existing quantitative methods such as confidence intervals or an F test. The predictive variance plays a key role in the sequential predictive probability method. The characteristics of the predictive variance are examined for the fundamental reservoir models.

Section 4.3 discusses the implementations of the sequential predictive probability method for effective use in model discrimination in well test analysis. Several practical

considerations need to be taken into account.

Section 4.4 discusses the modified sequential predictive probability method. The modified sequential predictive probability method is introduced as a way to reduce the computation cost.

Section 4.5 presents the application of the sequential predictive probability method to the simulated well test data to show how the method performs in discriminating between models.

4.1 Sequential Predictive Probability Method

As discussed in Chapter 3, confidence intervals are often convenient and good criteria to decide whether the model selected is acceptable or not. However, confidence intervals sometimes provide wrong results. Moreover, they can say little about discrimination between candidate reservoir models, since models with different numbers of parameters have the probability distributions of the parameters with different dimensions. As a result, different models are incomparable in the parameter space. This work describes a new quantitative method of model discrimination to overcome these difficulties. The new method is called the sequential predictive probability method. This method can be derived as a direct extension of the derivation of confidence intervals, based on Bayesian inference.

The idea behind this method is that a correct model should predict the pressure responses more accurately than other incorrect models. From a quantitative point of view, the probability of the actual pressure response occurring at future time under the correct model should be higher than that under the incorrect models. The predictive probability distribution of the pressure response under the model can be calculated from the probability distributions of the parameters.

This idea was proposed originally by Box and Hill (1967) in the field of applied statistics to construct an effective experimental design. Box and Hill (1967) considered, the cases where several models are possible and the results are inconclusive as to which model is best after several preliminary experimental runs, and proposed a criterion to construct the effective sequential design of further experiments for discriminating

between these models using the predictive probabilities of these models. The Box and Hill procedures (1967) have been modified depending on the nature of the model by several authors (Hill, Hunter and Wichern, 1968, Hill and Hunter, 1969, Atkinson, 1978).

Although different approaches for model discrimination have been proposed by several authors (Atkinson, 1969, Andrews, 1971, Atkinson and **COX**, 1974, Atkinson and Fedorov, 1975a, Atkinson and Fedrov, 1975b), Hill (1978) reported that the Box and Hill procedure (1967) had proved to be most effective in practice.

A form of the Box and Hill procedure (1967) is modified here for use in model discrimination in well test analysis.

The basic procedure of the sequential predictive probability method for model discrimination in well test analysis is as follows:

1. Select several candidate reservoir models which might be consistent with both the pressure data and the other available information.
2. Use the first several pressure data points to estimate the reservoir parameters and predict the probability distribution of the pressure at the next time point, under each of the reservoir models.
3. Calculate the probability by substituting the actual pressure data at the next time point into the predictive probability distribution under each reservoir model and update the joint probability under each reservoir model by multiplying the joint probability by the probability.
4. Repeat this procedure until a difference of the joint probability between reservoir models is obtained.
5. Discrimination between candidate reservoir models is performed according to this joint probability.

The predictive probability distribution for each model is derived by the following procedure. The derivation is made in the framework of Bayesian inference and the

parameters (θ) are regarded as random variables with specific probability distributions.

Suppose that variance σ^2 is known.

Uncertainty due to the model is described as follows.

Recall Eq. 3.55:

$$Prob(\theta|y_1, \dots, y_n) = \frac{|H|^{1/2}}{(2\pi\sigma^2)^{m/2}} \cdot \exp \left[-\frac{1}{2\sigma^2} (\theta - \hat{\theta})^T H (\theta - \hat{\theta}) \right] \quad (4.1)$$

The parameters θ form a multinormal distribution about $\hat{\theta}$ with the covariance matrix $\sigma^2 H^{-1}$, given n independent observed pressure data. θ is a vector with m dimensions and the probability distribution of the parameters has m dimensions. The mean (expected) values of the parameters ($\hat{\theta}$) are the least squares estimates of θ using n independent observed pressure data. The covariance matrix $\sigma^2 H^{-1}$ is an $m \times m$ matrix.

Using the parameter estimates ($\hat{\theta}$) based on the first n observations, the true pressure response of y_{n+1} at x_{n+1} (y_{n+1}^*) is expressed approximately as the following linear form using Taylor series expansion up to the first order:

$$\begin{aligned} y_{n+1}^* &= F(\theta, x_{n+1}) \\ &= F(\hat{\theta}, x_{n+1}) + \mathbf{g}^T (\theta - \hat{\theta}) \end{aligned} \quad (4.2)$$

where

$$\mathbf{g} = \begin{pmatrix} \frac{\partial F(\theta, x_{n+1})}{\partial \theta_1} \\ \vdots \\ \frac{\partial F(\theta, x_{n+1})}{\partial \theta_m} \end{pmatrix}_{\theta=\hat{\theta}} \quad (4.3)$$

The gradient of the model function ($\mathbf{g}$) is evaluated at x_{n+1} using the estimated values of the parameters based on the match to the first n observations ($\hat{\theta}$). The gradient $\mathbf{g}$ is a vector with m dimensions.

Setting $\hat{y}_{n+1} = F(\hat{\theta}, x_{n+1})$, Eq. 4.2 can be expressed as:

$$y_{n+1}^* - \hat{y}_{n+1} = \mathbf{g}^T (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) \quad (4.4)$$

Using Eq. 4.1, y_{n+1}^* is normally distributed about $\hat{y}_{n+1}$ with predictive variance:

$$\begin{aligned} \sigma_p^2 &= \mathbf{g}^T \cdot \sigma^2 H^{-1} \cdot \mathbf{g} \\ &= \mathbf{g}^T H^{-1} \mathbf{g} \cdot \sigma^2 \end{aligned} \quad (4.5)$$

$$\begin{aligned} Prob(y_{n+1}^* | \hat{y}_{n+1}) &= Prob(y_{n+1}^* | y_1, \dots, y_n) \\ &= \frac{1}{\sqrt{2\pi}\sigma_p} \cdot \exp \left[-\frac{1}{2\sigma_p^2} (y_{n+1}^* - \hat{y}_{n+1})^2 \right] \end{aligned} \quad (4.6)$$

Eq. 4.6 expresses uncertainty due to the model, which represents how reliable the model is, and this uncertainty is fully characterized by the mean (expected) value ($\hat{y}_{n+1}$) and the predictive variance (σ_p^2).

The predicted pressure response at x_{n+1} from the match to the first n observations ($\hat{y}_{n+1}$) represents the expected value of the true pressure response at x_{n+1} .

The predictive variance (σ_p^2) involves three important features. Firstly, Eq. 4.5 shows that the predictive variance contains the whole information about uncertainty involved in the parameters since it uses the covariance matrix $\sigma^2 H^{-1}$ for the probability distribution of the parameters. Secondly, the Schwartz inequality shows that the magnitude of the predictive variance is bounded by the determinant of the covariance matrix $\sigma^2 H^{-1}$:

$$\begin{aligned}
\mathbf{g}^T \cdot \sigma^2 H^{-1} \cdot \mathbf{g} &= |\mathbf{g}^T \cdot \sigma^2 H^{-1} \cdot \mathbf{g}| \\
&\leq |\mathbf{g}^T| |\sigma^2 H^{-1}| |\mathbf{g}| \\
&= |\sigma^2 H^{-1}| |\mathbf{g}|^2
\end{aligned} \tag{4.7}$$

As more information is obtained about the parameters and uncertainty involved in the parameters is reduced, the spreads of the multinormal distribution become narrower and the determinant of the covariance matrix becomes smaller. Consequently, the magnitude of the predictive variance also becomes smaller and uncertainty due to the model is reduced. Hence, as the multinormal distribution of the parameters approaches the Dirac delta function, the normal distribution of the true pressure response about the predicted (expected) pressure response also approaches the Dirac delta function. In other words, uncertainty involved in the parameters is transformed into uncertainty involved in the pressure response. Finally, and most importantly, the predictive variance is a scalar regardless of the dimension of the covariance matrix. Therefore, models with different numbers of parameters become comparable through the predictive variance.

Uncertainty due to the data is described as follows. It is assumed that the observed pressure data y_{n+1} is normally distributed about the true pressure response y_{n+1}^* with variance σ^2 :

$$Prob(y_{n+1}|y_{n+1}^*) = \frac{1}{\sqrt{2\pi}\sigma} \cdot \exp\left[-\frac{1}{2\sigma^2} (y_{n+1} - y_{n+1}^*)^2\right] \tag{4.8}$$

This assumption is the same as that used in the derivation of Eq. 4.1. Eq. 4.8 expresses uncertainty due to the data, which represents how well the model fits the data.

Here, the exact value of y_{n+1}^* is unknown, and the relationship between y_{n+1} and $\hat{y}_{n+1}$ is obtained by integrating out y_{n+1}^* .

Now the predictive probability distribution of y_{n+1} is expressed as:

$$\begin{aligned}
\text{Prob}(y_{n+1}|\hat{y}_{n+1}) &= \text{Prob}(y_{n+1}|y_1, \dots, y_n) \\
&= \int \text{Prob}(y_{n+1}|y_{n+1}^*) \cdot \text{Prob}(y_{n+1}^*|\hat{y}_{n+1}) dy_{n+1}^* \\
&= \frac{1}{\sqrt{2\pi(\sigma^2 + \sigma_p^2)}} \cdot \exp\left[-\frac{1}{2(\sigma^2 + \sigma_p^2)}(y_{n+1} - \hat{y}_{n+1})^2\right] \quad (4.9)
\end{aligned}$$

Therefore, the actual pressure response y_{n+1} is normally distributed about the predicted (expected) pressure response $\hat{y}_{n+1}$ with the overall predictive variance $\sigma^2 + \sigma_p^2 = (1 + \mathbf{g}^T H^{-1} \mathbf{g}) \cdot \sigma^2$. The overall predictive variance expresses both uncertainty due to the model (σ_p^2) and uncertainty due to the data (σ^2). The relationship between Eq. 4.6, Eq. 4.8, and Eq. 4.9 is shown in Fig. 4.1.

Substituting the actual observed pressure data y_{n+1} into Eq. 4.9 provides the probability of y_{n+1} occurring at x_{n+1} under the model based on the match to the first n observations. This procedure is described schematically in Fig. 4.2. In Fig. 4.2, X axis is time and Y axis is pressure. Small circles represent the current data (n data points) and the thick line, which could be a curve, represents the fitted model to the current data. The predictive probability distribution of y_{n+1} is constructed using the fitted model and the probability of y_{n+1} is expressed by a thick line within the predictive probability distribution.

Fig. 4.3 shows the predictive probability distributions for two models. Two models are named Model 1 and Model 2. In this case, the probability of y_{n+1} occurring under Model 1 is higher than that under Model 2. Hence, Model 1 is more favorable than Model 2.

Eq. 4.9 indicates that the probability of y_{n+1} occurring at x_{n+1} under the model becomes high when the overall predictive variance $\sigma^2 + \sigma_p^2 = (1 + \mathbf{g}^T H^{-1} \mathbf{g}) \cdot \sigma^2$ is small and the actual observed pressure data y_{n+1} is close to the predicted (expected) pressure response $\hat{y}_{n+1}$.

Fig. 4.4 shows three possible cases of two models. In Fig. 4.4 (a), the expected pressure responses under two models are the same and the overall predictive variances

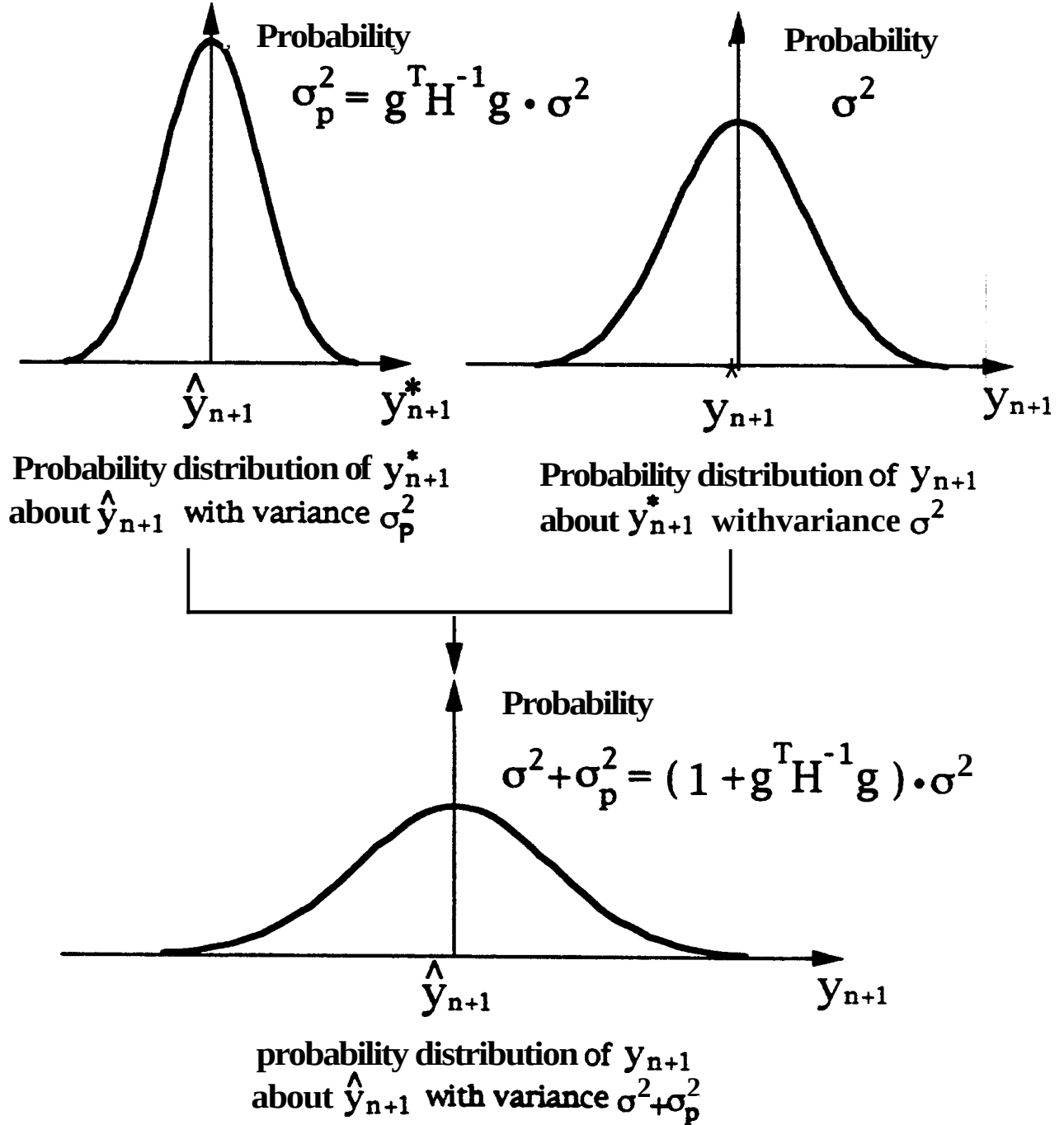


Figure 4.1: Relationship between $Prob(y_{n+1}^*|\hat{y}_{n+1})$, $Prob(y_{n+1}|y_{n+1}^*)$ and $Prob(y_{n+1}|\hat{y}_{n+1})$: $Prob(y_{n+1}|\hat{y}_{n+1})$ is obtained by integrating out y_{n+1}^* from $Prob(y_{n+1}^*|\hat{y}_{n+1})$ and $Prob(y_{n+1}|y_{n+1}^*)$

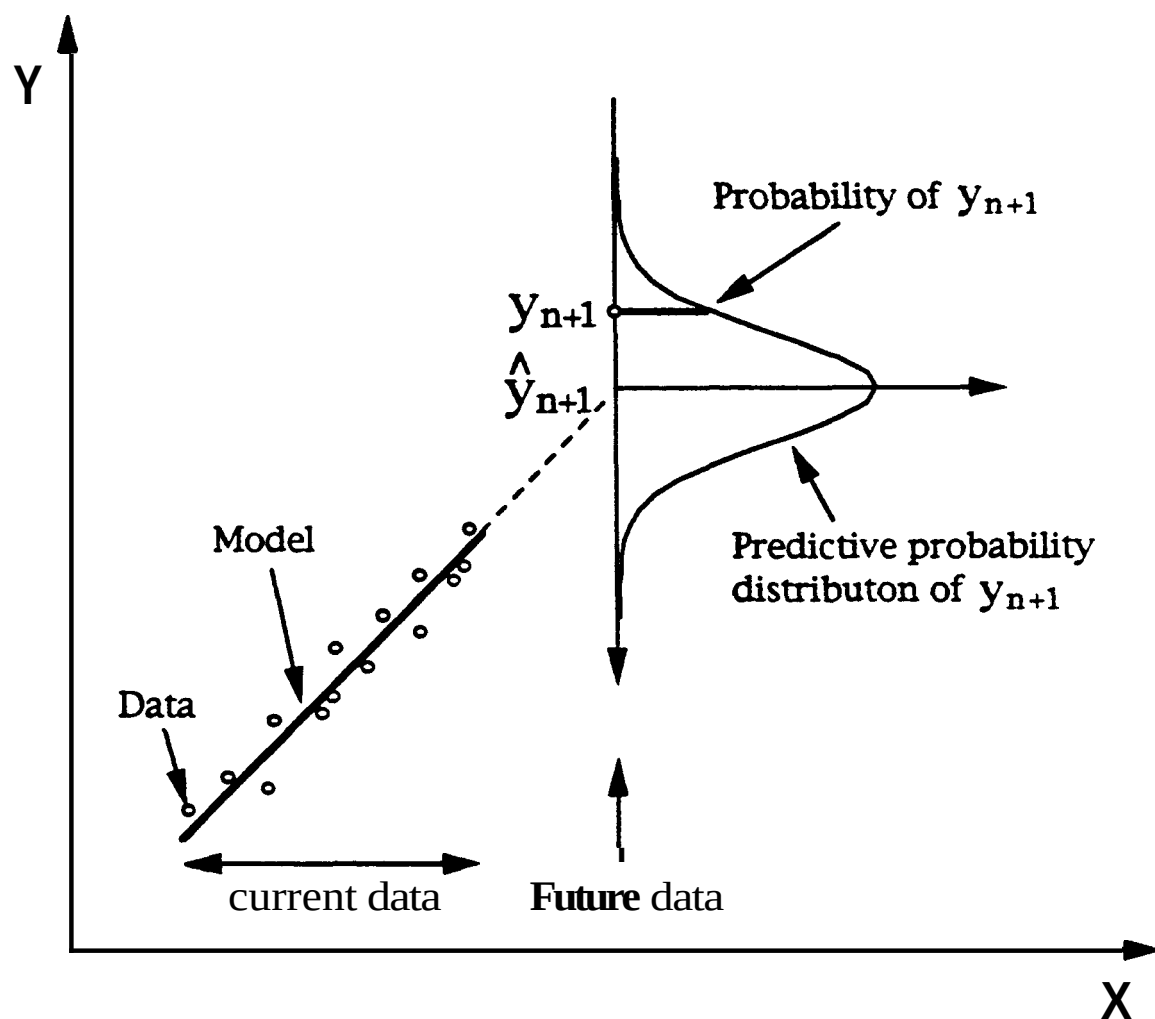


Figure 4.2: Schematic illustration of the predictive probability method: the probability of y_{n+1} under the model is calculated by substituting y_{n+1} into the predictive probability distribution of y_{n+1}

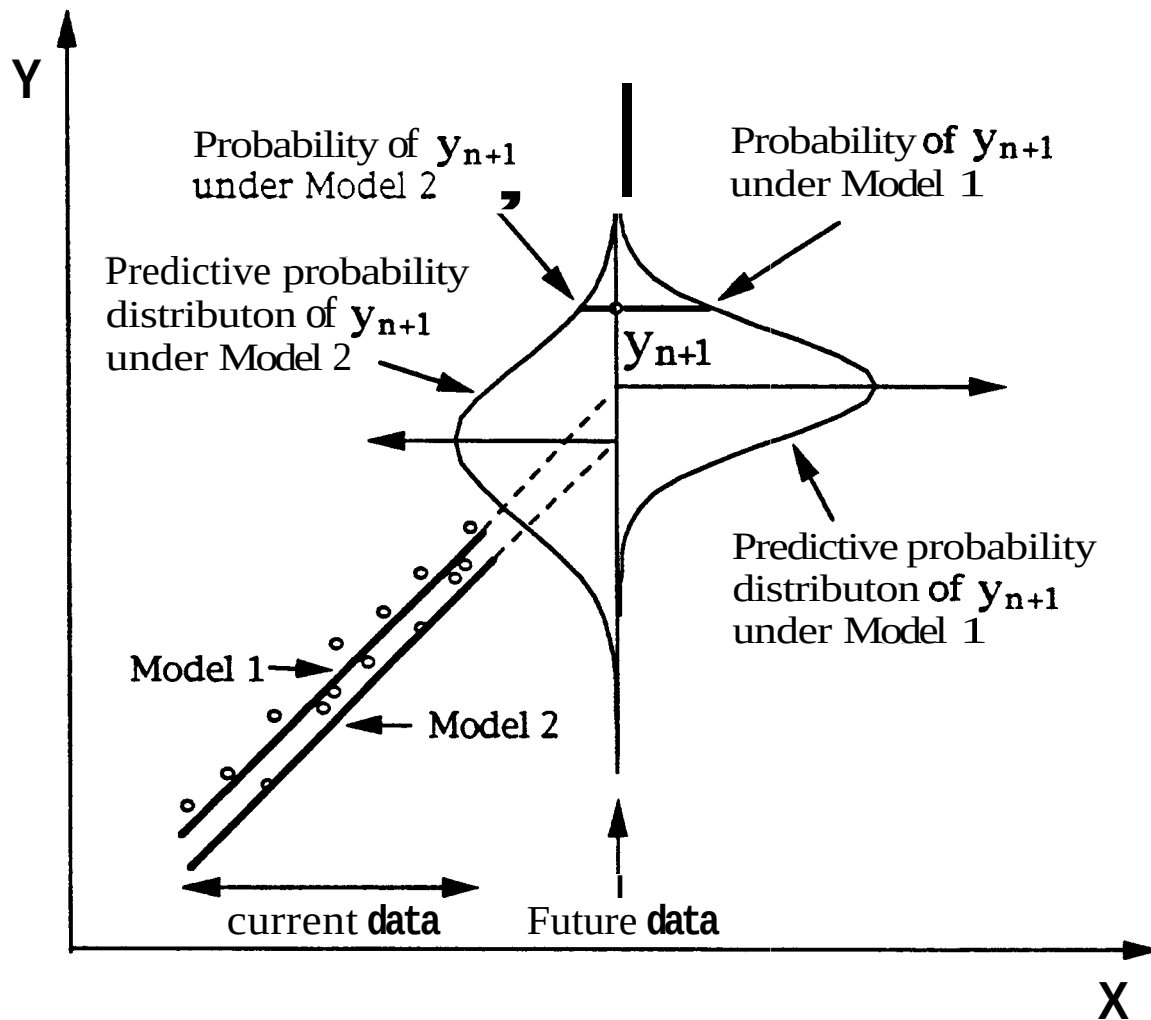
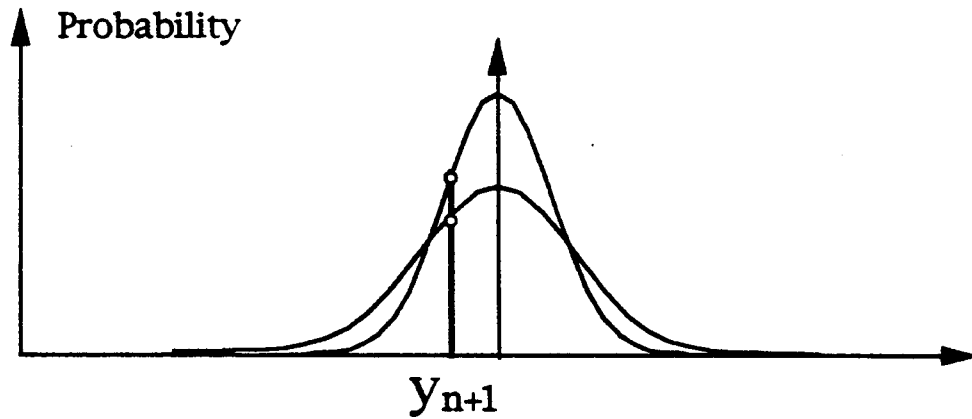
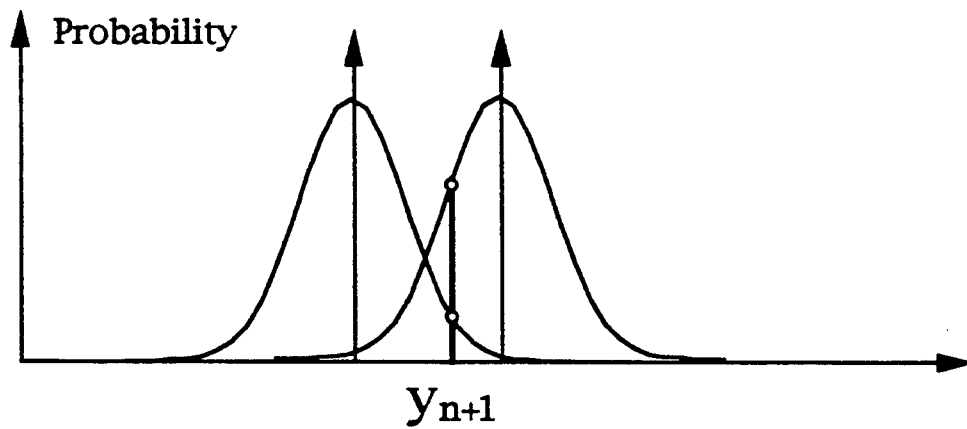


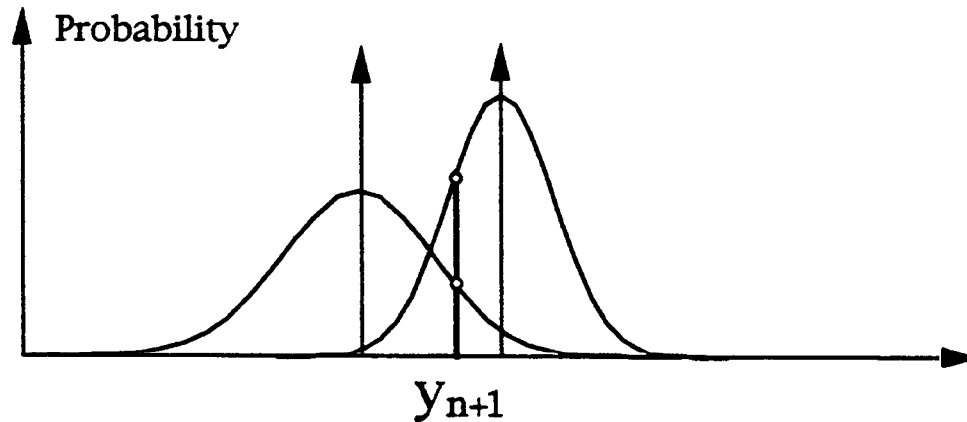
Figure 4.3: Schematic illustration of the predictive probability distributions for two models: the probability of y_{n+1} under Model 1 is higher than that under Model 2



(a) the expected responses are the same and the variances are different



(b) the expected responses are different and the variances are the same



(c) the expected responses are different and the variances are different

Figure 4.4: Schematic illustration of three possible cases of predictive probability distributions for two models

are different. In this case, the probability of y_{n+1} occurring under the model which has the smaller variance is higher. In Fig. 4.4 (b), the expected pressure responses under two models are different and the overall predictive variances are the same. In this case, the probability of y_{n+1} occurring under the model whose expected pressure response is closer to the actual pressure response is higher. In Fig. 4.4 (c), the expected pressure responses under two models are different and the overall predictive variances are different. In this case, the probability of y_{n+1} occurring under the model which has the smaller variance and whose expected pressure response is closer to the actual pressure response is higher.

However, in cases where the actual pressure response is close to the expected pressure response of the model with the larger overall predictive variance in Fig. 4.4 (c), this model can have higher probability of y_{n+1} occurring at x_{n+1} . This implies that the model with the larger overall predictive variance at the current stage may be a true model in the future. Therefore, the sequential procedure, which will be discussed in Section 4.3, is employed to examine whether this expectation is true or not.

In other words, substituting the actual pressure response into the predictive probability distribution under each model is a process of decision making as to which model is the most adequate at the current stage and the sequential procedure is the accumulation of decision making over all stages.

Hill and Hunter (1969) derived the modified expression of the predictive probability distribution defined by Eq. 4.9 in cases where variance σ^2 is unknown.

Recall Eq. 3.60:

$$Prob(\theta|y_1, \dots, y_n) = \frac{\Gamma(n/2)|H|^{1/2}s^{-m}}{[\Gamma(1/2)]^m\Gamma(\nu/2)(\sqrt{\nu})^m} \left[1 + \frac{(\theta - \hat{\theta})^T H (\theta - \hat{\theta})}{\nu s^2} \right]^{-(\nu+m)/2} \quad (4.10)$$

where

$$s^2 = \frac{SSR}{\nu} \quad (4.11)$$

$$\nu = n - m \quad (4.12)$$

$$SSR = \sum_{i=1}^n [y_i - F(\hat{\theta}, x_i)]^2 \quad (4.13)$$

In cases where σ^2 is unknown, the parameters θ form a multivariate student t distribution about $\hat{\theta}$ with the covariance matrix $s^2 H^{-1}$ with the $n - m$ degrees of freedom.

Using Eq. 4.10, Eq. 4.9 is modified as:

$$\begin{aligned} \text{Prob}(y_{n+1}|\hat{y}_{n+1}) = & \\ & \frac{\Gamma((\nu + 1)/2) \left(s\sqrt{1 + \mathbf{g}^T H^{-1} \mathbf{g}} \right)^{-1}}{\Gamma(1/2)\Gamma(\nu/2)\sqrt{\nu}} \left[1 + \frac{1}{\nu} \left(\frac{y_{n+1} - \hat{y}_{n+1}}{s\sqrt{1 + \mathbf{g}^T H^{-1} \mathbf{g}}} \right)^2 \right]^{-(\nu+1)/2} \end{aligned} \quad (4.14)$$

Therefore, in cases where variance σ^2 is unknown, y_{n+1} follows a student t distribution about $\hat{y}_{n+1}$ with the overall predictive variance $s^2 + s_p^2 = (1 + \mathbf{g}^T H^{-1} \mathbf{g}) \cdot s^2$ with the $n - m$ degrees of freedom.

Note that as n increases a student t distribution approaches a normal distribution and when $n - m$ is greater than 30 the normal distribution sufficiently replaces the student t distribution. This indicates that Eq. 4.14 is asymptotically the same as Eq. 4.9 and Eq. 4.9 can replace Eq. 4.14 when $n - m$ is greater than 30. Therefore, in this work Eq. 4.9 is used in cases where $n - m$ is greater than 30.

4.2 Theoretical Features

4.2.1 Overview

Theoretical features of the sequential predictive probability method are as follows:

1. This method provides a unified measure of model discrimination regardless of the number of parameters in reservoir models. Confidence intervals require the same number of measures as the number of parameters. Therefore, they cannot be applied directly to compare between reservoir models with different numbers of unknown parameters. For instance, it is not possible to compare directly the confidence intervals of permeability between two different models such as an infinite reservoir model with three parameters (k, S, C) and a sealing fault model with four parameters (k, S, C, r_e), since in the sealing fault model k and r_e have a strong negative correlation.
2. This method takes into account not only the diagonal elements of the inverse Hessian matrix but also the off-diagonal elements, and therefore this method uses the entire information available about the parameters through the predictive variance. Confidence intervals use the information from the diagonal elements of the inverse Hessian matrix only.
3. This method accommodates the requirements of both simplicity and complexity of the model. Theoretically, as the number of parameters is increased, the overall fitting of the model to the current data is improved due to the decrease of the degrees of freedom. The model complexity, however, forces the prediction to be more obscure. This relationship is quantitatively expressed by σ^2 and $\mathbf{g}^T \mathbf{H}^{-1} \mathbf{g}$, which are the components of uncertainty due to the model ($\mathbf{g}^T \mathbf{H}^{-1} \mathbf{g} \cdot \sigma^2$). In general, the magnitude of σ^2 decreases according to the complexity of the model, while that of $\mathbf{g}^T \mathbf{H}^{-1} \mathbf{g}$ decreases according to the simplicity of the model. The overall predictive variance is expressed as $\sigma^2 + \sigma_p^2 = (1 + \mathbf{g}^T \mathbf{H}^{-1} \mathbf{g}) \cdot \sigma^2$, and this value takes into account both simplicity and complexity of the model.
4. This method can compare any number of possible reservoir models simultaneously. Moreover, one model need not be a subset of another. As discussed in Section 3.5.3, an F test can be used to select the most appropriate reservoir model from pairs of models in a hierarchy. The primary constraint of the F test is that models should be subsets of each other. The sequential predictive probability method can relax this constraint.

4.2.2 Characteristics of the Predictive Variance

The predictive variance plays a key role in the sequential predictive probability method. In general the magnitude of $\mathbf{g}^T H^{-1} \mathbf{g}$ decreases according to the simplicity of the model. However, there is no general verification available. The characteristics of the predictive variance were examined for several models.

Firstly, Box and Hill (1967) investigated the predictive variances of the following four polynomial models which are nested and found that the predictive variances obtained from the models with fewer parameters are smaller.

$$\begin{aligned} y_1 &= \theta_{11}x \\ y_2 &= \theta_{21} + \theta_{22}x \\ y_3 &= \theta_{31} + \theta_{32}x + \theta_{33}x^2 \\ y_4 &= \theta_{41}x + \theta_{42}x^2 \end{aligned} \quad (4.15)$$

Secondly, Atkinson (1978) compared the predictive variances of the following two models which are not nested and concluded that the predictive variance obtained from the model with fewer parameters is smaller.

$$\begin{aligned} y_1 &= \frac{\theta_{11}x}{1+x} \\ y_2 &= \theta_{21} + \theta_{22}x + \theta_{23}x^2 \end{aligned} \quad (4.16)$$

In this work, the basic characteristics of the predictive variance for the following eight models were examined.

- infinite acting model (three parameters: l_0 , S , and C)
- sealing fault model (four parameters: l_0 , S , C , and r_e)
- no flow outer boundary model (four parameters: k , S , C , and r_e)
- constant pressure outer boundary model (four parameters: k , S , C , and r_e)

- double porosity model with pseudosteady state interporosity flow (five parameters: k , S , C , w , and λ)
- double porosity with pseudosteady state interporosity flow and sealing fault model (six parameters: k , S , C , w , A , and r_e)
- double porosity with pseudosteady state interporosity flow and no flow outer boundary model (six parameters: k , S , C , ω , A , and r_e)
- double porosity with pseudosteady state interporosity flow and constant pressure outer boundary model (six parameters: k , S , C , w , λ , and r_e)

Firstly, the basic characteristics of the predictive variance for the infinite acting model, the sealing fault model, the no flow outer boundary model, and the constant pressure outer boundary model were investigated. According to the number of parameters, the infinite acting model is regarded as a simple model and the other models are regarded as complex models. Fig. 4.5 shows typical pressure responses and the corresponding values of $\mathbf{g}^T H^{-1} \mathbf{g}$. Pressure responses were calculated using the reservoir and fluid data given in Table 3.2. The true reservoir parameters are $k = 50md$, $S = 10$, and $C = 0.01STB/psi$. For the boundary model, $r_e = 2000ft$. No errors were added to the pressure data. The total number of data points is 101 and the time period is from 0.01 hour (x_1) to 1000 hours (x_{101}). The inverse Hessian matrix (H^{-1}) was calculated using the data points from x_1 to x_n . The gradient $\mathbf{g}$ was calculated at x_{n+1} , which is the next point to x_n . x_n changes from 1.0 hour (x_{41}) to 891.251 hours (x_{100}). In calculating both the inverse Hessian matrix and the gradient, the parameter values were fixed to the true values. The boundary effects become significant after around 20 hours (x_{67}). Fig. 4.5 (lower) shows that the values of $\mathbf{g}^T H^{-1} \mathbf{g}$ for the complex models are fluctuating due to the truncation errors caused by numerical inversion of the Laplace transformation but are generally larger than that for the simple model (the infinite acting model) before the boundary effects become significant. The errors caused by numerical inversion will be discussed at the end of this section. Even after the boundary effects become significant, the

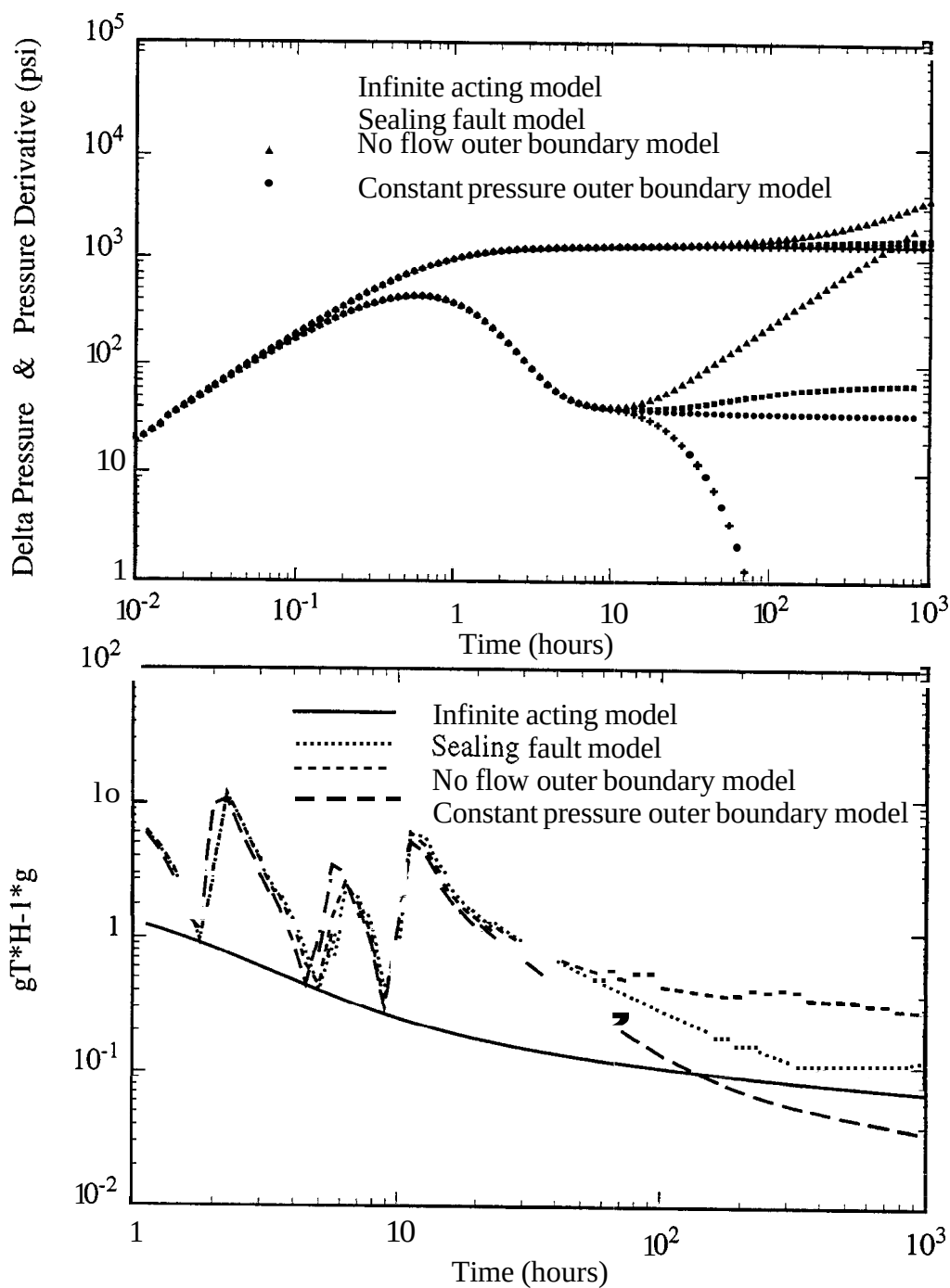


Figure 4.5: Typical pressure responses for the infinite acting model, the sealing fault model, the no flow outer boundary model, and the constant pressure outer boundary model (upper) and the corresponding values of $g^T H^{-1} g$ (lower)

values of $\mathbf{g}^T H^{-1} \mathbf{g}$ for the complex models except the constant pressure outer boundary model are always larger than that for the simple model. Hence, in these typical pressure responses the magnitude of $\mathbf{g}^T H^{-1} \mathbf{g}$ decreases according to the simplicity of the model for the infinite acting model, the sealing fault model, and the no flow outer boundary model, regardless of the time period. The only exception is the constant pressure outer boundary model, which has the smaller values of $\mathbf{g}^T H^{-1} \mathbf{g}$ than the infinite acting model after the boundary effect becomes significant.

Secondly, the basic characteristics of the predictive variance for the infinite acting model and the double porosity model were investigated. According to the number of parameters, the infinite acting model is regarded as a simple model and the double porosity model is regarded as a complex model. Fig. 4.6 shows typical pressure responses and the corresponding values of $\mathbf{g}^T H^{-1} \mathbf{g}$. Pressure responses were calculated using the reservoir and fluid data given in Table 3.2. The true reservoir parameters are $k = 50md$, $S = 10$, and $C = 0.01STB/psi$. For the double porosity model, four cases were considered: (a) $\omega = 0.1$ and $\lambda = 1.0 \times 10^{-8}$, (b) $w = 0.1$ and $\lambda = 1.0 \times 10^{-9}$, (c) $w = 0.01$ and $\lambda = 1.0 \times 10^{-8}$, and (d) $w = 0.01$ and $\lambda = 1.0 \times 10^{-9}$. No errors were added to the pressure data. The total number of data points is 101 and the time period is from 0.01 hour (x_1) to 1000 hours (x_{101}). The inverse Hessian matrix (H^{-1}) was calculated using the data points from x_1 to x_n . The gradient $\mathbf{g}$ was calculated at x_{n+1} , which is the next point to x_n . x_n changes from 1.0 hour (x_{41}) to 891.251 hours (x_{100}). In calculating both the inverse Hessian matrix and the gradient, the parameter values were fixed to the true values. Fig. 4.6 (lower) shows that the value of $\mathbf{g}^T H^{-1} \mathbf{g}$ for the double porosity model is always larger than that for the infinite acting model throughout the time period determined. Hence, in these typical pressure responses the magnitude of $\mathbf{g}^T H^{-1} \mathbf{g}$ decreases according to the simplicity of the model for the infinite acting model and the double porosity model regardless of the time period.

Finally, the basic characteristics of the predictive variance for the infinite acting model, the double porosity model, the double porosity and sealing fault model, the double porosity and no flow outer boundary model, and the double porosity and constant pressure outer boundary model were investigated. According to the number of parameters, the double porosity model is regarded as a simple model,

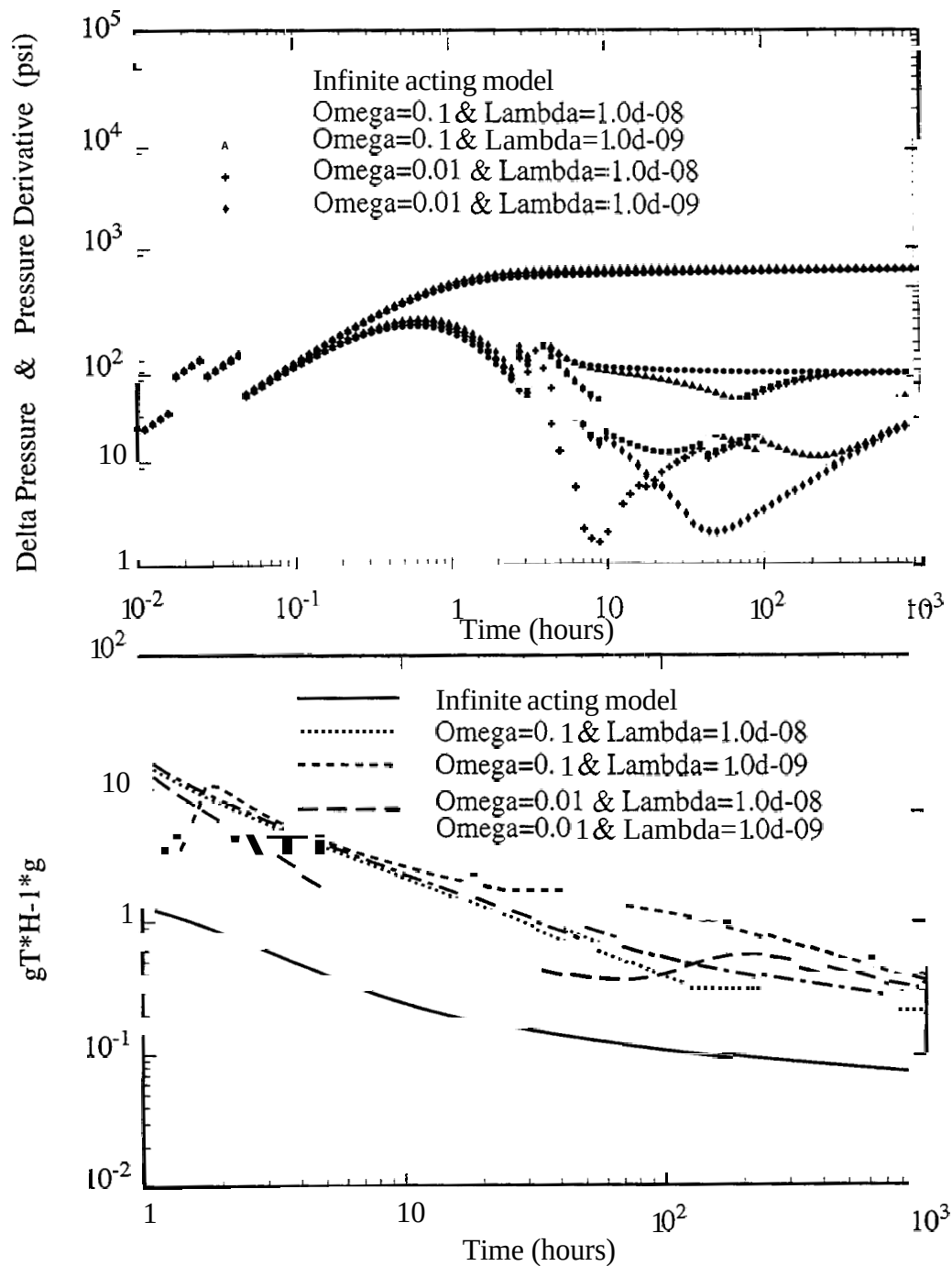


Figure 4.6: Typical pressure responses for the infinite acting model and the double porosity model (upper) and the corresponding values of $g^T H^{-1} g$ (lower)

while the double porosity and sealing fault model, the double porosity and no flow outer boundary model, and the double porosity and constant pressure outer boundary model are regarded as complex models. Fig. 4.7 shows typical pressure responses and the corresponding values of $\mathbf{g}^T H^{-1} \mathbf{g}$. Pressure responses were calculated using the reservoir and fluid data given in Table 3.2. The true reservoir parameters are $k = 50md$, $S = 10$, and $C = 0.01STB/psi$. For the double porosity model, $w = 0.1$ and $\lambda = 1.0 \times 10^{-8}$. For the double porosity and boundary model, $w = 0.1$, $\lambda = 1.0 \times 10^{-8}$, and $r_e = 4000ft$. Hence, the boundary effects become significant after the double porosity effects. No errors were added to the pressure data. The total number of data points is 101 and the time period is from 0.01 hour (x_1) to 1000 hours (x_{101}). The inverse Hessian matrix (H^{-1}) was calculated using the data points from x_1 to x_n . The gradient $\mathbf{g}$ was calculated at x_{n+1} , which is the next point to x_n . x_n changes from 1.0 hour (x_{41}) to 891.251 hours (x_{100}). In calculating both the inverse Hessian matrix and the gradient, the parameter values were fixed to the true values. Fig. 4.7 (lower) shows that the values of $\mathbf{g}^T H^{-1} \mathbf{g}$ for the double porosity and boundary models are fluctuating due to the truncation errors caused by numerical inversion of the Laplace transformation but are the values of $\mathbf{g}^T H^{-1} \mathbf{g}$ for the double porosity and boundary models except the double porosity and constant pressure outer boundary model are always larger than that for the simpler models. The errors caused by numerical inversion will also be discussed at the end of this section. Hence, in these typical pressure responses the magnitude of $\mathbf{g}^T H^{-1} \mathbf{g}$ decreases according to the simplicity of the model for the infinite acting model, the double porosity model, the double porosity and sealing fault model, and the double porosity model and no flow outer boundary model, regardless of the time period. The only exception is the double porosity and constant pressure outer boundary model, which has the smaller values of $\mathbf{g}^T H^{-1} \mathbf{g}$ than the double porosity model after the boundary effect becomes significant.

Therefore, from Fig. 4.5, Fig. 4.6, and Fig. 4.7 it may be concluded that in typical pressure responses the magnitude of $\mathbf{g}^T H^{-1} \mathbf{g}$ generally decreases according to the simplicity of the model.

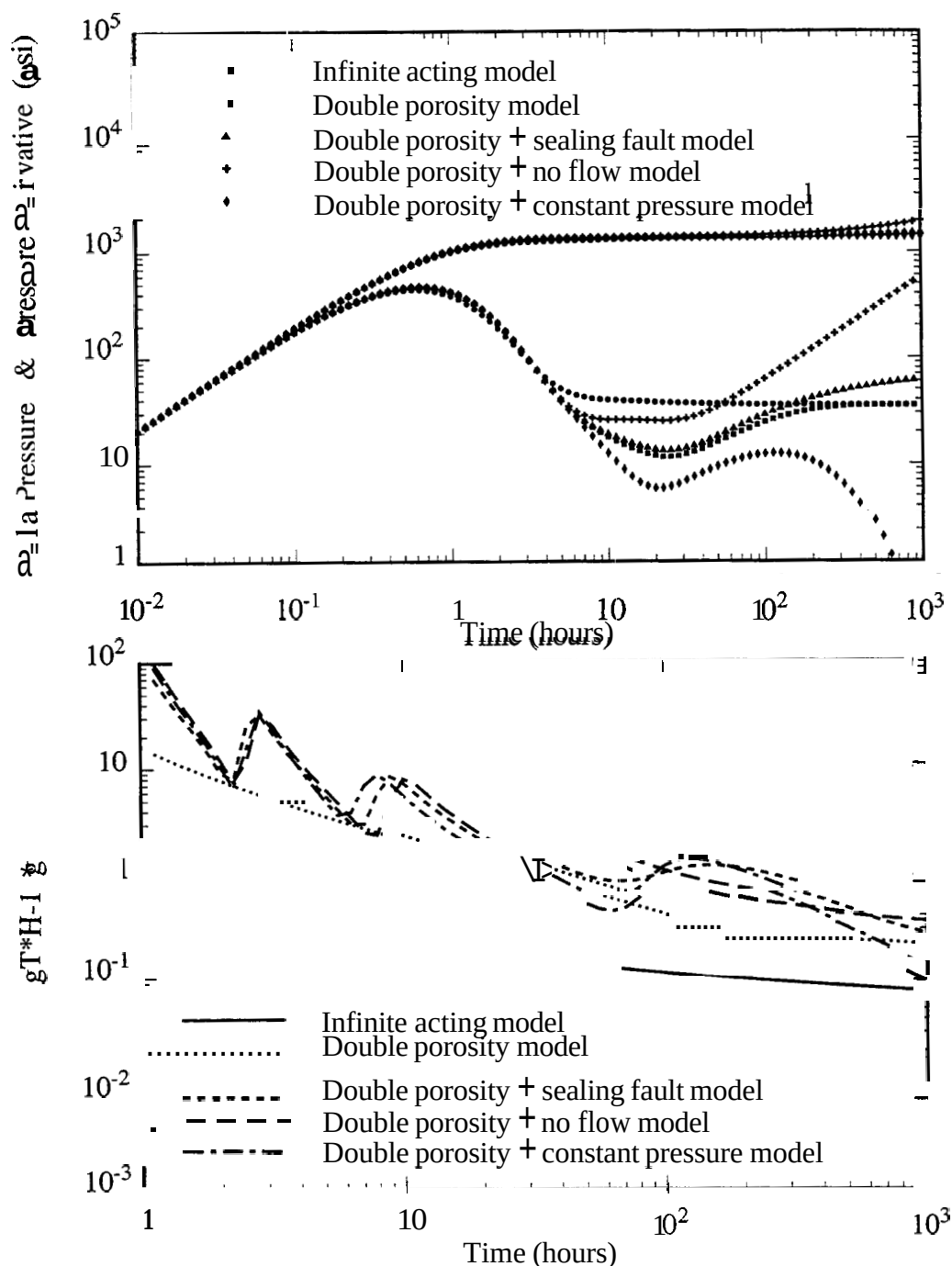


Figure 4.7: Typical pressure responses for the infinite acting model, the double porosity model, the double porosity and sealing fault model, the double porosity and no flow outer boundary model, and the double porosity and constant pressure outer boundary model (upper) and the corresponding values of $g^T H^{-1} g$ (lower)

As mentioned above, the values of $\mathbf{g}^T H^{-1} \mathbf{g}$ for the boundary models are fluctuating due to the truncation errors caused by numerical inversion of the Laplace transformation before the boundary effects become significant. In this work the Stehfest algorithm was employed, since this algorithm is popular in the Laplace transformation in automated well test analysis. The numerical accuracy of this algorithm depends on the sampling numbers of A_4 and N . In general, as the numbers of A_4 and N increase, the numerical accuracy becomes higher. Meanwhile, the amount of the computation cost increases proportional to the sampling numbers of M and N . Following the paper by Rosa and Horne (1983), the values of $M = 7$ and $N = 8$ were employed for Fig. 4.5, Fig. 4.6, and Fig. 4.7.

Fig. 4.8 shows that the partial derivatives of the sealing fault model with respect to the boundary ($\frac{\partial F}{\partial r_e}$) for different sampling numbers of M and N and the corresponding values of $\mathbf{g}^T H^{-1} \mathbf{g}$. Pressure responses were calculated using the reservoir and fluid data given in Table 3.2. The true reservoir parameters are $k = 50md$, $S = 10$, $C = 0.01STB/psi$, and $r_e = 2000ft$. No errors were added to the pressure data. The fluctuations of $\mathbf{g}^T H^{-1} \mathbf{g}$ before the boundary effects become significant are mainly caused by the inaccuracy of $\frac{\partial F}{\partial r_e}$. The inaccuracy of $\frac{\partial F}{\partial r_e}$ is caused by the low orders of $\frac{\partial F}{\partial r_e}$, which are less than the order of 10^{-4} before the boundary effects become significant. Therefore, as shown in Fig. 4.8 (upper), even if the sampling numbers of M and N increase, the inaccuracy of $\frac{\partial F}{\partial r_e}$ is not improved significantly and, moreover, the additional fluctuations are introduced after the boundary effects become significant in cases of $A_4 = 15$ and $N = 16$. As for the computation cost, the sampling numbers of $M = 7$ and $N = 8$ are 1.5 times faster than those of $A_4 = 11$ and $N = 12$. Hence: throughout the following study, the sampling numbers of $M = 7$ and $N = 8$ are used exclusively.

4.3 Several Practical Considerations

The main objective of the work by Box and Hill (1967) was to construct the effective sequential design of further experiments for discriminating between several models using the predictive probabilities of these models. The situation that Box and Hill

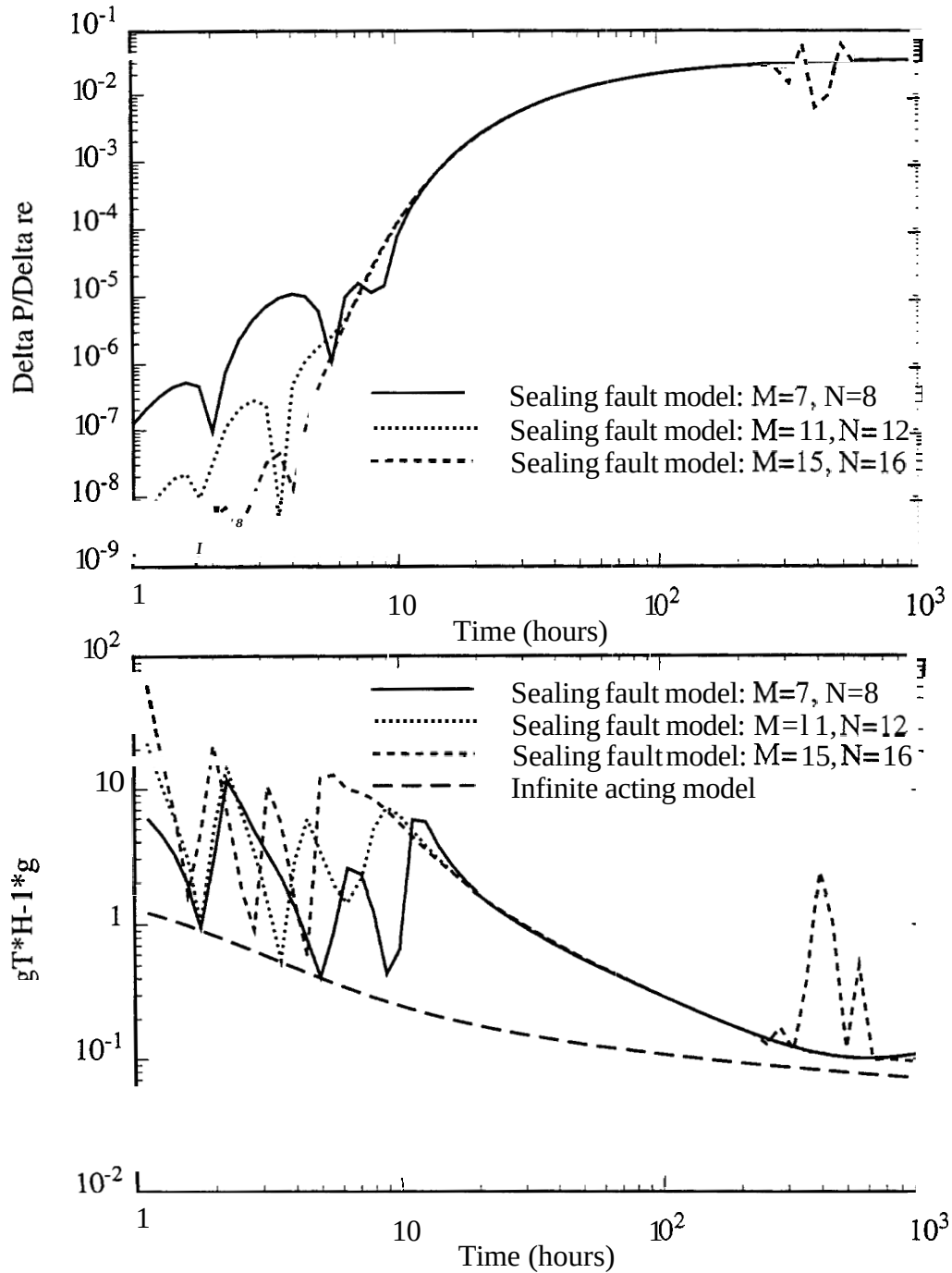


Figure 4.8: $\frac{\partial F}{\partial re}$ for the sealing fault model (upper) and the corresponding values of $g^T H^{-1} g$ (lower)

(1967) considered was that several preliminary experimental runs are given and no further experimental runs have been observed. The total number of observations required by the sequential procedure is not predetermined, since at any stage of the experiment the decision of terminating the procedure depends on the results of the observations previously made. The procedure that Box and Hill (1967) used is that the most effective next observation point, which has not yet been observed, is selected using the predictive probabilities of the models based on the previous observations, and the current standing of each model is determined by calculating its corresponding probability after the experimental run is actually performed at that point. This procedure is repeated until the results indicate that one model is clearly superior to the others. The next observation points are selected one at a time.

On the other hand, in general the total number of well test data is fixed. It is not feasible to construct a sequential design of further experiments without any constraint, since all data have already been observed. Therefore, in order to utilize the sequential predictive probability method for model discrimination in well test analysis, it is necessary to consider several points as follows.

4.3.1 Selection of Candidate Reservoir Models

The described method relates to the model verification problem, not the model recognition problem. Therefore, this method requires the selection of possible reservoir models in advance. This method selects the most appropriate model to the given data among these selected models. This implies that an insufficient selection of possible reservoir models could lead to the wrong result. Thus, a sufficient selection of possible reservoir models is crucial for the success of the sequential predictive probability method. In this work eight basic models were employed, since these models may cover most of all possible models in actual well test analysis. As discussed in Section 3.2.1, the conventional graphical procedure using the pressure derivative plot for model recognition was employed in this work.

4.3.2 Selection of Starting Point

It is a difficult problem to decide where the procedure should be started. The decision should consider the following two aspects of well test analysis described in Chapter 2. While wellbore storage effects dominate the surface flow, there is little sand face flow occurring and, as a result, no input signal is being sent to the reservoir. Therefore, during this period it is not possible to obtain any information about the reservoir from the pressure data. On the other hand, once the input signal is applied to the reservoir, theoretically the output signal involves all the information about the reservoir such as the average permeability, skin, the boundary effect, the heterogeneity effect and so on, since pressure changes propagate throughout the reservoir at an infinite velocity due to their diffusive nature. To accommodate these two aspects, the starting point is selected just after the wellbore storage hump in the pressure derivative plot.

4.3.3 Selection of Next Time Step

Theoretically, it is possible to select time steps in any order. In this work, the time steps are selected chronologically. Boundary effects are clearly detected after a specific time and these effects become more and more significant on the pressure responses as time goes by. This chronological selection of time steps can make this method more sensitive in the detection of the boundary effect. As will be shown later, this selection is consistent with the concepts of radius of investigation and stabilization time. This chronological selection also reveals the characteristics of the pressure difference between the observed data and the calculated data based on a time-dependent model. These characteristics provide useful additional information to decide whether the model is acceptable or not. The advantage of the chronological selection will be discussed later. During this work, other time step sequences (for example, starting with alternate points) were also investigated, however the chronological sequence was found to be most effective.

4.3.4 Number of Data Points To Predict

Even though Box and Hill (1967) predicted only one observation point at a time, it is theoretically possible to predict any number of pressure responses at any points based on the fixed number of data points. In other words, based on the first n observations, it is possible to predict the pressure responses not only at x_{n+1} but also at x_{n+2} at a time. However, in this work, only one pressure response is predicted at each step of the procedure. The reason is described in Section 4.3.8.

4.3.5 Number of Data Points To Use

At each step of the procedure, the whole data from the first point to the current investigating point are used to predict the pressure response at the next time step. Every time the procedure is repeated, the number of data to use increases by one. The reason is also described in Section 4.3.8. This sequential procedure is described in Fig. 4.9.

4.3.6 Parameter Estimates at Each Time Step

Parameter estimates are updated at each time step. New pressure data contain additional information about the parameters as long as the model is correct. As discussed in Section 3.3, the Gauss-Marquardt method with penalty function and interpolation and extrapolation technique was employed as the nonlinear regression algorithm for parameter estimation.

4.3.7 Probability at the Starting Point

At the starting point, there is no information available about which model is better, and when the number of possible models is n_m , the probability associated with each model is set to be equal to $\frac{1}{n_m}$.

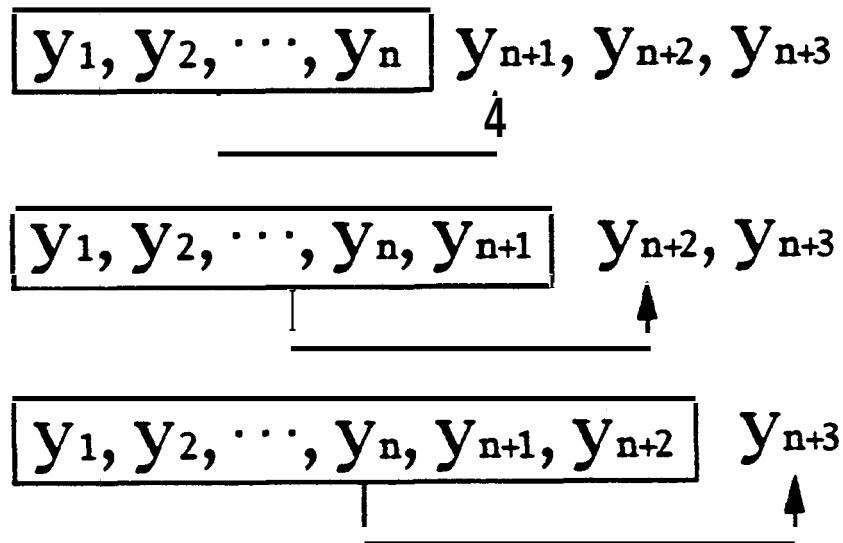


Figure 4.9: Sequential procedure: the whole data from the first point to the current investigating point are used to predict the pressure response at the next time step

4.3.8 Joint Probability

Joint probability should be used to confirm the adequacy of model fits over the whole time range. The joint probability for each model can be easily calculated by sequential multiplication of the conditional probability at each prediction procedure.

From the definition of conditional probability:

$$Prob(B|A) = \frac{Prob(A \cap B)}{Prob(A)} \quad (4.17)$$

Hence,

$$Prob(y_{n+1}|y_1, \dots, y_n) = \frac{Prob(y_1, \dots, y_n, y_{n+1})}{Prob(y_1, \dots, y_n)} \quad (4.18)$$

This leads to:

$$\begin{aligned} & Prob(y_{n+2}|y_1, \dots, y_n, y_{n+1}) \cdot Prob(y_{n+1}|y_1, \dots, y_n) \\ &= \frac{Prob(y_1, \dots, y_n, y_{n+1}, y_{n+2})}{Prob(y_1, \dots, y_n, y_{n+1})} \cdot \frac{Prob(y_1, \dots, y_n, y_{n+1})}{Prob(y_1, \dots, y_n)} \\ &= \frac{Prob(y_1, \dots, y_n, y_{n+1}, y_{n+2})}{Prob(y_1, \dots, y_n)} \end{aligned} \quad (4.19)$$

Hence, a product of conditional probabilities represents the joint probability of $y_1, \dots, y_n, y_{n+1}, y_{n+2}$ occurring under the model. This joint probability represents the adequacy of model fitting over the whole time range.

Suppose that pressure res, both x_{n+1} and x_{n+2} are predicted using the first n observations. The sequential product of conditional probabilities is expressed as:

$$\begin{aligned} & Prob(y_{n+2}|y_1, \dots, y_n, y_{n+1}) \cdot Prob(y_{n+1}|y_1, \dots, y_n) \\ &= \frac{Prob(y_1, \dots, y_n, y_{n+1}, y_{n+2})}{Prob(y_1, \dots, y_n, y_{n+1})} \cdot \frac{Prob(y_1, \dots, y_n, y_{n+1})}{Prob(y_1, \dots, y_n)} \end{aligned} \quad (4.20)$$

Eq. 4.20 indicates that a product of conditional probabilities does not represent the joint probability of $y_1, \dots, y_n, y_{n+1}, y_{n+2}$ occurring under the model.

Therefore, only one pressure response should be predicted at each procedure to calculate the joint probability.

Eq. 4.19 implies one more important insight that after pressure responses at both x_{n+1} and x_{n+2} are observed, $Prob(y_{n+2}|y_1, \dots, y_n, y_{n+1}) \cdot Prob(y_{n+1}|y_1, \dots, y_n)$ can be recovered from $Prob(y_1, \dots, y_n, y_{n+1}, y_{n+2})$ and $Prob(y_1, \dots, y_n)$. This idea can greatly reduce the computational cost. This idea is discussed fully in Section 4.4.

4.3.9 Comparison of Joint Probability

The joint probabilities associated with the models are normalized, so that they sum up to one. The emphasis is put on the relative magnitudes of the joint probabilities associated with the reservoir models rather than the absolute ones. Normalization does not change the relative magnitudes of the joint probabilities. The range of the normalized joint probability is from zero to one. When the normalized joint probability of a model tends to be close to one during the sequential procedure, it can be concluded that this model is the most appropriate model. In cases where no model is clearly superior to the others, there are two interpretations of the data possible: one is that another model should be used and the other is that the given data is not sufficient to discriminate between models.

Box and Hill (1967) used the concept of entropy to measure the amount of information supplied by a communication system where entropy is defined as:

$$S = - \sum_{j=1}^{n_m} \Pi_j \ln \Pi_j \quad (4.21)$$

where Π_j is the joint probability associated with a model j and the number of models is n_m . The least possible information corresponds to maximum entropy, that is, when $\Pi_1 = \Pi_2 = \dots = \Pi_{n_m} = \frac{1}{n_m}$. This is the situation where no information is available about the reservoir parameters before well testing. To discriminate between n , possible reservoir models, it is more desirable to have a situation where $\Pi_j \gg \Pi_i$ for $i \neq j$. In the case where the maximum information is deduced from well test data, $\Pi_j = 1$ and $\Pi_i = 0$ for $i \neq j$. In this case it can be concluded that the model j is the most appropriate one.

Box and Hill (1967) selected the next observation point where the expected change in entropy should be a maximum.

After n observations, the joint probabilities associated with the n_m models are $\Pi_1^n, \Pi_2^n, \dots, \Pi_{n_m}^n$. The joint probabilities obtained by taking the $n+1$ -th observation are $\Pi_1^{n+1}, \Pi_2^{n+1}, \dots, \Pi_{n_m}^{n+1}$. The normalized joint probability associated with the j -th model is

$$\Pi_j^{n+1} = \frac{\Pi_j^n P_j^{n+1}}{\sum_{j=1}^{n_m} \Pi_j^n P_j^{n+1}} \quad (4.22)$$

where P_j^{n+1} is the predictive probability of the $n+1$ -th observation occurring under the model j based on the match to the n observations.

$$P_j^{n+1} = \text{Prob}(y_{n+1}|y_1, \dots, y_n) \quad (4.23)$$

4.3.10 Outliers

Outliers might make the probability extremely small, since the predictive pressure is far from the outlier (recall Eq. 4.9). However it is difficult to decide which data points are outliers before a true reservoir model is selected. Thus, in this work a lower limit of probability is set, so that less possible data points are automatically excluded. In practice, in cases where the calculated predictive probability lies outside the 95 % interval, the probability at the 95 % interval is used as a lower limit, and the calculated probability is replaced with that at the 95 % interval. Mathematically, this modification is expressed as follows:

$$\begin{aligned} \text{Prob}(y_{n+1}|\hat{y}_{n+1}) &= \text{Prob}(\hat{y}_{n+1} - 1.96 \cdot \sqrt{\sigma^2 + \sigma_p^2}|\hat{y}_{n+1}) \\ &\quad \text{if } y_{n+1} < \hat{y}_{n+1} - 1.96 \cdot \sqrt{\sigma^2 + \sigma_p^2} \\ \text{Prob}(y_{n+1}|\hat{y}_{n+1}) &= \text{Prob}(\hat{y}_{n+1} + 1.96 \cdot \sqrt{\sigma^2 + \sigma_p^2}|\hat{y}_{n+1}) \\ &\quad \text{if } y_{n+1} > \hat{y}_{n+1} + 1.96 \cdot \sqrt{\sigma^2 + \sigma_p^2} \end{aligned} \quad (4.24)$$

This modification makes this method robust to the outliers.

4.3.11 Initial Estimates of Parameter Values

When the parameters are evaluated at the next step, the estimated values of the parameters at the previous step are used as the initial guesses for the parameters. In other words, when the parameters are evaluated using $n + 1$ data points, the estimated values of the parameters using n data points serve as the initial estimates.

4.4 Modified Sequential Predictive Probability Method

In this section, the modified sequential predictive probability method is discussed. The weak point of the sequential predictive probability method is the computation cost. At each step of the sequential procedure, the parameters need to be evaluated using nonlinear regression. Nonlinear regression requires iterations to obtain the optimal values of the parameters. In general, the computation cost increases proportional to the number of sequential steps, although in practice the iterations may converge quickly, since each sequential step can begin with the parameter estimates from the previous step.

The modified sequential predictive probability method is introduced as a way to reduce the computation cost. This method is derived by combining the sequential reestimation method with probability theory. The sequential reestimation method is one of the methods of nonlinear regression. The sequential reestimation method considers the situation where the optimal values of the parameters have been obtained using n data points. When the $n + 1$ -th data point is available, the optimal values of the parameters for $n + 1$ data points are calculated directly using the optimal values of the parameters and the Hessian matrix for n data points. This method does not require iteration and can reduce the computation cost compared to the nonlinear regression using $n + 1$ data points. In the context of well test analysis, Padmanabhan and Woo (1976) and Padmanabhan (1979) discussed the sequential reestimation method for parameter estimation.

Here the modified sequential predictive probability method is derived as follows.

Recall Eq. 4.18:

$$Prob(y_{n+1}|y_1, \dots, y_n) = \frac{Prob(y_1, \dots, y_n, y_{n+1})}{Prob(y_1, \dots, y_n)} \quad (4.25)$$

This leads to:

$$\begin{aligned} & Prob(y_{n+2}|y_1, \dots, y_n, y_{n+1}) \cdot Prob(y_{n+1}|y_1, \dots, y_n) \\ &= \frac{Prob(y_1, \dots, y_n, y_{n+1}, y_{n+2})}{Prob(y_1, \dots, y_n, y_{n+1})} \cdot \frac{Prob(y_1, \dots, y_n, y_{n+1})}{Prob(y_1, \dots, y_n)} \\ &= \frac{Prob(y_1, \dots, y_n, y_{n+1}, y_{n+2})}{Prob(y_1, \dots, y_n)} \end{aligned} \quad (4.26)$$

Eq. 4.26 shows that a product of conditional probabilities is numerically identical to the joint probability. After pressure responses at both x_{n+1} and x_{n+2} are observed, $Prob(y_{n+2}|y_1, \dots, y_n, y_{n+1}) \cdot Prob(y_{n+1}|y_1, \dots, y_n)$ can be recovered by the values of $Prob(y_1, \dots, y_n)$ and $Prob(y_1, \dots, y_n, y_{n+1}, y_{n+2})$. This idea makes it possible to reduce greatly the computation cost.

First of all, $Prob(y_1, \dots, y_n)$ and $Prob(y_1, \dots, y_n, y_{n+1})$ need to be calculated. $Prob(y_1, \dots, y_n)$ can be calculated from Eq. 3.53 using the locally uniform prior probability distributions (noninformative prior probability distributions):

$$\begin{aligned} Prob(y_1, \dots, y_n) &= \int_m L(\theta|y_1, \dots, y_n) Prob(\theta) d\theta \\ &= \int_m L(\theta|y_1, \dots, y_n) d\theta \end{aligned} \quad (4.27)$$

where

$$\begin{aligned} L(\theta|y_1, \dots, y_n) &= \\ &= \frac{1}{(\sqrt{2\pi}\sigma)^n} \cdot \exp \left[-\frac{1}{2\sigma^2} \left(\mathbf{R}_n^T \mathbf{R}_n + (\theta - \hat{\theta}^{(n)})^T H_n (\theta - \hat{\theta}^{(n)}) \right) \right] \end{aligned} \quad (4.28)$$

$$\mathbf{R}_n = \begin{pmatrix} y_1 - F(\hat{\boldsymbol{\theta}}^{(n)}, x_1) \\ \vdots \\ y_n - F(\hat{\boldsymbol{\theta}}^{(n)}, x_n) \end{pmatrix} \quad (4.29)$$

$$H_n = \begin{pmatrix} \sum_{i=1}^n \left(\frac{\partial F}{\partial \theta_1} \right) \left(\frac{\partial F}{\partial \theta_1} \right) & \cdots & \sum_{i=1}^n \left(\frac{\partial F}{\partial \theta_1} \right) \left(\frac{\partial F}{\partial \theta_m} \right) \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^n \left(\frac{\partial F}{\partial \theta_m} \right) \left(\frac{\partial F}{\partial \theta_1} \right) & \cdots & \sum_{i=1}^n \left(\frac{\partial F}{\partial \theta_m} \right) \left(\frac{\partial F}{\partial \theta_m} \right) \end{pmatrix} \bigg|_{\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}^{(n)}} \quad (4.30)$$

The components of both $\mathbf{R}_n$ and H_n are evaluated using the estimated values of the parameters based on the match to the n observations ($\hat{\boldsymbol{\theta}}^{(n)}$).

By definition of the multinormal distribution (recall **Eq. 3.54**):

$$\int_m \frac{|H|^{1/2}}{(\sqrt{2\pi}\sigma)^m} \cdot \exp \left[-\frac{1}{2\sigma^2} (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}})^T H (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}) \right] d\boldsymbol{\theta} = 1 \quad (4.31)$$

Hence, **Eq. 4.27** reduces to:

$$Prob(y_1, \dots, y_n) = \frac{(\sqrt{2\pi}\sigma)^m}{|H_n|^{1/2}} \frac{1}{(\sqrt{2\pi}\sigma)^n} \cdot \exp \left[-\frac{1}{2\sigma^2} \mathbf{R}_n^T \mathbf{R}_n \right] \quad (4.32)$$

Similarly, $Prob(y_1, \dots, y_n, y_{n+1})$ can be calculated by:

$$\begin{aligned} Prob(y_1, \dots, y_n, y_{n+1}) &= \int_m L(\boldsymbol{\theta} | y_1, \dots, y_n, y_{n+1}) Prob(\boldsymbol{\theta}) d\boldsymbol{\theta} \\ &= \int_m L(\boldsymbol{\theta} | y_1, \dots, y_n, y_{n+1}) d\boldsymbol{\theta} \end{aligned} \quad (4.33)$$

where

$$L(\theta|y_1, \dots, y_n, y_{n+1}) = \frac{1}{(\sqrt{2\pi}\sigma)^n} \cdot \exp \left[-\frac{1}{2\sigma^2} \left(\mathbf{R}_{n+1}^T \mathbf{R}_{n+1} + \left(\theta - \hat{\theta}^{(n+1)} \right)^T H_{n+1} \left(\theta - \hat{\theta}^{(n+1)} \right) \right) \right] \quad (4.34)$$

The components of both $\mathbf{R}_{n+1}$ and H_{n+1} are evaluated using the estimated values of the parameters based on the match to the $n+1$ observations ($\hat{\theta}^{(n+1)}$).

Eq. 4.33 reduces to:

$$Prob(y_1, \dots, y_n, y_{n+1}) = \frac{(\sqrt{2\pi}\sigma)^n}{|H_{n+1}|^{1/2}} \frac{1}{(\sqrt{2\pi}\sigma)^{n+1}} \cdot \exp \left[-\frac{1}{2\sigma^2} \mathbf{R}_{n+1}^T \mathbf{R}_{n+1} \right] \quad (4.35)$$

Dividing Eq. 4.35 by Eq. 4.33 leads to:

$$\frac{Prob(y_1, \dots, y_n, y_{n+1})}{Prob(y_1, \dots, y_n)} = \frac{1}{\sqrt{2\pi}\sigma} \frac{|H_n|^{1/2}}{|H_{n+1}|} \cdot \exp \left[-\frac{1}{2\sigma^2} \left(\mathbf{R}_{n+1}^T \mathbf{R}_{n+1} - \mathbf{R}_n^T \mathbf{R}_n \right) \right] \quad (4.36)$$

Here the relationship between H_n and H_{n+1} and that between $\mathbf{R}_n^T \mathbf{R}_n$ and $\mathbf{R}_{n+1}^T \mathbf{R}_{n+1}$ should be investigated. The procedure of investigation owes much to the derivation of the sequential reestimation method.

For simplicity, the objective function (Eq. 3.8) is modified as:

$$E_n(\theta) = \sum_{i=1}^n [y_i - F(\theta, x_i)]^2 \quad (4.37)$$

and $\hat{\theta}^{(n)}$ denotes the estimated values of the parameters based on the match to the n observations.

According to this definition, $\mathbf{R}_n^T \mathbf{R}_n$ and $\mathbf{R}_{n+1}^T \mathbf{R}_{n+1}$ are expressed as:

$$\mathbf{R}_n^T \mathbf{R}_n = \mathbf{E} \left(\hat{\theta}^{(n)} \right) \quad (4.38)$$

and

$$\begin{aligned}
\mathbf{R}_{n+1}^T \mathbf{R}_{n+1} &= E_{n+1} \left(\hat{\boldsymbol{\theta}}^{(n+1)} \right) \\
&= E_n \left(\hat{\boldsymbol{\theta}}^{(n+1)} \right) + e_{n+1}^2 \left(\hat{\boldsymbol{\theta}}^{(n+1)} \right)
\end{aligned} \tag{4.39}$$

where

$$e_{n+1} \left(\hat{\boldsymbol{\theta}}^{(n+1)} \right) = y_{n+1} - F \left(\hat{\boldsymbol{\theta}}^{(n+1)}, x_{n+1} \right) \tag{4.40}$$

Firstly, the relationship between H_n and H_{n+1} is examined as follows.

Suppose that for a region in the parameter space sufficiently close to the parameter estimates the objective functions can be expressed as:

$$E_n(\boldsymbol{\theta}) = E_n(\hat{\boldsymbol{\theta}}^{(n)}) + (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}^{(n)})^T H_n (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}^{(n)}) \tag{4.41}$$

and

$$E_{n+1}(\boldsymbol{\theta}) = E_{n+1}(\hat{\boldsymbol{\theta}}^{(n+1)}) + (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}^{(n+1)})^T H_{n+1} (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}^{(n+1)}) \tag{4.42}$$

For a region in the parameter space (m dimensions) sufficiently close to the parameter estimates $(\hat{\boldsymbol{\theta}}^{(n)})$, the model function can be expressed as:

$$F(\boldsymbol{\theta}, x_{n+1}) = F(\hat{\boldsymbol{\theta}}^{(n)}, x_{n+1}) + \mathbf{g}_{n+1}^T (\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}^{(n)}) \tag{4.43}$$

where

$$\mathbf{g}_{n+1} = \begin{bmatrix} \frac{\partial F(\boldsymbol{\theta}, x_{n+1})}{\partial \theta_1} \\ \vdots \\ \frac{\partial F(\boldsymbol{\theta}, x_{n+1})}{\partial \theta_m} \end{bmatrix}_{\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}^{(n)}} \tag{4.44}$$

Hence,

$$e_{n+1}^2(\boldsymbol{\theta}) = [y_{n+1} - F(\boldsymbol{\theta}, x_{n+1})]^2$$

$$\begin{aligned}
 &= e_{n+1}^2 \left(\hat{\theta}^{(n)} \right) - 2e_{n+1} \left(\hat{\theta}^{(n)} \right) \cdot \mathbf{g}_{n+1}^T \left(\theta - \hat{\theta}^{(n)} \right) \\
 &\quad + \left(\theta - \hat{\theta}^{(n)} \right)^T \mathbf{g}_{n+1} \mathbf{g}_{n+1}^T \left(\theta - \hat{\theta}^{(n)} \right)
 \end{aligned} \tag{4.45}$$

where

$$e_{n+1} \left(\hat{\theta}^{(n)} \right) = y_{n+1} - F \left(\hat{\theta}^{(n)}, x_{n+1} \right) \tag{4.46}$$

Combining Eq. 4.41 with Eq. 4.45 leads to:

$$\begin{aligned}
 E_{n+1}(\theta) &= E_n(\theta) + e_{n+1}^2(\theta) \\
 &= E_n(\theta^{(n)}) + \left(\theta - \hat{\theta}^{(n)} \right)^T H_n \left(\theta - \right. \\
 &\quad \left. + e_{n+1} \left(\hat{\theta}^{(n)} \right) - 2e_{n+1} \left(\hat{\theta}^{(n)} \right) \cdot \mathbf{g}_{n+1}^T \left(\theta - \hat{\theta}^{(n)} \right) \right. \\
 &\quad \left. + \left(\theta - \hat{\theta}^{(n)} \right)^T \mathbf{g}_{n+1} \mathbf{g}_{n+1}^T \left(\theta - \hat{\theta}^{(n)} \right) \right)
 \end{aligned} \tag{4.47}$$

The minimization of $E_{n+1}(\theta)$ calls for its derivatives with respect to θ to be zero at a stationary point:

$$\frac{\partial E_{n+1}(\theta)}{\partial \theta} = 0 \tag{4.48}$$

Substituting Eq. 4.47 into Eq. 4.48 leads to:

$$\hat{\theta}^{(n+1)} = \theta + A^{-1} \mathbf{g}_{n+1} \cdot e_{n+1} \left(\hat{\theta}^{(n)} \right) \tag{4.49}$$

where

$$A = H_n + \mathbf{g}_{n+1} \mathbf{g}_{n+1}^T \tag{4.50}$$

Now it is necessary to prove that $A = H_{n+1}$.

From Eq. 4.47,

$$E_{n+1}(\theta) - E_{n+1} \left(\hat{\theta}^{(n+1)} \right) = \left(\theta - \hat{\theta}^{(n)} \right)^T A \left(\theta - \hat{\theta}^{(n)} \right)$$

$$\begin{aligned}
& -2e_{n+1} \left(\hat{\theta}^{(n)} \right) \cdot \mathbf{g}_{n+1}^T \left(\theta - \hat{\theta}^{(n)} \right) \\
& - \left(\hat{\theta}^{(n+1)} - \hat{\theta}^{(n)} \right)^T A \left(\hat{\theta}^{(n+1)} - \hat{\theta}^{(n)} \right) \\
& + 2e_{n+1} \left(\hat{\theta}^{(n)} \right) \cdot \mathbf{g}_{n+1}^T \left(\hat{\theta}^{(n+1)} - \hat{\theta}^{(n)} \right) \quad (4.51)
\end{aligned}$$

Eq. 4.49 indicates that:

$$e_{n+1} \left(\hat{\theta}^{(n)} \right) \cdot \mathbf{g}_{n+1}^T = \left(\hat{\theta}^{(n+1)} - \hat{\theta}^{(n)} \right)^T A \quad (4.52)$$

Now substituting Eq. 4.52 into Eq. 4.51 leads to:

$$\begin{aligned}
E_{n+1}(\theta) - E_{n+1} \left(\hat{\theta}^{(n+1)} \right) &= \left(\theta - \hat{\theta}^{(n)} \right)^T A \left(\theta - \hat{\theta}^{(n)} \right) \\
& - 2 \left(\hat{\theta}^{(n+1)} - \hat{\theta}^{(n)} \right)^T A \left(\theta - \hat{\theta}^{(n)} \right) \\
& - \left(\hat{\theta}^{(n+1)} - \hat{\theta}^{(n)} \right)^T A \left(\hat{\theta}^{(n+1)} - \hat{\theta}^{(n)} \right) \\
& + 2 \left(\hat{\theta}^{(n+1)} - \hat{\theta}^{(n)} \right)^T A \left(\hat{\theta}^{(n+1)} - \hat{\theta}^{(n)} \right) \\
&= \left(\theta - \hat{\theta}^{(n+1)} \right)^T A \left(\theta - \hat{\theta}^{(n+1)} \right) \quad (4.53)
\end{aligned}$$

Comparison between Eq. 4.42 and Eq. 4.53 shows that $\mathbf{A} = H_{n+1}$:

$$H_{n+1} = H_n + \mathbf{g}_{n+1} \mathbf{g}_{n+1}^T \quad (4.54)$$

Hence, the relationship between the determinant of H_n and that of H_{n+1} can be expressed as:

$$\begin{aligned}
|H_{n+1}| &= |H_n + \mathbf{g}_{n+1} \mathbf{g}_{n+1}^T| \\
&= |H_n| \left\{ 1 + \mathbf{g}_{n+1}^T H_n^{-1} \mathbf{g}_{n+1} \right\} \quad (4.55)
\end{aligned}$$

Eq. 4.55 leads to:

$$\frac{|H_n|^{1/2}}{|H_{n+1}|^{1/2}} = \frac{1}{\sqrt{1 + \mathbf{g}_{n+1}^T H_n^{-1} \mathbf{g}_{n+1}}} \quad (4.56)$$

Secondly, the relationship between $\mathbf{R}_n^T \mathbf{R}_n$ and $\mathbf{R}_{n+1}^T \mathbf{R}_{n+1}$ is examined as follows. Suppose that $\hat{\mathbf{g}}^{(n+1)}$ are close to $\hat{\mathbf{g}}^{(n)}$ sufficiently enough to hold Eq. 4.41:

$$E_n \left(\hat{\mathbf{g}}^{(n+1)} \right) - E_n \left(\hat{\mathbf{g}}^{(n)} \right) = \left(\hat{\boldsymbol{\theta}}^{(n+1)} - \hat{\boldsymbol{\theta}}^{(n)} \right)^T H_n \left(\hat{\boldsymbol{\theta}}^{(n+1)} - \hat{\boldsymbol{\theta}}^{(n)} \right) \quad (4.57)$$

Combining Eq. 4.38, Eq. 4.39, and Eq. 4.45 with Eq. 4.57 leads to:

$$\begin{aligned} & \mathbf{R}_{n+1}^T \mathbf{R}_{n+1} - \mathbf{R}_n^T \mathbf{R}_n \\ &= E_{n+1} \left(\hat{\boldsymbol{\theta}}^{(n+1)} \right) - E_n \left(\hat{\boldsymbol{\theta}}^{(n)} \right) \\ &= E_n \left(\hat{\boldsymbol{\theta}}^{(n+1)} \right) + e_{n+1}^2 \left(\hat{\boldsymbol{\theta}}^{(n+1)} \right) - E_n \left(\hat{\boldsymbol{\theta}}^{(n)} \right) \\ &= \left(\hat{\boldsymbol{\theta}}^{(n+1)} - \hat{\boldsymbol{\theta}}^{(n)} \right)^T H_n \left(\hat{\boldsymbol{\theta}}^{(n+1)} - \hat{\boldsymbol{\theta}}^{(n)} \right) \\ &\quad + e_{n+1}^2 \left(\hat{\boldsymbol{\theta}}^{(n)} \right) - 2e_{n+1} \left(\hat{\boldsymbol{\theta}}^{(n)} \right) \cdot \mathbf{g}_{n+1}^T \left(\hat{\boldsymbol{\theta}}^{(n+1)} - \hat{\boldsymbol{\theta}}^{(n)} \right) \\ &\quad + \left(\hat{\boldsymbol{\theta}}^{(n+1)} - \hat{\boldsymbol{\theta}}^{(n)} \right) \mathbf{g}_{n+1} \mathbf{g}_{n+1}^T \left(\hat{\boldsymbol{\theta}}^{(n+1)} - \hat{\boldsymbol{\theta}}^{(n)} \right) \\ &= \left(\hat{\boldsymbol{\theta}}^{(n+1)} - \hat{\boldsymbol{\theta}}^{(n)} \right)^T H_{n+1} \left(\hat{\boldsymbol{\theta}}^{(n+1)} - \hat{\boldsymbol{\theta}}^{(n)} \right) \\ &\quad + e_{n+1}^2 \left(\hat{\boldsymbol{\theta}}^{(n)} \right) - 2e_{n+1} \left(\hat{\boldsymbol{\theta}}^{(n)} \right) \cdot \mathbf{g}_{n+1}^T \left(\hat{\boldsymbol{\theta}}^{(n+1)} - \hat{\boldsymbol{\theta}}^{(n)} \right) \end{aligned} \quad (4.58)$$

From Eq. 4.49,

$$\hat{\boldsymbol{\theta}}^{(n+1)} - \hat{\boldsymbol{\theta}}^{(n)} = H_{n+1}^{-1} \mathbf{g}_{n+1} \cdot e_{n+1} \left(\hat{\boldsymbol{\theta}}^{(n)} \right) \quad (4.59)$$

Substituting Eq. 4.59 into Eq. 4.58 leads to:

$$\begin{aligned} & \mathbf{R}_{n+1}^T \mathbf{R}_{n+1} - \mathbf{R}_n^T \mathbf{R}_n \\ &= \mathbf{g}_{n+1}^T H_{n+1}^{-1} \mathbf{g}_{n+1} \cdot e_{n+1}^2 \left(\hat{\boldsymbol{\theta}}^{(n)} \right) \\ &\quad + e_{n+1}^2 \left(\hat{\boldsymbol{\theta}}^{(n)} \right) - 2\mathbf{g}_{n+1}^T H_{n+1}^{-1} \mathbf{g}_{n+1} \cdot e_{n+1}^2 \left(\hat{\boldsymbol{\theta}}^{(n)} \right) \\ &= \left(1 - \mathbf{g}_{n+1}^T H_{n+1}^{-1} \mathbf{g}_{n+1} \right) \cdot e_{n+1}^2 \left(\hat{\boldsymbol{\theta}}^{(n)} \right) \end{aligned} \quad (4.60)$$

H_{n+1}^{-1} can be calculated by H_n^{-1} and $\mathbf{g}_{n+1}$ as follows:

$$H_{n+1}^{-1} = H_n^{-1} - \frac{H_n^{-1} \mathbf{g}_{n+1} (H_n^{-1} \mathbf{g}_{n+1})^T}{1 + \mathbf{g}_{n+1}^T H_n^{-1} \mathbf{g}_{n+1}} \quad (4.61)$$

This equation is verified easily by multiplying Eq. 4.54 by Eq. 4.61.

Substituting Eq. 4.61 into Eq. 4.60 leads to:

$$\mathbf{R}_{n+1}^T \mathbf{R}_{n+1} - \mathbf{R}_n^T \mathbf{R}_n = \frac{1}{1 + \mathbf{g}_{n+1}^T H_n^{-1} \mathbf{g}_{n+1}} \left(y_{n+1} - F \left(\hat{\boldsymbol{\theta}}^{(n)}, x_{n+1} \right) \right)^2 \quad (4.62)$$

Finally, substituting Eq. 4.56 and Eq. 4.62 into Eq. 4.36 reduces to:

$$\frac{\text{Prob}(y_1, \dots, y_n, y_{n+1})}{\text{Prob}(y_1, \dots, y_n)} = \frac{1}{\sqrt{2\pi\sigma^2(1 + \mathbf{g}_{n+1}^T H_n^{-1} \mathbf{g}_{n+1})}} \cdot \exp \left[-\frac{1}{2\sigma^2(1 + \mathbf{g}_{n+1}^T H_n^{-1} \mathbf{g}_{n+1})} \left(y_{n+1} - F \left(\hat{\boldsymbol{\theta}}^{(n)}, x_{n+1} \right) \right)^2 \right] \quad (4.63)$$

This equation is numerically identical to Eq. 4.9 but it is necessary to mention the fundamental difference between Eq. 4.9 and Eq. 4.63. In deriving Eq. 4.9, it is assumed that n observations $(y_1, \dots, y_n)$ are given and the $n+1$ -th observation (y_{n+1}) is not obtained. After the predictive probability of y_{n+1} occurring at x_{n+1} is calculated based on the match to the n observations, the $n+1$ -th observation (y_{n+1}) is given and the actual probability is calculated by substituting the actual value of y_{n+1} into Eq. 4.9. On the other hand, in deriving Eq. 4.63, it is assumed that $n+1$ observations $(y_1, \dots, y_n, y_{n+1})$ are available in advance. The joint probability of $y_1, \dots, y_n$ is calculated based on the match to the n observations and the joint probability of $y_1, \dots, y_n, y_{n+1}$ is calculated based on the match to the $n+1$ observations. In other words, Eq. 4.9 is derived before the $n+1$ -th observation is available, while Eq. 4.63 is derived after the $n+1$ -th observation is available. The crucial assumption to connect Eq. 4.9 with Eq. 4.63 is that the estimated values of the parameters based on the match to the $n+1$ observations $(\hat{\boldsymbol{\theta}}^{(n+1)})$ are sufficiently close to those based on the match to the n observations $(\hat{\boldsymbol{\theta}}^{(n)})$ (Eq. 4.2 and Eq. 4.43).

Suppose that the number of data at the starting stage is n and the number of data at the final stage is $n + k$. Assume that $n + k$ data are available in advance.

Eq. 4.26 implies that:

$$\begin{aligned}
 & Prob(y_{n+k}|y_1, \dots, y_n, y_{n+1}, \dots, y_{n+k-1}) \cdots Prob(y_{n+1}|y_1, \dots, y_n) \\
 &= \frac{Prob(y_1, \dots, y_n, y_{n+1}, \dots, y_{n+k})}{Prob(y_1, \dots, y_n, y_{n+1}, \dots, y_{n+k-1})} \cdots \frac{Prob(y_1, \dots, y_n, y_{n+1})}{Prob(y_1, \dots, y_n)} \\
 &= \frac{Prob(y_1, \dots, y_n, y_{n+1}, \dots, y_{n+k})}{Prob(y_1, \dots, y_n)} \quad (4.64)
 \end{aligned}$$

In this case, the product of conditional probabilities requires the calculations of k conditional probabilities, while the joint probability requires only two calculations which are $Prob(y_1, \dots, y_n)$ and $Prob(y_1, \dots, y_n, y_{n+1}, \dots, y_{n+k})$. As the number of k increases, the reduction of the computation cost becomes significant. It should be noted that this reduction of the computation cost was not discussed by Box and Hill (1967), since in the procedure proposed by Box and Hill (1967) only n data are available at the beginning and the number of data increases by one at each prediction step.

In cases where variance σ^2 is known, Eq. 4.36 indicates that:

$$\begin{aligned}
 & \frac{Prob(y_1, \dots, y_n, y_{n+1}, \dots, y_{n+k})}{Prob(y_1, \dots, y_n)} = \\
 & \frac{1}{(\sqrt{2\pi}\sigma)^k} \frac{|H_n|^{1/2}}{|H_{n+k}|^{1/2}} \cdot \exp \left[-\frac{1}{2\sigma^2} (\mathbf{R}_{n+k}^T \mathbf{R}_{n+k} - \mathbf{R}_n^T \mathbf{R}_n) \right] \quad (4.65)
 \end{aligned}$$

In cases where variance σ^2 is unknown and $n - m$ is greater than 30, slight modifications are applied to Eq. 4.65:

$$\begin{aligned}
 & \frac{Prob(y_1, \dots, y_n, y_{n+1}, \dots, y_{n+k})}{Prob(y_1, \dots, y_n)} = \\
 & \frac{(\sqrt{2\pi}s_n)^{n-m}}{(\sqrt{2\pi}s_{n+k})^{n+k-m}} \frac{|H_n|^{1/2}}{|H_{n+k}|^{1/2}} \cdot \exp \left[-\frac{1}{2s_{n+k}^2} \mathbf{R}_{n+k}^T \mathbf{R}_{n+k} + \frac{1}{2s_n^2} \mathbf{R}_n^T \mathbf{R}_n \right] \quad (4.66)
 \end{aligned}$$

where

$$s_n^2 = \frac{SSR_n}{n - m} \quad (4.67)$$

$$\begin{aligned} SSR_n &= \mathbf{R}_n^T \mathbf{R}_n \\ &= \sum_{i=1}^n \left[y_i - F\left(\hat{\boldsymbol{\theta}}^{(n)}, x_i\right) \right]^2 \end{aligned} \quad (4.68)$$

and

$$s_{n+k}^2 = \frac{SSR_{n+k}}{n + k - m} \quad (4.69)$$

$$\begin{aligned} SSR_{n+k} &= \mathbf{R}_{n+k}^T \mathbf{R}_{n+k} \\ &= \sum_{i=1}^{n+k} \left[y_i - F\left(\hat{\boldsymbol{\theta}}^{(n+k)}, x_i\right) \right]^2 \end{aligned} \quad (4.70)$$

Hence, Eq. 4.66 is expressed as:

$$\begin{aligned} \frac{Prob(y_1, \dots, y_n, y_{n+1}, \dots, y_{n+k})}{Prob(y_1, \dots, y_n)} &= \\ \frac{(\sqrt{2\pi}s_n)^{n-m}}{(\sqrt{2\pi}s_{n+k})^{n+k-m}} \frac{|H_n|^{1/2}}{|H_{n+k}|^{1/2}} \cdot \exp\left[-\frac{k}{2}\right] \end{aligned} \quad (4.71)$$

Here the joint probability associated with the j -th model at the final stage should be evaluated using either Eq. 4.65 or Eq. 4.71.

Recall Eq. 4.22 and **Eq.** 4.23:

$$\Pi_j^{n+1} = \frac{\Pi_j^n P_j^{n+1}}{\sum_{j=1}^{n_m} \Pi_j^n P_j^{n+1}} \quad (4.72)$$

where P_j^{n+1} is the predictive probability of the $n + 1$ -th observation occurring under the model j based on the match to the n observations.

From Eq. 4.72, the joint probability associated with the j -th model at the final stage is calculated by:

$$\Pi_j^{n+k} = \Pi_j^n \cdot \frac{\text{Prob}(y_1, \dots, y_n, y_{n+1}, \dots, y_{n+k})}{\text{Prob}(y_1, \dots, y_n)} \quad (4.73)$$

The normalized joint probability associated with the j -th model at the final stage is calculated by dividing Eq. 4.73 by the summation of Eq. 4.73 over the number of models.

4.5 Example

In this example, simulated drawdown data were used to demonstrate the basic behavior of the sequential predictive probability method.

The purpose in using simulated data is to test the procedure on a problem in which the reservoir model is known in advance. Pressure data were calculated using a no flow outer boundary model and random errors were added (Fig. 4.10). Reservoir and fluid data were the same as given in Table 3.2. The true parameter values are $k = 50 \text{ md}$, $S = 10$, $C = 0.01 \text{ STB/psi}$, and $r_e = 2000 \text{ ft}$.

A random number generator was used to simulate random normal errors with zero mean and a variance of 1.0 psi^2 (Fig 4.11). The total number of data points is 81 and the time period is from 0.01 hour to 100 hours (Table 4.1 and Table 4.2).

In this example, the graphical analysis using the pressure derivative plot suggests a no flow outer boundary model, since the boundary effect becomes significant after around 20 hours and thereafter a steep rising straight line with a unit slope is recognized on the pressure derivative plot. However, in order to show how this method works, an infinite acting model was selected as a competitive model. For simplicity, name the infinite acting model Model 1 and the no flow outer boundary model Model 2. Model 1 has three parameters, which are k , S , and C , while Model 2 has four parameters, which are k , S , C , and r_e .

The final estimated values of the parameters, which were evaluated using the whole data (81 data points), are given in Table 4.3. Final matches, which were calculated using the final estimated values of the parameters, are shown in Fig. 4.12.

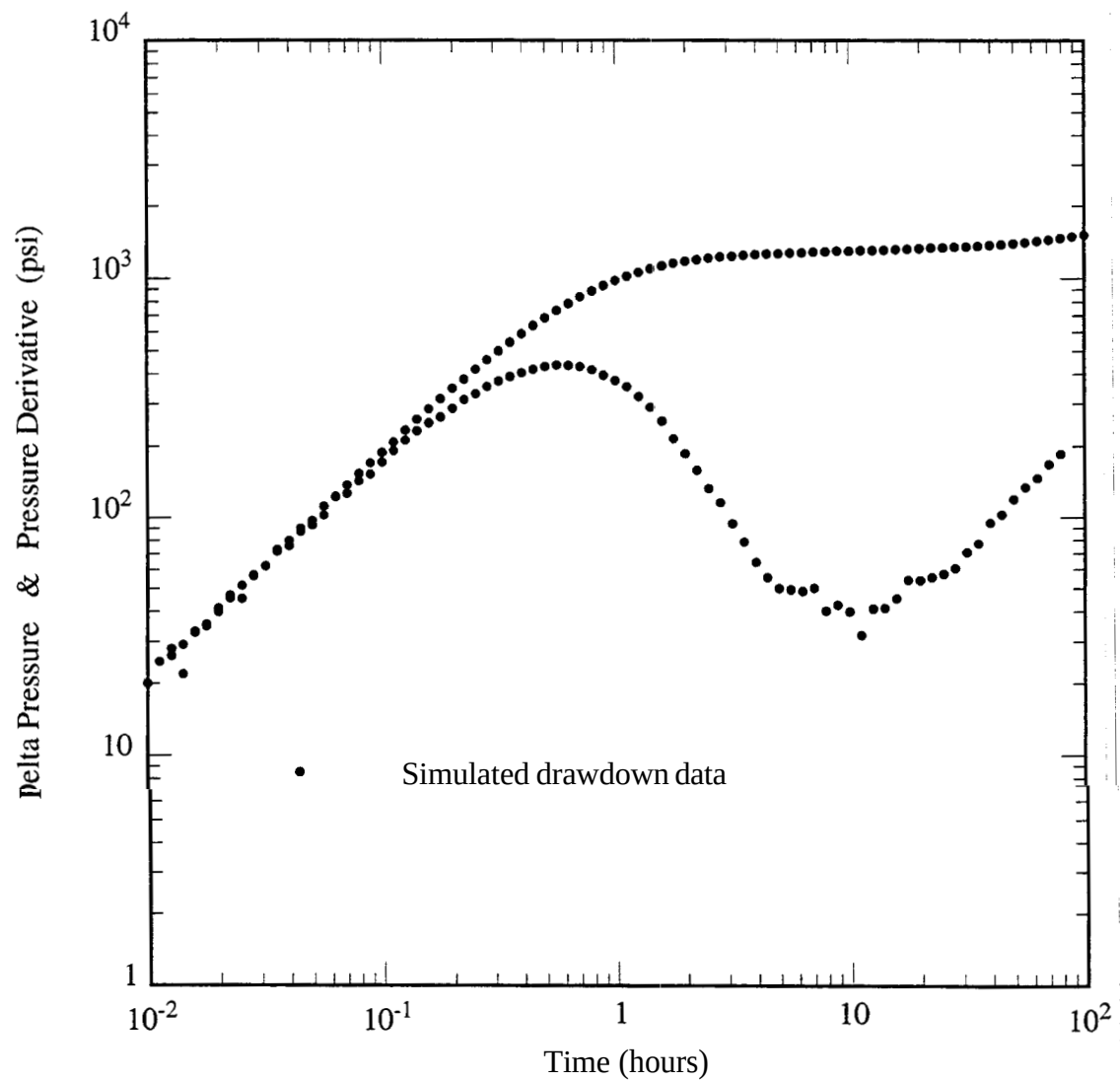


Figure 4.10: Simulated drawdown data using a no flow outer boundary model with normal random errors

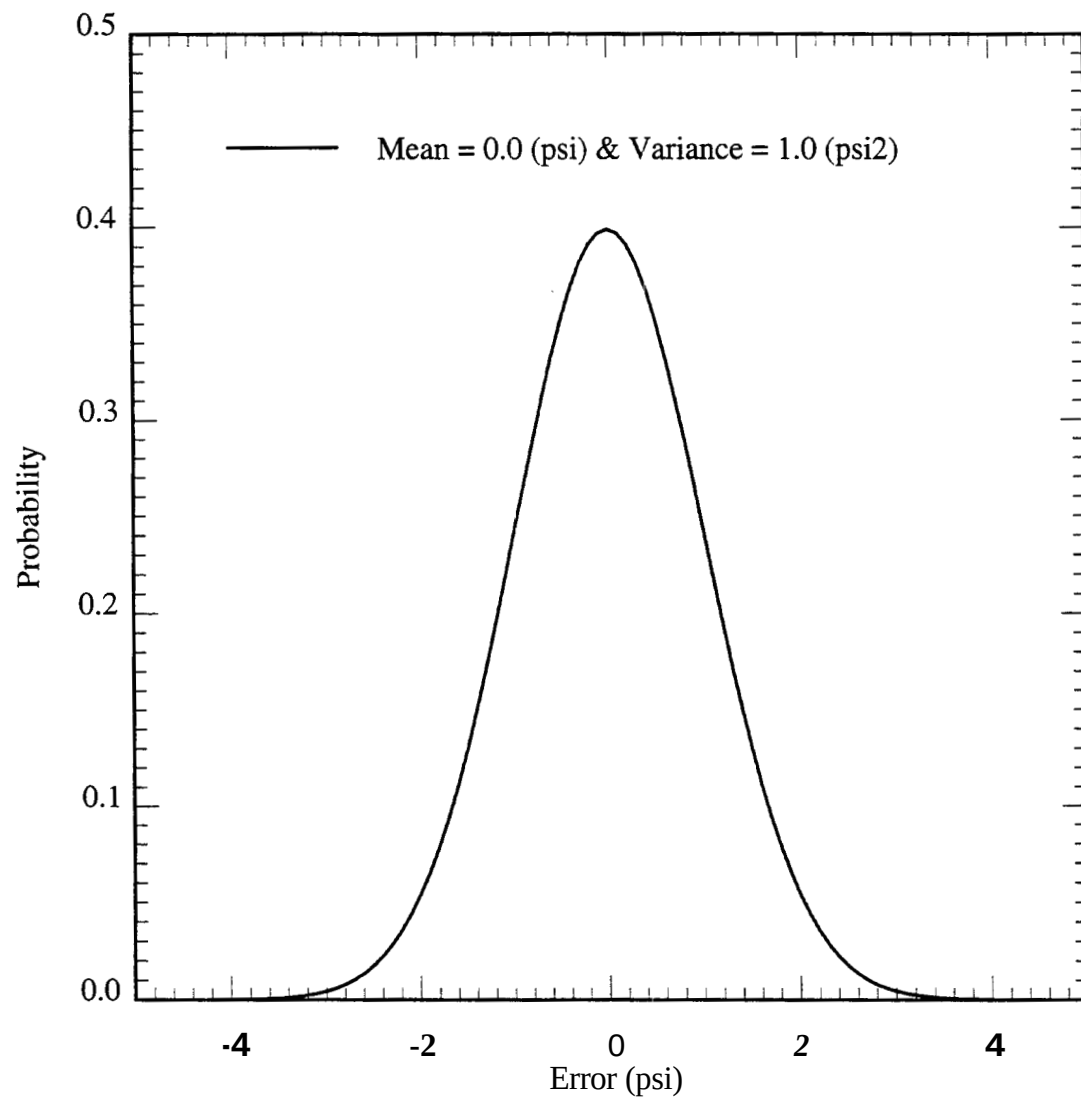


Figure 4.11: Normal distribution with zero mean and a variance of 1.0psi^2

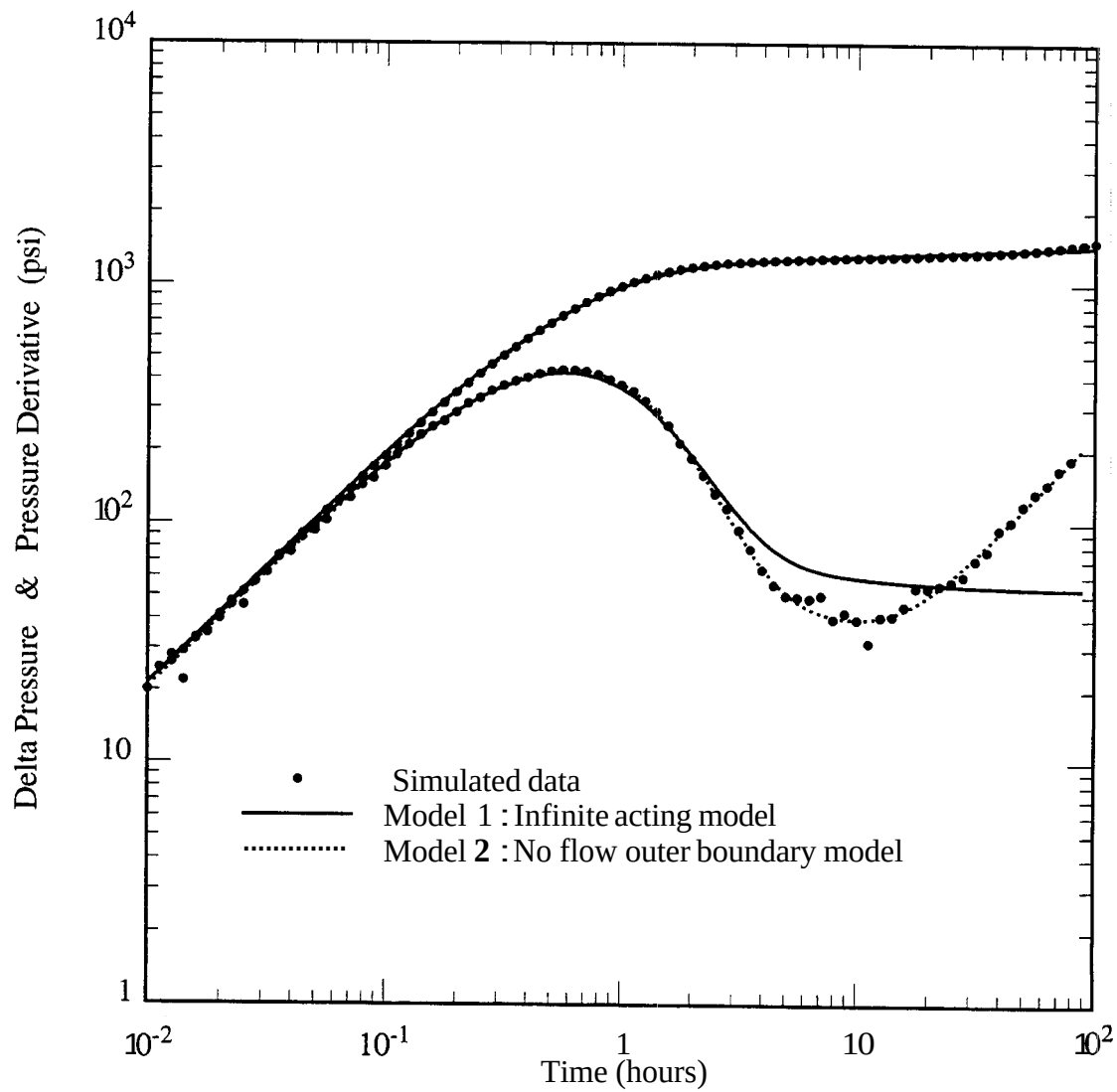


Figure 4.12: Final matches of Model 1 and Model 2 to the data

Table 4.1: Pressure data calculated using a no flow outer boundary model with normal random errors (1)

Number of data	Time (hours)	Pressure (psi)	Number of data	Time (hours)	Pressure (psi)
1	0.0100	20.1152	21	0.1000	188.7886
2	0.0112	24.7672	22	0.1122	207.8700
3	0.0126	26.2986	23	0.1259	233.0306
4	0.0141	29.1696	24	0.1413	258.7846
5	0.0158	33.0225	25	0.1585	286.2793
6	0.0178	34.8778	26	0.1778	314.4368
7	0.0200	41.3793	27	0.1995	348.1626
8	0.0224	45.5047	28	0.2239	380.5104
9	0.0251	51.3825	29	0.2512	418.4551
10	0.0282	56.4313	30	0.2818	458.6909
11	0.0316	62.2886	31	0.3162	500.3290
12	0.0355	71.7095	32	0.3548	544.0076
13	0.0398	79.9113	33	0.3981	590.4244
14	0.0447	90.0302	34	0.4467	638.3126
15	0.0501	97.2079	35	0.5012	686.9263
16	0.0562	111.8982	36	0.5623	736.8065
17	0.0631	122.7691	37	0.6310	788.7568
18	0.0708	137.3560	38	0.7079	839.5027
19	0.0794	153.9514	39	0.7943	887.8142
20	0.0891	170.3791	40	0.8913	934.8115

Table 4.2: Pressure data calculated using a no flow outer boundary model with normal random errors (2)

Number of data	Time (hours)	Pressure (psi)	Number of data	Time (hours)	Pressure (psi)
41	1.0000	980.8302	61	10.0000	1300.2422
42	1.1220	1021.7939	62	11.2202	1307.3192
43	1.2589	1061.1836	63	12.5893	1310.0145
44	1.4125	1097.9647	64	14.1254	1313.7211
45	1.5849	1129.5849	65	15.8489	1319.1840
46	1.7783	1155.6681	66	17.7828	1326.3575
47	1.9953	1178.3133	67	19.9526	1330.8975
48	2.2387	1196.9707	68	22.3872	1338.7042
49	2.5119	1215.3349	69	25.1189	1344.0554
50	2.8184	1228.7332	70	28.1838	1351.9353
51	3.1623	1239.5291	71	31.6228	1357.3901
52	3.5481	1250.2988	72	35.4813	1366.6963
53	3.9811	1258.6538	73	39.8107	1376.7514
54	4.4668	1264.8830	74	44.6684	1387.5264
55	5.0119	1269.1952	75	50.1187	1401.1340
56	5.6234	1275.8646	76	56.2341	1414.1836
57	6.3096	1281.7214	77	63.0957	1431.9028
58	7.0795	1287.6542	78	70.7946	1449.5032
59	7.9433	1291.5970	79	79.4328	1468.7515
60	8.9125	1299.0021	80	89.1251	1491.6053
			81	100.0000	1517.3806

Table 4.3: Final parameter estimates evaluated using the 81 data points

Parameters	Units	Model 1	Model 2	True values
k	md	0.3356×10^2	0.4978×10^2	0.5000×10^2
S		0.4283×10^1	0.9929×10^1	0.1000×10^2
C	STB/psi	0.9631×10^{-2}	0.1001×10^{-1}	0.1000×10^{-1}
r_e	ft		0.2006×10^4	0.2000×10^4

The procedure of the sequential predictive probability method is described as follows.

The data at 1.0 hour was selected as the starting point just after the peak of the pressure derivative plot. The number of data points from the beginning (0.01 hour) to 1.0 hour is 41. This number is greater than 30 and the normal distribution is used to replace the student t distribution sufficiently. The parameters are evaluated by nonlinear regression using the first 41 data points ($y_1, \dots, y_{41}$). The variance (σ^2) and the inverse Hessian matrix (H^{-1}) are also evaluated using the first 41 data points. The next data point is observed at $x_{42} = 1.1220$ hours from Table 4.2. The gradient (g) is evaluated at x_{42} using the estimated values of the parameters based on the match to the first 41 data points.

The overall predictive variance in Model 1 is 1.8241 psi^2 , while that in Model 2 is 6.0908 psi^2 . The expected value of the pressure response at x_{42} under Model 1 ($\hat{y}_{42}$) is 1020.7414 psi , while that under Model 2 ($\hat{y}_{42}$) is 1020.4509 psi .

Hence, from Eq. 4.9 the predictive probability distribution of y_{42} under Model 1 is expressed as:

$$\begin{aligned} & \text{Prob}(y_{42}|\hat{y}_{42}) \\ &= \frac{1}{\sqrt{2\pi \cdot 1.8241}} \cdot \exp \left[-\frac{1}{2 \cdot 1.8241} (y_{42} - 1020.7414)^2 \right] \end{aligned} \quad (4.74)$$

On the other hand, the predictive probability distribution of y_{42} under Model 2 is expressed as:

$$\begin{aligned} & \text{Prob}(y_{42}|\hat{y}_{42}) \\ &= \frac{1}{\sqrt{2\pi \cdot 6.0908}} \cdot \exp \left[-\frac{1}{2 \cdot 6.0908} (y_{42} - 1020.4509)^2 \right] \end{aligned} \quad (4.75)$$

Now the observed pressure value (y_{42}) is 1021.7941 psi from Table 4.2. Substituting this value into Eq. 4.74 and Eq. 4.75 shows that the probability of y_{42} occurring under Model 1 (P_1^{42}) is 0.2180 and that under Model 2 (P_2^{42}) is 0.1394.

In this example the number of possible models is two, and the probability associated with each model is set to be equal at the starting point of the procedure

($\Pi_1^{41} = \Pi_2^{41} = 0.5$).

Therefore, the normalized joint probability associated with Model 1 after taking the y_{42} data point is calculated using Eq. 4.22:

$$\begin{aligned}\Pi_1^{42} &= \frac{\Pi_1^{41} P_1^{42}}{\Pi_1^{41} P_1^{42} + \Pi_2^{41} P_2^{42}} \\ &= \frac{0.5 \cdot 0.2180}{0.5 \cdot 0.2180 + 0.5 \cdot 0.1394} \\ &= 0.6100\end{aligned}\tag{4.76}$$

The normalized joint probability associated with Model 2 after taking the y_{42} data point is also calculated using Eq. 4.22:

$$\begin{aligned}\Pi_2^{42} &= \frac{\Pi_2^{41} P_2^{42}}{\Pi_1^{41} P_1^{42} + \Pi_2^{41} P_2^{42}} \\ &= \frac{0.5 \cdot 0.1394}{0.5 \cdot 0.2180 + 0.5 \cdot 0.1394} \\ &= 0.3900\end{aligned}\tag{4.77}$$

After taking the y_{42} data point, the normalized joint probability associated with Model 1 ($\Pi_1^{42} = 0.6100$) is a little greater than that associated with Model 2 ($\Pi_2^{42} = 0.3900$). This indicates that Model 1 is a little more favorable after taking the y_{42} data point. Here one step of the procedure of the sequential predictive probability method is completed. After taking y_{42} , the results are not conclusive and the next step of the procedure is required.

At the next step of the procedure, the parameters are evaluated by nonlinear regression using the 42 data points ($y_1, \dots, y_{41}, y_{42}$). The estimated values of the parameters, which have been obtained using the first 41 data points at the previous step, are used as the initial estimates of the parameters. The variance (σ^2) and the inverse Hessian matrix (H^{-1}) are evaluated using the 42 data points. The next data point is observed at $x_{43} = 1.2589$ hours from Table 4.2. The gradient (g) is evaluated at x_{43} using the estimated values of the parameters based on the match to the first 42 data points. The predictive probability distribution of y_{43} under each model is constructed and the probability of y_{43} occurring under each model is calculated by substituting

the observed pressure value (y_{43}) into this predictive probability distribution. The normalized joint probability associated with each model is updated using Eq. 4.22.

Thereafter, this procedure is repeated until the end of the data point at x_{81} .

In Fig. 4.13, as the procedure is repeated, the normalized joint probability associated with Model 1 increases gradually up to one. During the period between 10 hours and 20 hours, the normalized joint probability associated with Model 1 is almost one, which indicates that Model 1 is clearly superior to Model 2 and Model 1 can be selected as the best model to represent the reservoir. However, after 20 hours the normalized joint probability associated with Model 2 increases rapidly to one. At the end of the procedure, the normalized joint probability associated with Model 2 is 1.0000, while that associated with Model 1 is 0.0000. Hence, at the end of the procedure the normalized joint probability indicates clearly that Model 2 is superior to Model 1. This means that the existence of the no flow boundary is required to represent the reservoir adequately.

From a physical point of view, before the boundary effect becomes significant, there are not enough data available to support the existence of the boundary and Model 1 is selected. However, once the boundary effect appears, this effect becomes more and more apparent as time goes by. After 40 hours enough data are available to support the existence of the boundary and, as a result, Model 2 is selected. Hence, before the boundary effect becomes significant, the sequential predictive probability method selects an infinite acting model and then later selects a no flow outer boundary model. This is consistent with the concept of radius of investigation.

The time period between 20 hours, when the boundary effect first becomes significant, and 40 hours, when this method clearly indicates the existence of boundary, demonstrates the inability of the data to illuminate the cause of the discrepancy between the observed data and the calculated data based on Model 1, either by the errors or by a boundary effect.

The result obtained from this example indicates one of the requirements of the sequential predictive probability method for model discrimination in well test analysis, which is that the procedure should be repeated until the end of data. If the procedure

Table 4.4: Normalized joint probabilities, step 41 to 60

Number of step	Time (hours)	Model 1 P_1	Model2 P_2	Model 1 Π_1	Model 2 Π_2
41	1.0000			0.5000	0.5000
42	1.1220	0.2180	0.1394	0.6100	0.3900
43	1.2589	0.2954	0.1812	0.7183	0.2817
44	1.4125	0.0770	0.1638	0.5452	0.4548
45	1.5849	0.1775	0.1759	0.5475	0.4525
46	1.7783	0.2710	0.2531	0.5644	0.4356
47	1.9953	0.1917	0.1695	0.5944	0.4056
48	2.2387	0.1076	0.1008	0.6100	0.3900
49	2.5119	0.2308	0.1341	0.7292	0.2708
50	2.8184	0.3208	0.1848	0.8237	0.1763
51	3.1623	0.3506	0.2308	0.8766	0.1234
52	3.5481	0.1660	0.1857	0.8639	0.1361
53	3.9811	0.1858	0.1955	0.8578	0.1422
54	4.4668	0.3637	0.2016	0.9159	0.0841
55	5.0119	0.1193	0.0355	0.9731	0.0269
56	5.6234	0.3776	0.3511	0.9749	0.0251
57	6.3096	0.2878	0.2735	0.9761	0.0239
58	7.0795	0.0813	0.1833	0.9478	0.0522
59	7.9433	0.3497	0.1561	0.9760	0.0240
60	8.9125	0.0578	0.1914	0.9247	0.0754

Table 4.5: Normalized joint probabilities, step 61 to 81

Number of step	Time (hours)	Model 1 P_1	Model 2 P_2	Model 1	Model 2 Π_2
61	10.0000	0.2585	0.0420	0.9869	0.0131
62	11.2202	0.1119	0.24412	0.9719	0.0281
63	12.5893	0.3464	0.0383	0.9968	0.0032
64	14.1254	0.2063	0.2096	0.9968	0.0032
65	15.8489	0.3750	0.3054	0.9974	0.0026
66	17.7828	0.0558	0.0902	0.9957	0.0042
67	19.9526	0.0534	0.2540	0.9801	0.0199
68	22.3872	0.0519	0.1369	0.9492	0.0508
69	25.1189	0.0447	0.2319	0.7826	0.2174
70	28.1838	0.0398	0.2897	0.3307	0.6693
71	31.6228	0.0329	0.0444	0.2684	0.7316
72	35.4813	0.0290	0.1712	0.0585	0.9415
73	39.8107	0.0238	0.2634	0.0056	0.9944
74	44.6684	0.0189	0.2668	0.0004	0.9996
75	50.1187	0.0150	0.2545	0.0000	1.0000
76	56.2341	0.0118	0.3074	0.0000	1.0000
77	63.0957	0.0095	0.1025	0.0000	1.0000
78	70.7946	0.0075	0.2063	0.0000	1.0000
79	79.4328	0.0061	0.3284	0.0000	1.0000
80	89.1251	0.0051	0.2910	0.0000	1.0000
81	100.0000	0.0042	0.2146	0.0000	1.0000

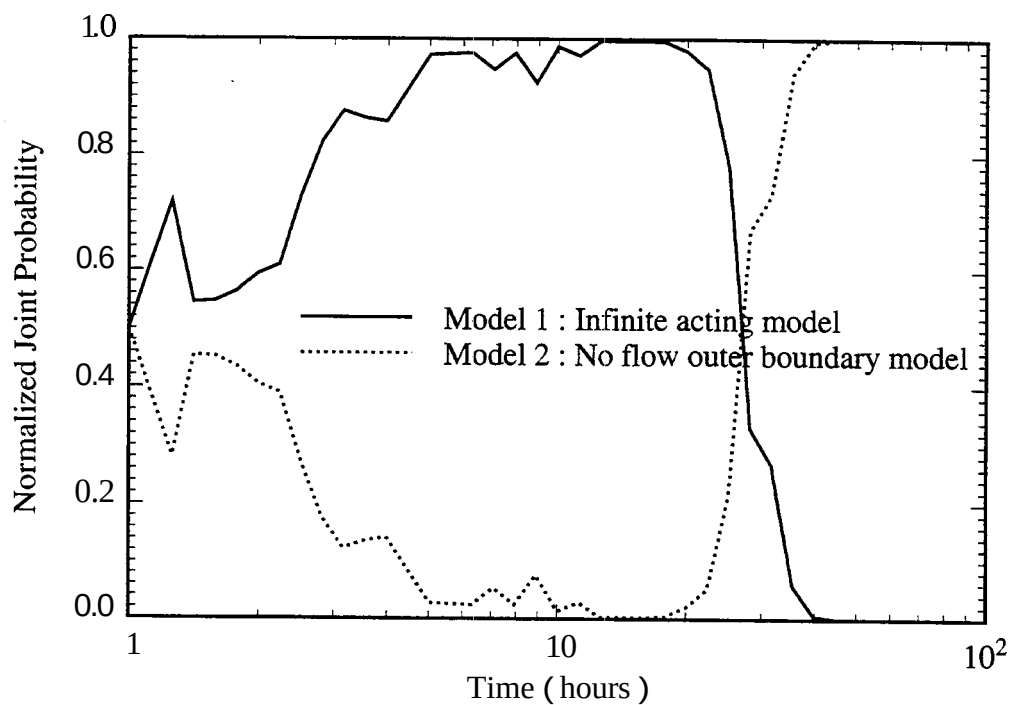
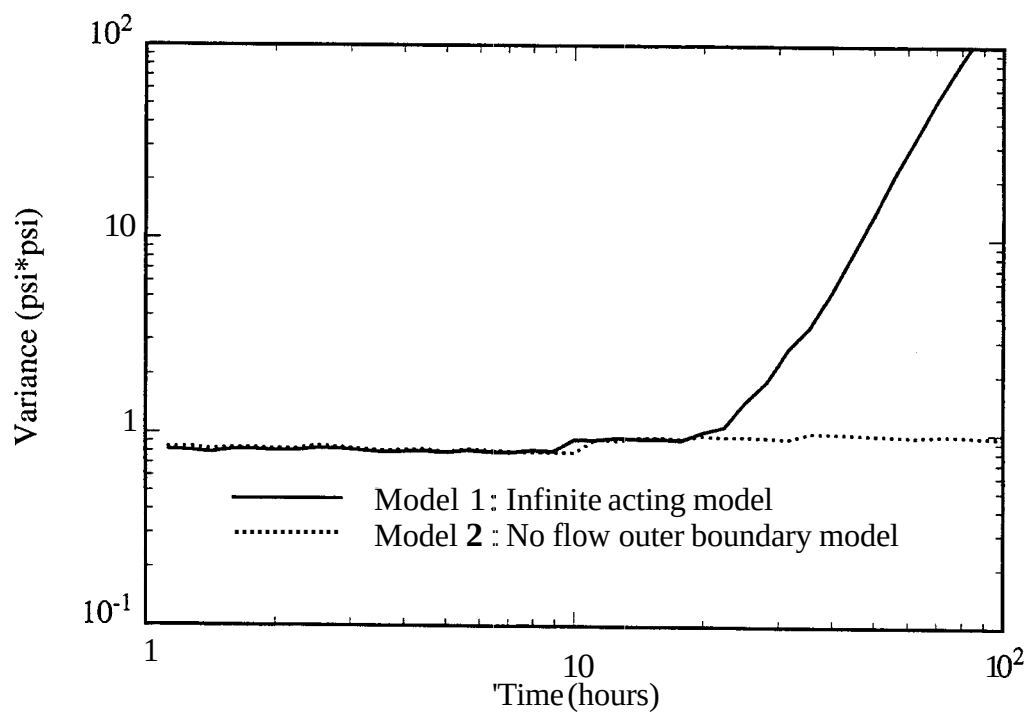


Figure 4.13: Normalized joint probability associated with the model

Figure 4.14: Estimated variance (σ^2)

is terminated at $x_{67} = 19.9526$ hours, an infinite acting model is selected according to the normalized joint probability associated with Model 1 of 0.9801. Actually, this model explains the data quite well from the beginning to around 20 hours but does not explain the data during the period in which the boundary effect is apparent. In general, the boundary effects appear at late time. Therefore, in order to decide whether the boundary exists or not, the procedure should be performed to the end of data.

It is also possible to deduce additional valuable information for model discrimination from the behavior of the normalized joint probabilities in Fig. 4.13. In cases where the original data were truncated at $x_{70} = 28.1838$ hours, the final normalized joint probability associated with Model 2 is 0.6693 and that with Model 1 is 0.3307, Model 2 is a little more favorable but the results are not conclusive. However, the normalized joint probability associated with Model 2 has an increasing trend after 20 hours. This suggests the possibility of the existence of the boundary. Therefore, it is necessary to investigate not only the final values of the normalized joint probabilities but also the behavior of the normalized joint probabilities during the sequential procedure.

Next, it is determined how each component contributes to the normalized joint probability.

In Fig. 4.14, the estimated variances σ^2 are almost the same as the true variance ($\sigma^2 = 1.0 \text{ psi}^2$) until 20 hours and thereafter the variance in Model 1 increases rapidly. This indicates that the model fits to the data are almost identical and both models match the data adequately before the boundary effect becomes significant. However, once the boundary effect becomes significant, Model 1 no longer represents the data adequately, since the estimated variance is far from the true variance. The estimated variance of Model 2 is still close to the true variance and Model 2 represents the data sufficiently well after the boundary effect becomes significant.

The magnitude of $\mathbf{g}^T H^{-1} \mathbf{g}$ in Model 2 is always larger than that in Model 1 throughout the time in Fig. 4.15. This indicates the expense of adding one more parameter (r_e) into the model.

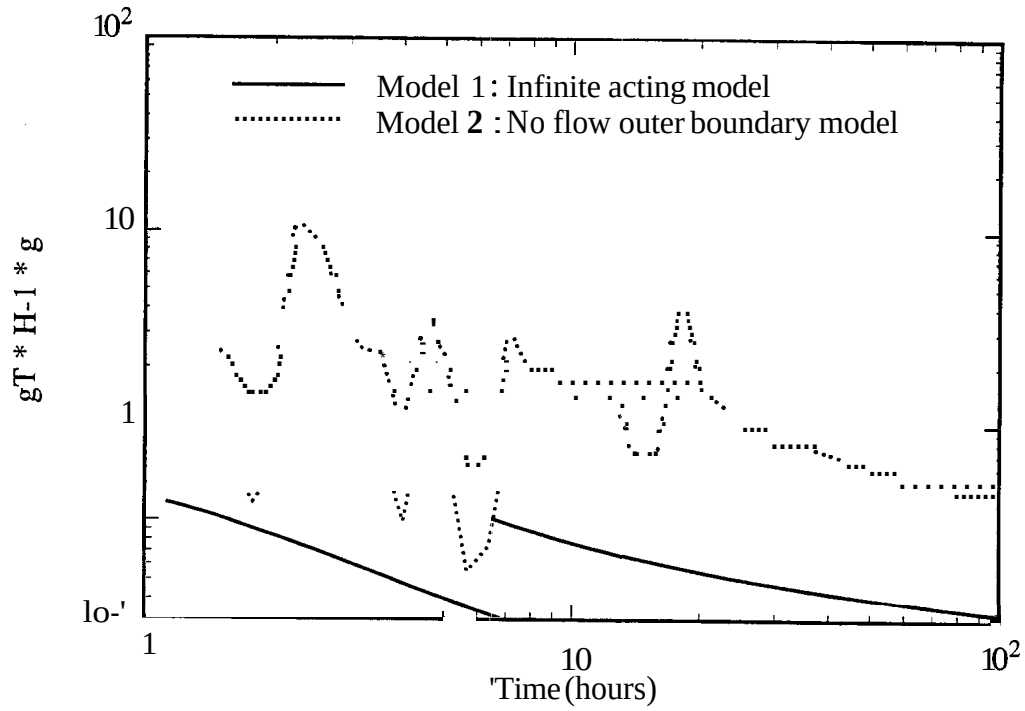
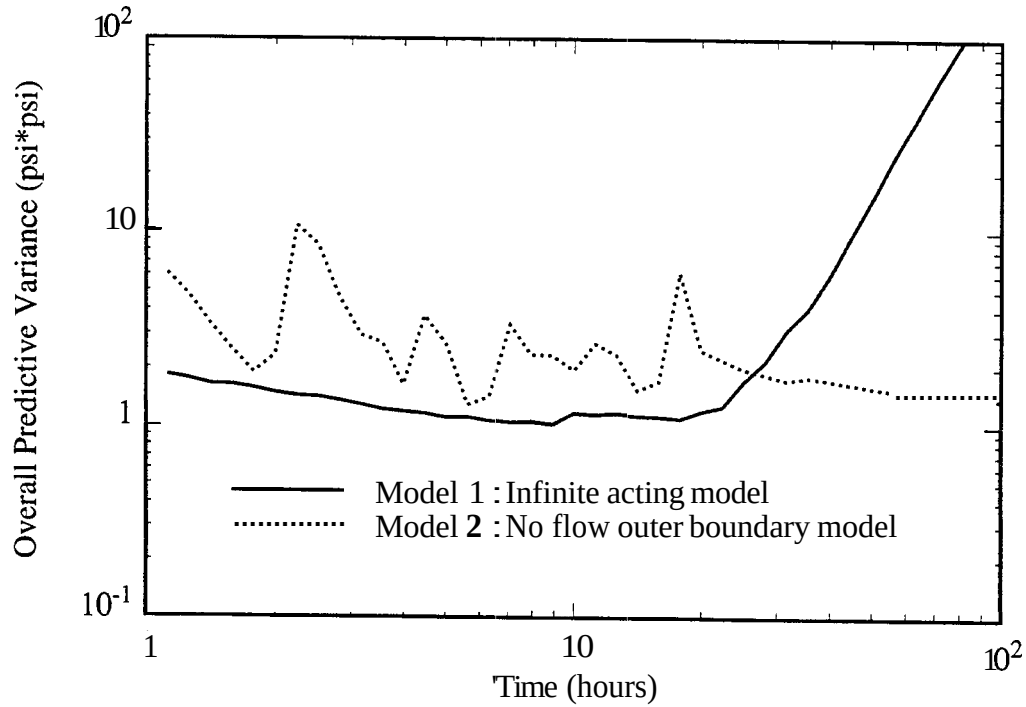
The overall predictive variance $(1 + \mathbf{g}^T H^{-1} \mathbf{g}) \cdot \sigma^2$ in Model 1, therefore, becomes smaller than that of Model 2 before around 25 hours in Fig. 4.16. This indicates that even though the model fits to the data are almost identical, the overall predictive variance in Model 2 is larger than that in Model 1 due to the expense of adding one more parameter (r_e) into the model before the boundary effect becomes significant. As the boundary effect becomes significant, the improvement of the model fits by adding one more parameter (r_e) into the model overcomes the expense of adding one more parameter (r_e) into the model and the overall predictive variance in Model 2 becomes smaller than that in Model 1.

In Fig. 4.17, the pressure difference between the observed pressure response and the expected pressure response based on Model 1 is almost identical to that based on Model 2 before 20 hours and both are distributed randomly around 0 *psi*. Note that the pressure difference should be distributed randomly around 0 *psi* as long as the model is correct. However, the magnitude of the pressure difference based on Model 1 becomes increasingly larger after 20 hours, while it is bounded in Model 2.

Recall that the probability of y_{n+1} occurring at x_{n+1} under a specific model becomes high when the overall predictive variance $\sigma^2 + \sigma_p^2 = (1 + \mathbf{g}^T H^{-1} \mathbf{g}) \cdot \sigma^2$ is small and the observed pressure response y_{n+1} is close to the expected pressure response $\hat{y}_{n+1}$.

Hence, the probability in Model 1 is generally larger than that in Model 2 before 20 hours in Fig. 4.18, since even though both model have similar pressure difference between the observed pressure response and the expected pressure response, the overall predictive variance in Model 1 is smaller than that in Model 2. After 20 hours the probability Model 1 becomes smaller than that in Model 2 in Fig. 4.18, since Model 1 has the larger overall predictive variance and the observed pressure response is far from the expected pressure response based on Model 1. Since the normalized joint probability is proportional to the product of the probabilities, the normalized joint probability in Model 1 is high before 20 hours and becomes low after 40 hours.

It is interesting to examine the estimated values and the confidence intervals of each parameter. Fig. 4.19 shows the estimated values of permeability in Model 1 and

Figure 4.15: $g^T H^{-1} g$ Figure 4.16: Overall predictive variance ($\sigma^2 + \sigma_p^2 = (1 + g^T H^{-1} g) \cdot \sigma^2$)

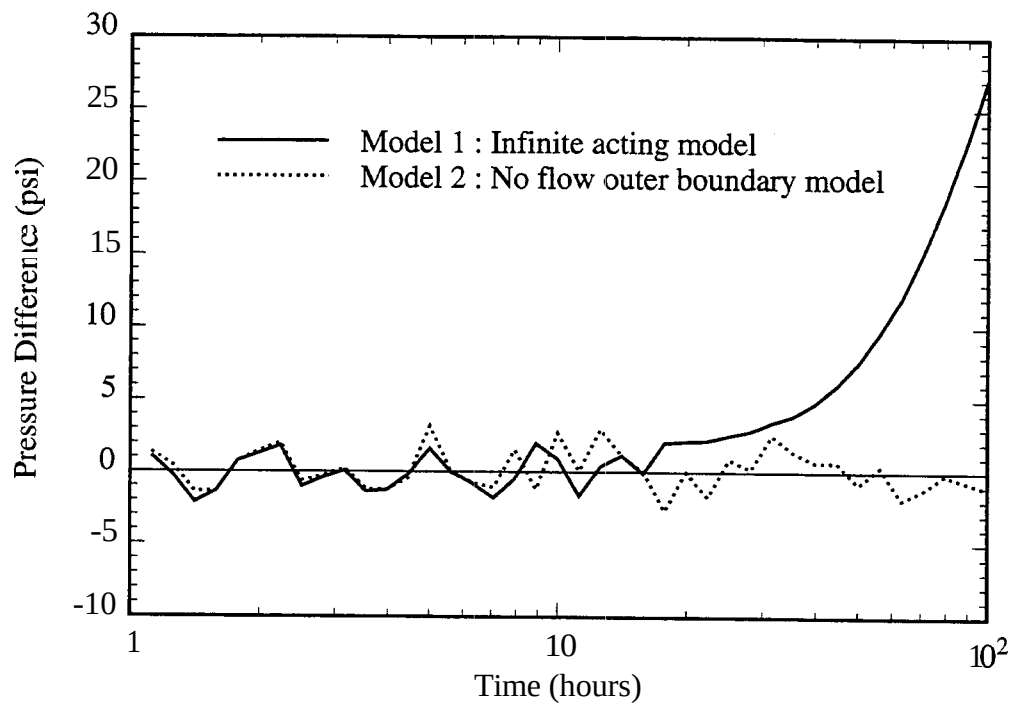


Figure 4.17: Pressure difference between the observed pressure response and the expected pressure response based on the model

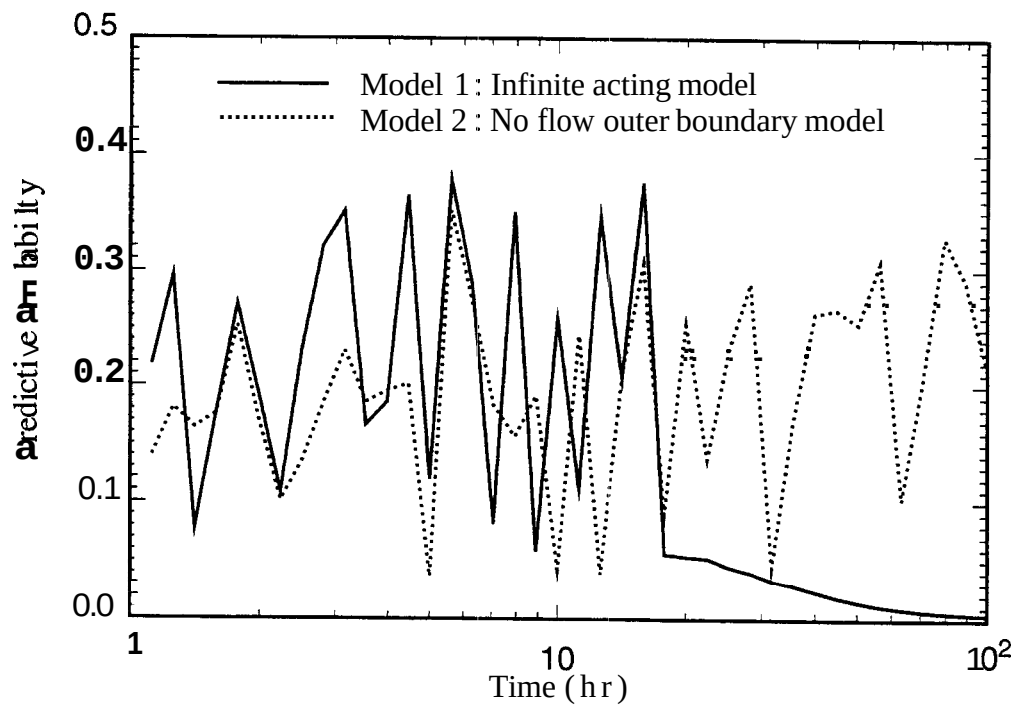


Figure 4.18: Probability associated with the model

Model 2. In the beginning, the estimated values are fluctuating due to the shortage of data. During the period between 3 hours and 20 hours, the estimated values are almost identical and they are almost the same as the true value ($k = 50md$). However, once the boundary effect becomes significant, the estimated value in Model 2 still remains the same but that in Model 1 deviates from the true value. In Fig. 4.20, the relative confidence intervals of the permeability estimate decrease in both models according to the increase of the data points used before the boundary effect becomes significant. Moreover, the relative confidence interval of the permeability estimate in Model 1 is narrower than that in Model 2. This indicates that uncertainty involved in the permeability estimate in Model 1 is lower than that in Model 2. After 20 hours, Model 1 no longer represents the data adequately, and the relative confidence interval of the permeability estimate in Model 1 starts increasing and becomes wider than that in Model 2. This indicates that uncertainty involved in the permeability estimate in Model 1 is higher than that in Model 2. This confidence intervals evaluation is consistent with the result obtained from the sequential predictive probability method.

However, it should be mentioned that the relative confidence interval of the permeability estimate in Model 1 is relatively narrow even at 100 hours. This is because a relatively small number of data is available in which the boundary effect is significant]

The estimated values and the confidence intervals of skin in Model 1 and Model 2 show similar behavior to those for permeability (Fig. 4.21 and Fig. 4.22). This is because in general permeability and skin have a relatively strong correlation.

The wellbore storage constant in Model 1 shows an interesting behavior in Fig. 4.23 and Fig. 4.24. Theoretically, after wellbore storage effects end, the data points contain no information about the wellbore storage constant and therefore have no effect on the estimated value of the wellbore storage constant. However, after the boundary effect becomes significant, Model 1 does not represent the data adequately and, as a result, the model fit to the data becomes worse (the variance becomes larger). Hence, the relative confidence interval of the wellbore storage constant in Model 1 becomes wider after the boundary effect becomes significant.

Fig. 4.26 shows the estimated values of the distance to the boundary. These Values are fluctuating before 20 hours and thereafter they are converging to the true

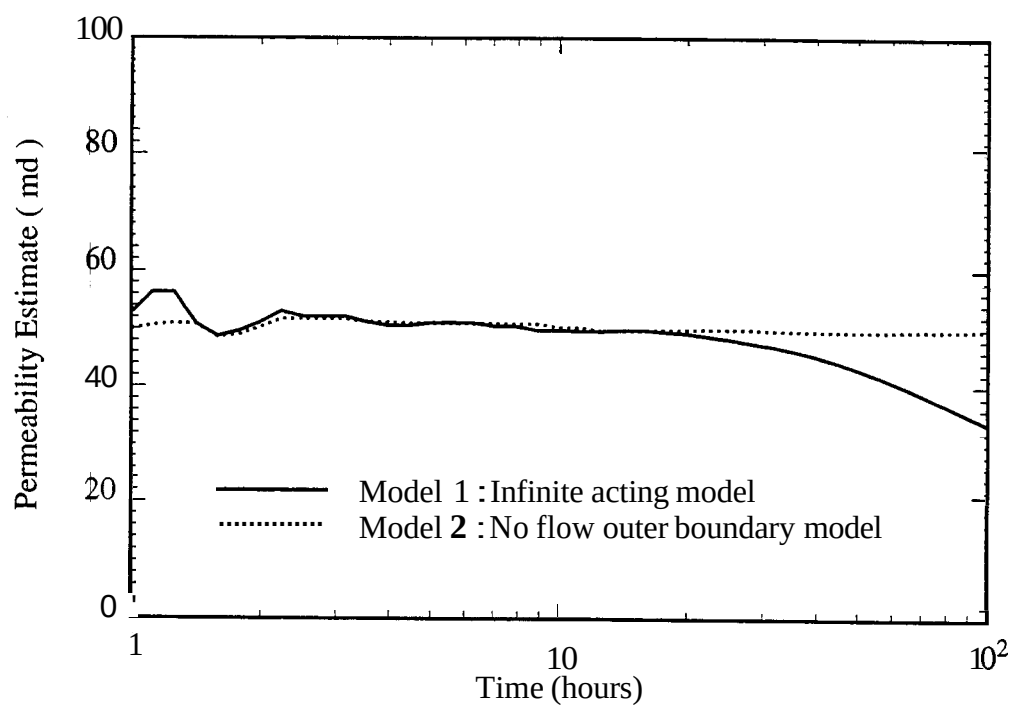


Figure 4.19: Permeability estimate

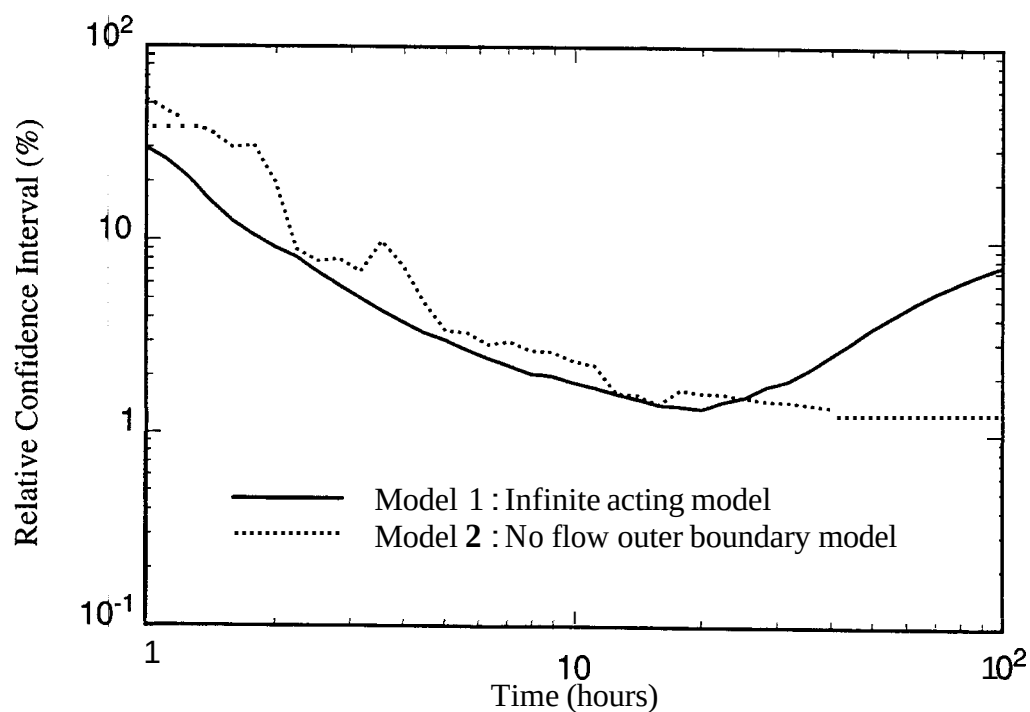


Figure 4.20: Relative confidence interval of permeability

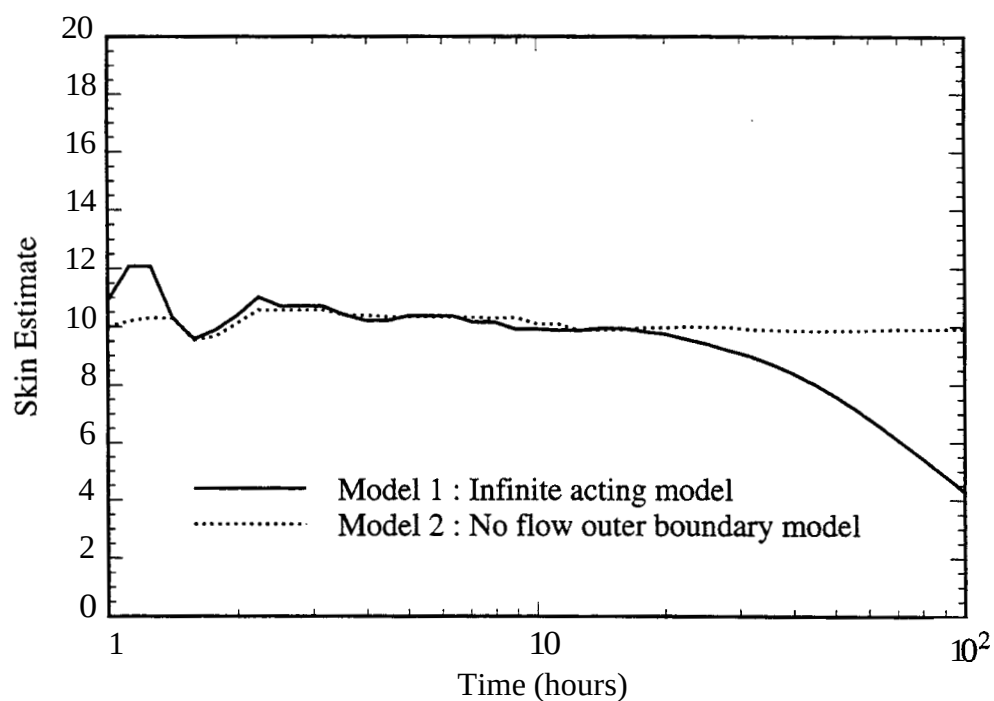
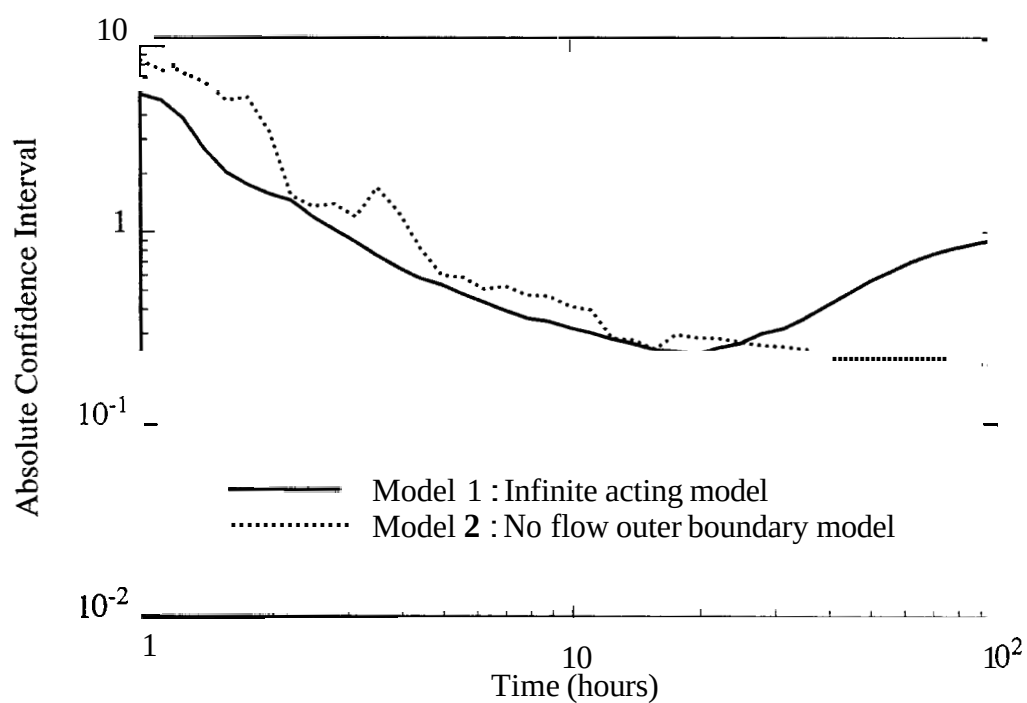
Figure 4.21: **Skin** estimate

Figure 4.22: Absolute confidence interval of skin

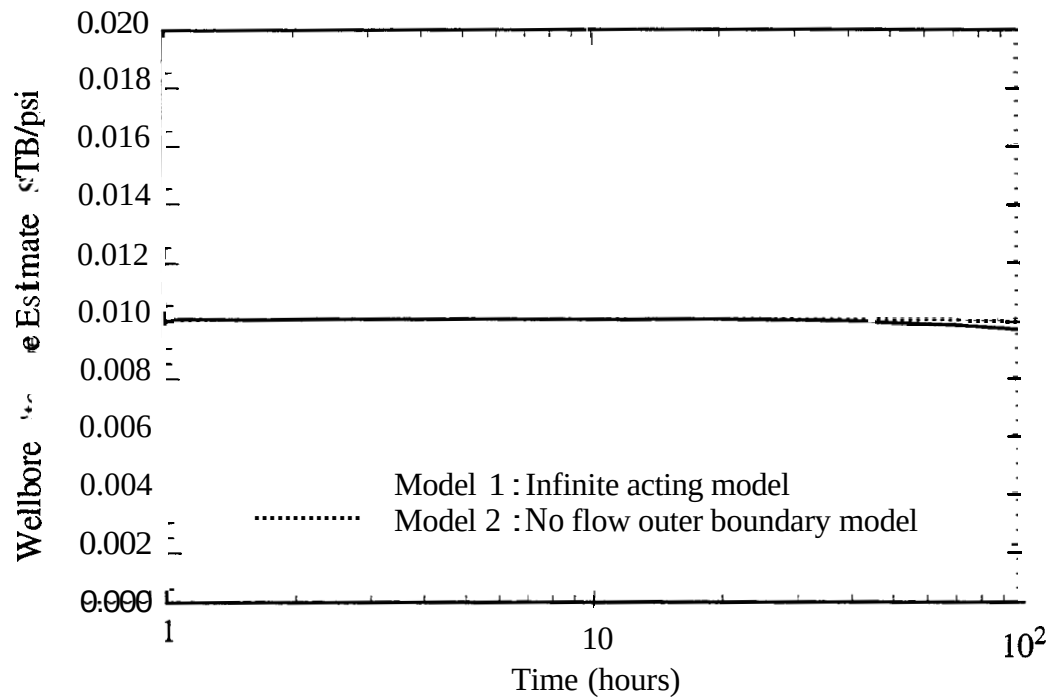


Figure 4.23: Wellbore storage constant estimate

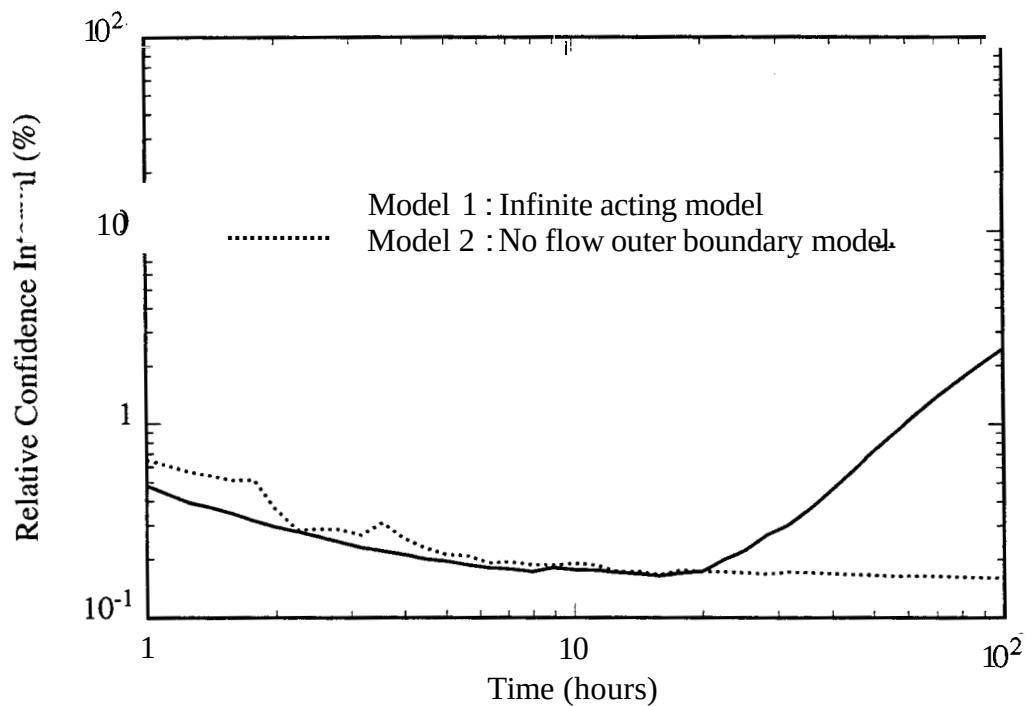


Figure 4.24: Relative confidence interval of wellbore storage constant

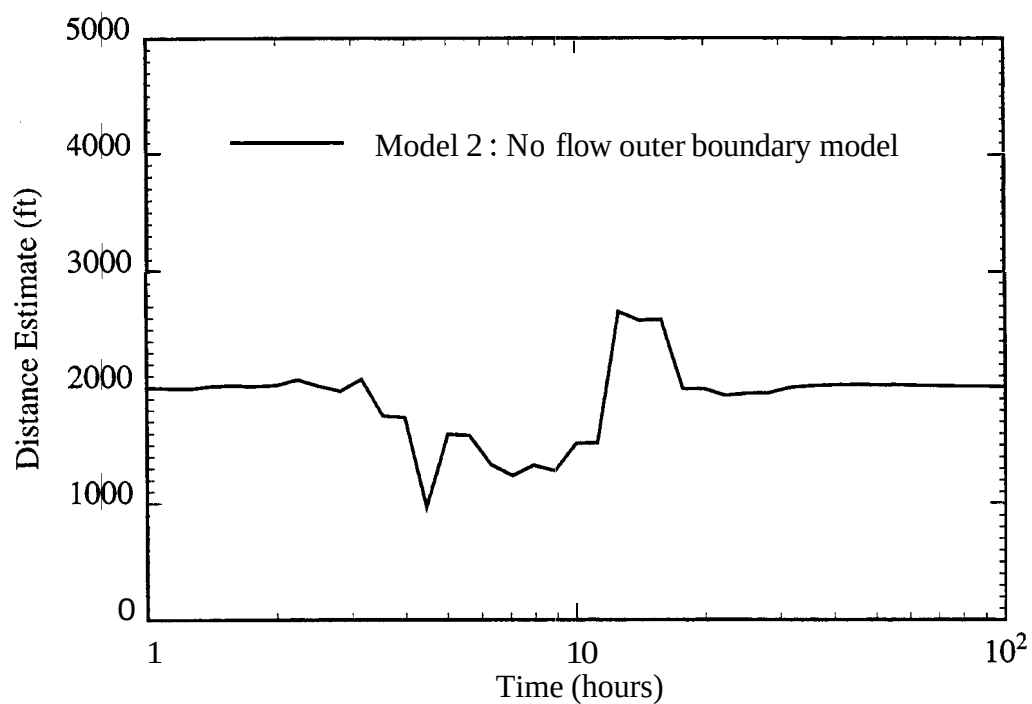


Figure 4.25: Distance to the boundary estimate

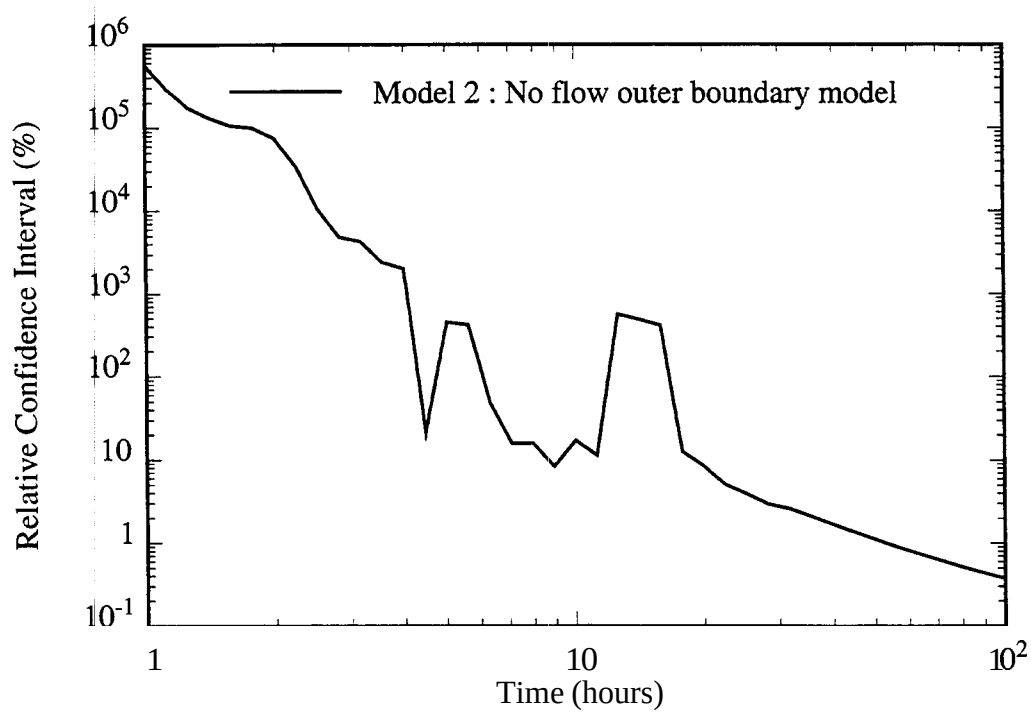


Figure 4.26: Relative confidence interval of distance to the boundary

value ($r_e = 2000\text{ft}$). Comparison between Fig. 4.20 and Fig. 4.26 in Model 2 reveals the basic characteristics of the relative confidence interval of permeability and the distance to the boundary. In Fig. 4.26 the relative confidence interval of the distance to the boundary in Model 2 is extremely wide before the boundary effect becomes significant. This is because the data contain little information about the distance to the boundary before the boundary effect becomes significant. Once the boundary effect becomes significant, the data include valuable information about the distance to the boundary and the relative confidence interval of the distance to the boundary in Model 2 becomes narrower and narrower. On the other hand, the relative confidence interval of the permeability estimate in Model 2 is relatively narrow up to 20 hours in Fig. 4.20, since the data contain valuable information about the permeability. After the boundary effect becomes significant, the reservoir condition approaches pseudosteady state. Under the pseudosteady state condition, the pressure response is influenced not by the permeability but by the reservoir size (the distance to the boundary). Therefore, as the reservoir condition approaches pseudosteady state, information about the permeability within the data becomes smaller and the rate of decrease of the relative confidence interval of permeability in Model 2 becomes slower.

Now it should be mentioned that matches of the models to the data at each step of the sequential procedure may change, since at each step of the sequential procedure the estimated values of the parameters may be different as shown in Fig. 4.191 Fig. 4.21, Fig. 4.23, and Fig. 4.25. Fig. 4.12 shows that Model 1 starts deviating from the data on the pressure derivative plot after around 2 hours, since final matches of Model 1 to the data were calculated using the final estimated values of the parameters, which were evaluated using the whole data (81 data points). However, as shown in Fig. 4.21, Fig. 4.23, and Fig. 4.25, the estimated values of the parameters in Model 1 are close to the true values until the sequential procedure reaches the data point at $x_{67} = 19.9526$ hours. This means that even though final matches of Model 1 to the data, which were calculated using the parameter estimates evaluated with 81 data points, are not appropriate, matches of the model 1 to the data, which were calculated using the parameter estimates evaluated with 67 data points, are appropriate.

Next, the computation cost related with the sequential predictive probability method is discussed. Table 4.6 shows the number of iterations in nonlinear regression at each time step. The total number of iterations required for Model 1 and Model 2 is 149 and 191, respectively. As discussed before, the modified sequential probability method has the potential to reduce the computation cost. This section illustrates how the modified sequential predictive probability method works. Two cases were considered; one in which the data were truncated at $x_{67} = 19.9520$ hours and the other in which the whole data were available. In the former case, the sequential predictive probability method shows that the normalized joint probability associated with Model 1 is 0.9801 in Table 4.5. In the latter case, the sequential predictive probability method shows that the normalized joint probability associated with Model 1 is 0.0000 in Table 4.5.

Firstly, the case where the data were truncated at $x_{67} = 19.9520$ hours is considered. Using the modified sequential predictive probability method, the joint probability for Model 1 is calculated using Eq. 4.71:

$$\begin{aligned} & \frac{Prob(y_1, \dots, y_{41}, y_{42}, \dots, y_{67})}{Prob(y_1, \dots, y_{41})} \\ &= \frac{(\sqrt{2\pi} \cdot 0.9042)^{38} |0.5985 \times 10^{12}|^{1/2}}{(\sqrt{2\pi} \cdot 1.0416)^{64} |0.3259 \times 10^{17}|^{1/2}} \cdot \exp\left[-\frac{26}{2}\right] \\ &= 6.3076 \times 10^{-22} \end{aligned} \quad (4.78)$$

Multiplying Eq. 4.78 by $\Pi_1^{41} = 0.5$ produces the joint probability associated with Model 1 as:

$$\begin{aligned} \Pi_1^{41} \cdot \frac{Prob(y_1, \dots, y_{41}, y_{42}, \dots, y_{67})}{Prob(y_1, \dots, y_{41})} &= 0.5 \cdot 6.3076 \times 10^{-22} \\ &= 3.1538 \times 10^{-22} \end{aligned} \quad (4.79)$$

Similarly, the joint probability associated with Model 2 is 5.1588×10^{-25} . Hence, the normalized joint probability associated with Model 1 is given by:

$$\begin{aligned}
\Pi_1^{67} &= \frac{3.1538 \times 10^{-22}}{3.1538 \times 10^{-22} + 5.1588 \times 10^{-25}} \\
&= 0.9984
\end{aligned} \tag{4.80}$$

This value is close to 0.9801, which is obtained from the sequential predictive probability method. The total number of iterations required for the modified sequential predictive probability method is 7 for Model 1 and 10 for Model 2, respectively.

Secondly, the case where the whole data were used is considered. The joint probability associated with Model 1 is 3.5894×10^{-124} and that with Model 2 is 3.9715×10^{-35} . Hence, the normalized joint probability associated with Model 2 is 1.0000. This value is also the same as that obtained from the sequential predictive probability method. The total number of iterations required for the modified sequential predictive probability method is 11 for Model 1 and 8 for Model 2, respectively, and these numbers are much smaller than those required for the sequential predictive probability method.

Therefore, the modified sequential predictive probability method can recover the results from the sequential predictive probability method, even though the computation cost is greatly reduced. However, the modified sequential predictive probability method does not provide the behavior of the normalized joint probability. In cases where the results obtained from the modified sequential predictive probability method are not conclusive, the sequential predictive probability method is required to investigate the trends of the normalized joint probability.

The findings in this example are concluded as follows:

1. The sequential predictive probability method selects the true model, which is consistent with graphical analysis and confidence intervals evaluation.
2. Before the boundary effect becomes significant, the sequential predictive probability method selects an infinite acting model and then later selects a no flow boundary model. This is consistent with the concept of radius of investigation,
3. There is a time period between the time when the boundary effect becomes significant and the time when this method clearly indicates the existence of the

boundary. This time period is inherent in the uncertainty in discriminating between the effects of errors in the data and the effects of reservoir boundaries.

4. In order to decide whether the boundary exists or not, the procedure should be performed until the end of data.
5. Even in cases where the results are not conclusive, the behavior of the normalized joint probability provides some information about model discrimination.
6. Matches of the models to the data at each step of the sequential procedure may change, since at each step of the sequential procedure the estimated values of the parameters may be different.
7. The modified sequential predictive probability method can produce similar values to the sequential predictive probability method. However, in cases where the results obtained from the modified sequential predictive probability method are not conclusive, the sequential predictive probability method is required to investigate the trends of the normalized joint probability.

Table 4.6: Number of iterations in evaluating the estimated values of the parameters

i	Time (hours)	Model 1	Model 2	i	Time (hours)	Model 1	Model 2
42	1.1220	6	3	62	11.2202	1	8
43	1.2589	6	5	63	12.5893	5	1
44	1.4125	1	5	64	14.1254	1	13
45	1.5849	7	2	65	15.8489	5	7
46	1.7783	6	4	66	17.7828	1	7
47	1.9953	5	6	67	19.9526	1	7
48	2.2387	5	5	68	22.3872	1	1
49	2.5119	5	5	69	25.1189	5	5
50	2.8184	6	1	70	28.1838	1	3
51	3.1623	1	1	71	31.6228	5	1
52	3.5481	1	1	72	35.4813	4	4
53	3.9811	6	22	73	39.8107	1	3
54	4.4668	6	5	74	44.6684	1	3
55	5.0119	1	6	75	50.1187	5	3
56	5.6234	5	7	76	56.2341	5	5
57	6.3096	1	1	77	63.0957	5	1
58	7.0795	1	9	78	70.7946	5	5
59	7.9433	6	3	79	79.4328	5	5
60	8.9125	1	3	80	89.1251	5	5
61	10.0000	6	5	81	100.0000	5	5

Table 4.7: Modified sequential predictive probability method

Model	m	k	$n + k$	s	$ H $
Model 1	3	0	41	0.9042	0.5985×10^{12}
		26	67	1.0416	0.3259×10^{17}
		40	81	15.8304	0.1882×10^{19}
Model 2	4	0	41	0.9179	0.2300×10^{-1}
		26	67	0.9793	0.3937×10^{13}
		40	81	0.9797	0.5998×10^{16}

Chapter 5

Results and Discussion

This chapter demonstrates the utility of the sequential predictive probability method for model discrimination in well test analysis. Application to simulated and actual field well test data is discussed.

Section 5.1 discusses various factors affecting the use of the sequential predictive probability method. In Sections 4.2 and 4.3, the theoretical features and several practical considerations are discussed, however, some of them need further investigation for effective use of the sequential predictive probability method in model discrimination. Firstly, several methods of selecting the time steps are discussed. The advantage of the chronological selection is demonstrated. Secondly, the effect of the starting point is discussed. It was found that the starting point should be selected in the vicinity of the wellbore storage hump in the pressure derivative plot to obtain reliable results. Thirdly, the effect of the number of data points is discussed. Finally, the effect of the magnitude of errors is discussed. It was found that the sequential predictive probability method can discriminate between possible reservoir models even in cases where the data are noisy.

Section 5.2 presents the advantages of the sequential predictive probability method over confidence interval analysis and graphical analysis. The sequential predictive probability method can discriminate between possible reservoir models even in cases where the other two methods fail.

Section 5.3 discusses application of the sequential predictive probability method

to simulated well test data. Several possible situations were examined.

Section 5.4 discusses application of the sequential predictive probability method to actual field well test data.

5.1 Factors Affecting the Sequential Predictive Probability Method

In this section, factors affecting the sequential predictive probability method are discussed. These factors need to be investigated to determine the adequacy of the sequential predictive probability method as a robust method for model discrimination. Several ways of selecting the time steps are discussed. The effects of the starting point, the number of data points and the magnitude of the errors are also discussed.

For illustration purposes in this section, the same pressure data were used as in Section 4.3. These data were generated using a no flow outer boundary model: Random normal errors with zero mean and a variance of 1.0 psi^2 were added to the pressure data except in Section 5.1.4. The starting time of pseudosteady state is around 20 hours. The infinite acting model and the no flow outer boundary model were selected as possible candidate models. For simplicity, the infinite acting model is named Model 1 and the no flow outer boundary model is named Model 2. Model 1 has three parameters, which are k , S , and C , while Model 2 has four parameters, which are k , S , C , and r_e .

5.1.1 Effect of the Time Step Sequences

As discussed in Section 4.3.3, the chronological (forward) selection of the time steps is used as the standard procedure. The chronological selection always uses the subsequent data point as the next time step, starting with the data points at early time. The effectiveness of the chronological (forward) selection is demonstrated by comparing with other possible time selections using the simulated data. Two other time selections were tried as alternatives: one is the backward selection and the other is the alternating points selection. The backward selection uses the previous data point

as the next time step, starting with the data points at late time. The alternating points selection uses an alternating data point as the next time step, starting with a sparse subset of the total time sequence. For example, suppose that 81 data points $(y_1, \dots, y_{81})$ are available in chronological order. The chronological (forward) selection selects y_{42} as the next time step starting with $y_1, \dots, y_{41}$. The backward selection selects y_{40} as the next time step starting with $y_{81}, \dots, y_{41}$. The alternating points selection selects an even point y_2 as the next time step when the starting sequence is made up of odd points $y_1, y_3, \dots, y_{81}$.

Two cases were considered to compare these three different methods of selecting the time steps. One is the case where the boundary effect is not significant (infinite acting state) and the other is the case where the boundary effect is significant (pseudosteady state).

In the case where the boundary effect is not significant, the total number of data points is 61 and the time period is from 0.01 hour to 10 hours. This case is called Case 1. In the case where the boundary effect is significant, the total number of data points is 81 and the time period is from 0.01 hour to 100 hours. This case is called Case 2. Matches of the models to the data and the corresponding normalized joint probabilities are shown in Fig. 5.1, Fig. 5.2, Fig. 5.3, Fig. 5.4, Fig. 5.5, and Fig. 5.6. Matches of the models to the data were calculated using the final estimated values of the parameters, which were evaluated using all of the data points.

Fig. 5.1 shows the results of Case 1 obtained by the chronological selection. The procedure starts at 1.0 hour with 41 data points and thereafter the data points are selected chronologically up to 10 hours. Fig. 5.2 shows the results of Case 2 obtained by the chronological selection. The procedure starts at 1.0 hour with 41 data points and thereafter the data points are selected chronologically up to 100 hours. Fig. 5.3 shows the results of Case 1 obtained by the backward selection. The procedure starts at 3.16 hours with 31 data points and thereafter the data points are selected backwards up to 0.01 hour. Fig. 5.4 shows the results of Case 2 obtained by the backward selection. The procedure starts at 1.0 hour with 41 data points and thereafter the data points are selected backwards up to 0.01 hour. Fig. 5.5 shows the results of Case 1 obtained by the alternating points selection. The procedure starts at 0.0112 hour

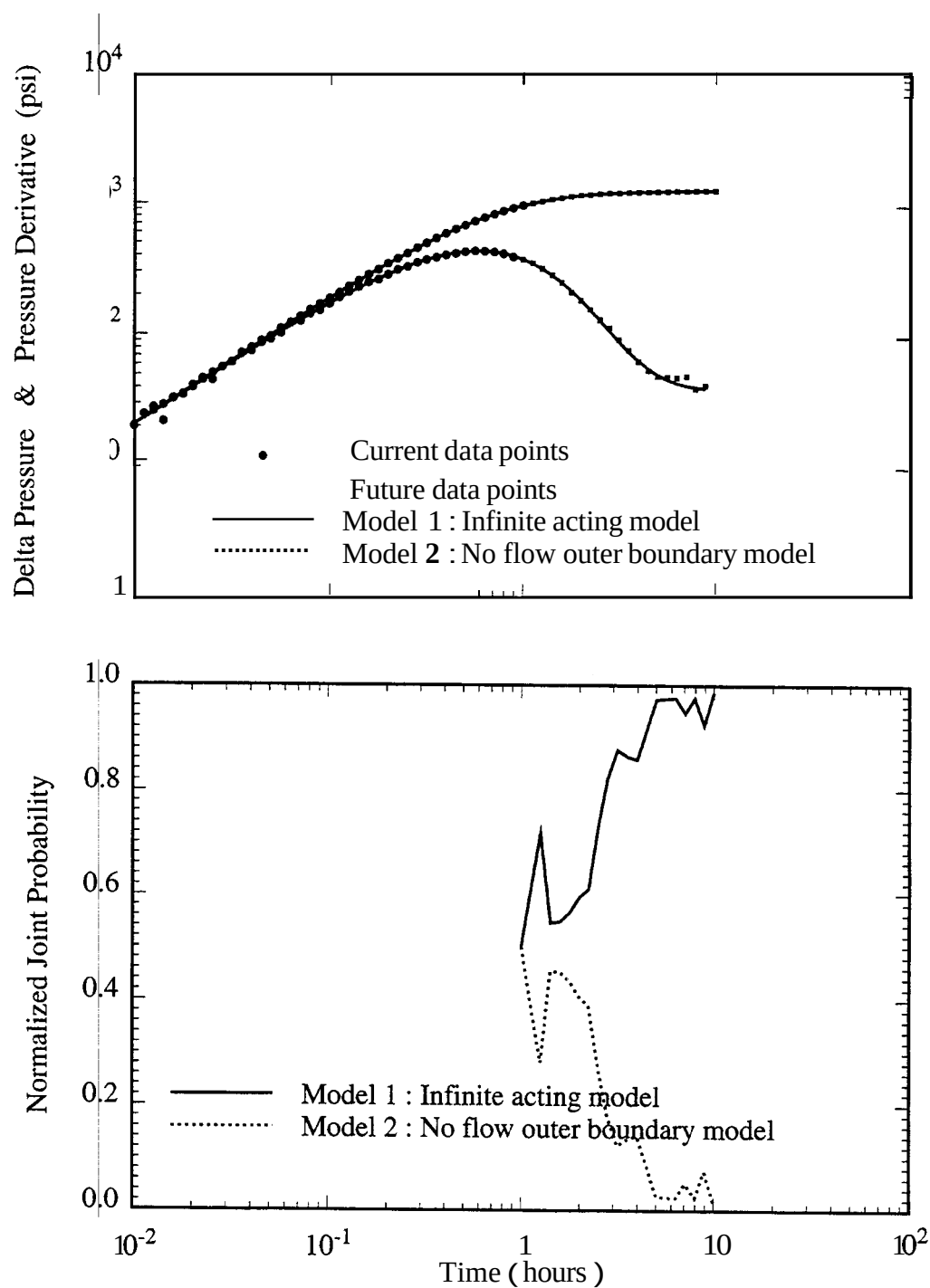


Figure 5.1: Chronological (forward) selection: matches to the 61 data points (upper) and the corresponding normalized joint probability (lower)

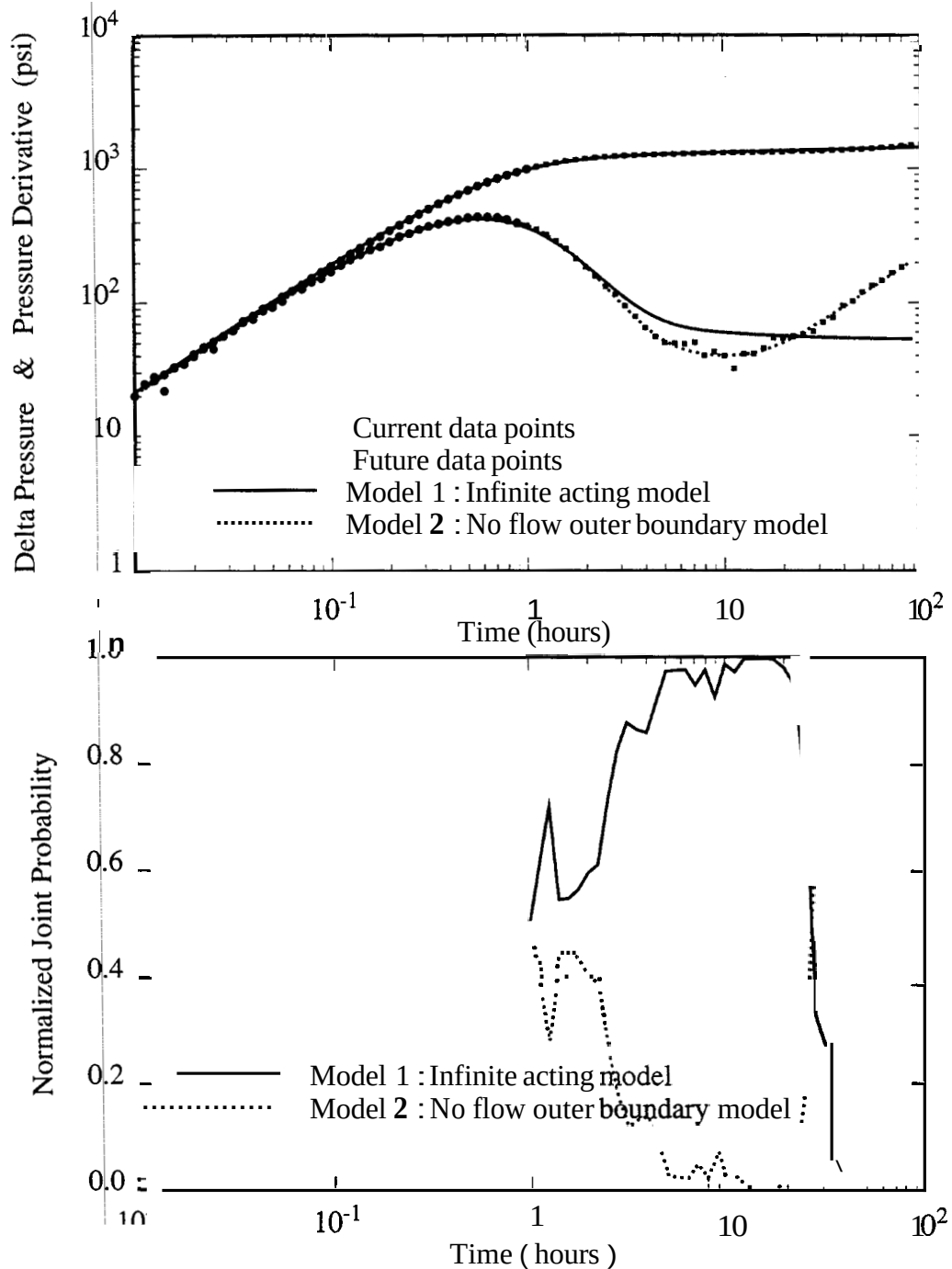


Figure 5.2: Chronological (forward) selection: matches to the 81 data points (upper) and the corresponding normalized joint probability (lower)

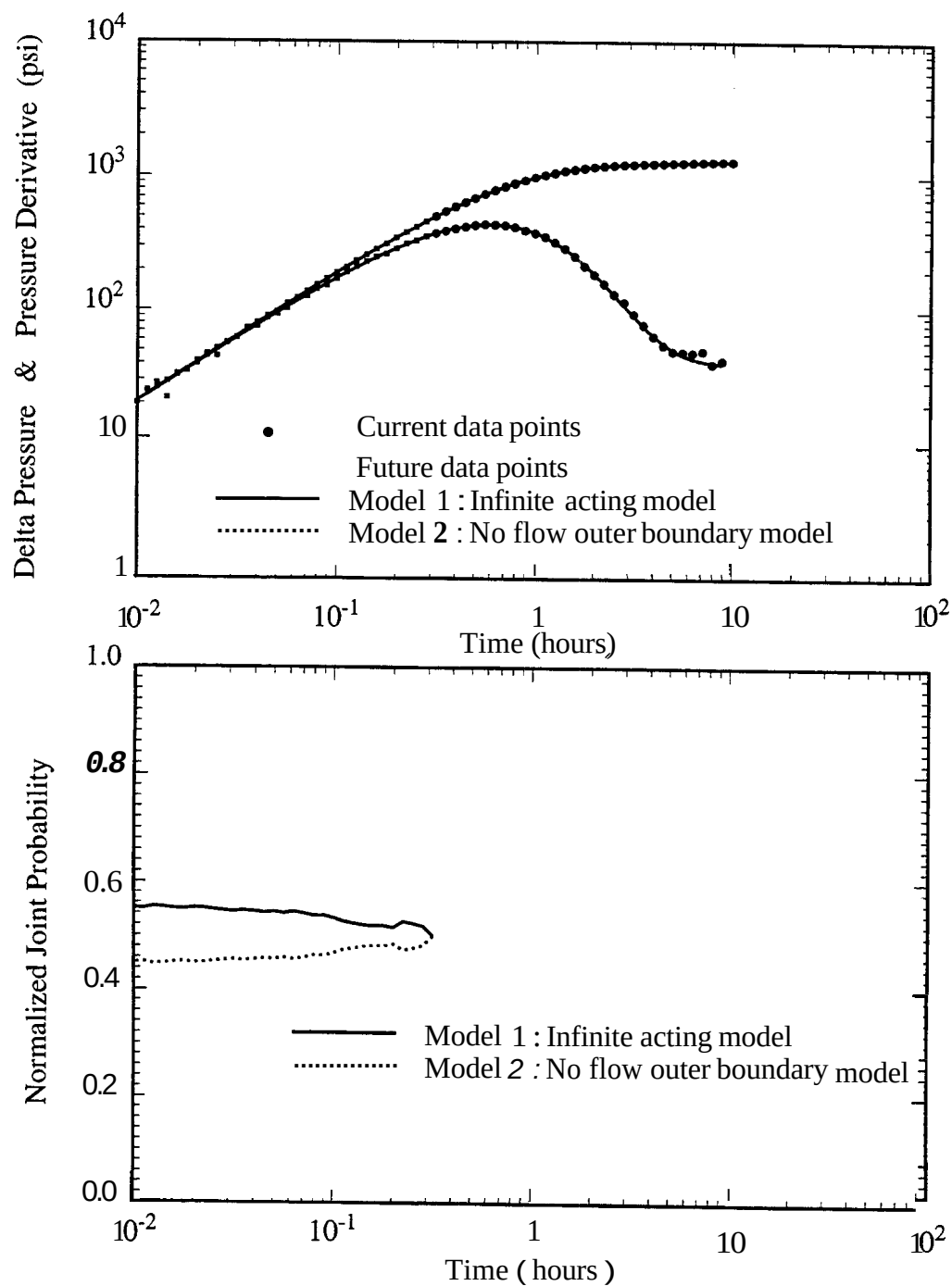


Figure 5.3: Backward selection: matches to the 61 data points (upper) and the corresponding normalized joint probability (lower)

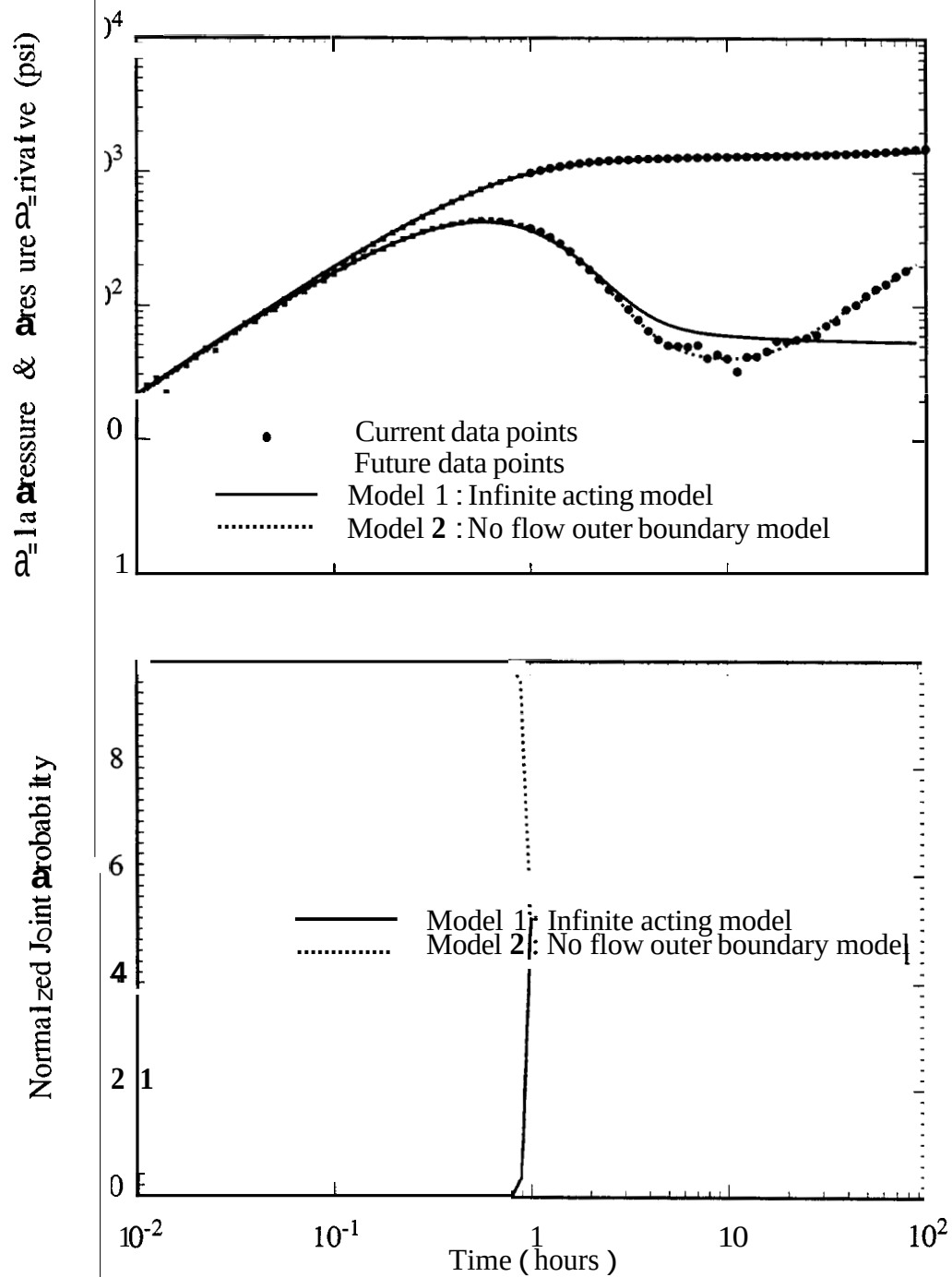


Figure 5.4 Backward selection: matches to the 81 data points (upper) and the corresponding normalized joint probability (lower)

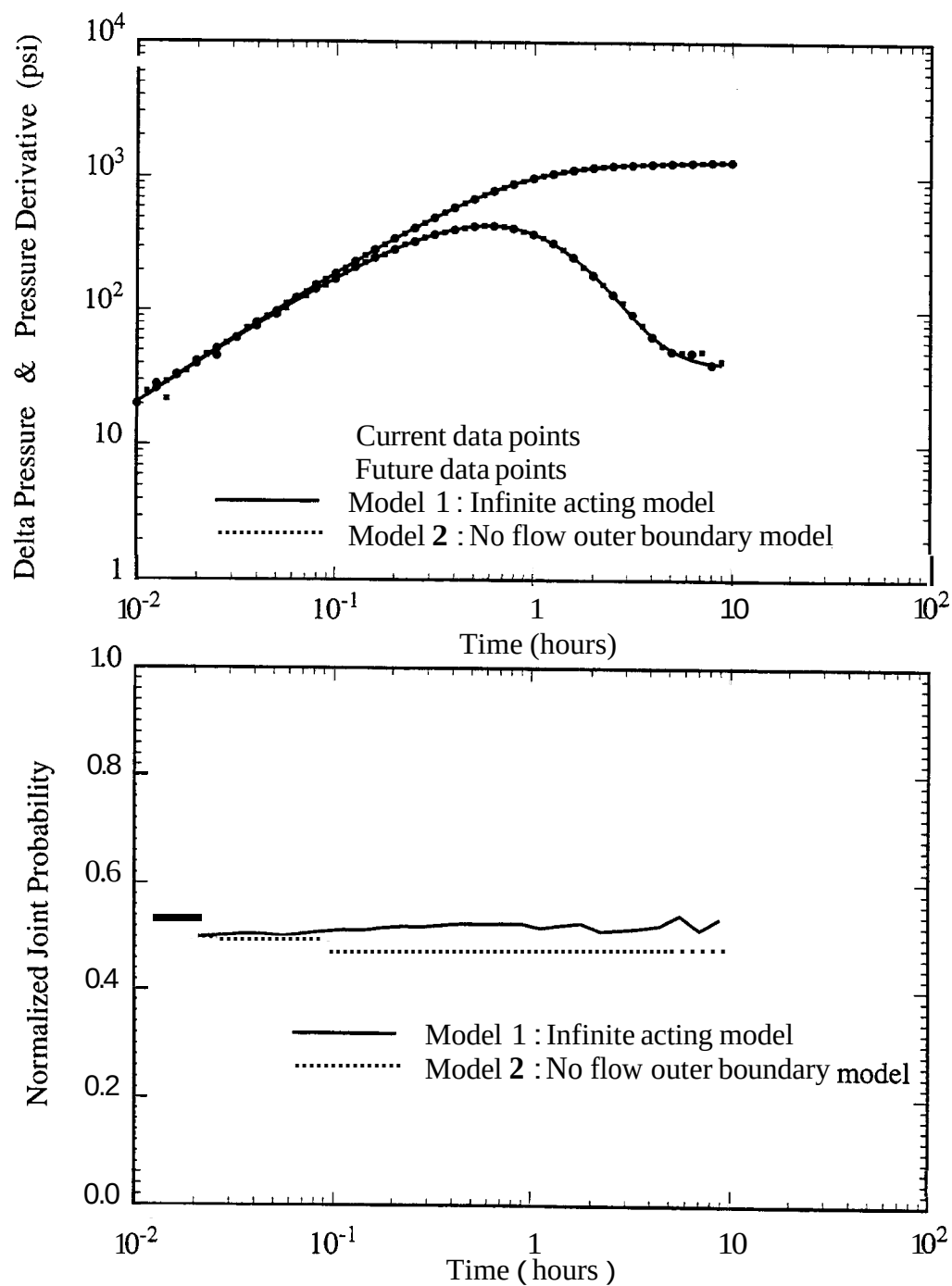


Figure 5.5: Alternating points selection: matches to the 61 data points (upper) and the corresponding normalized joint probability (lower)

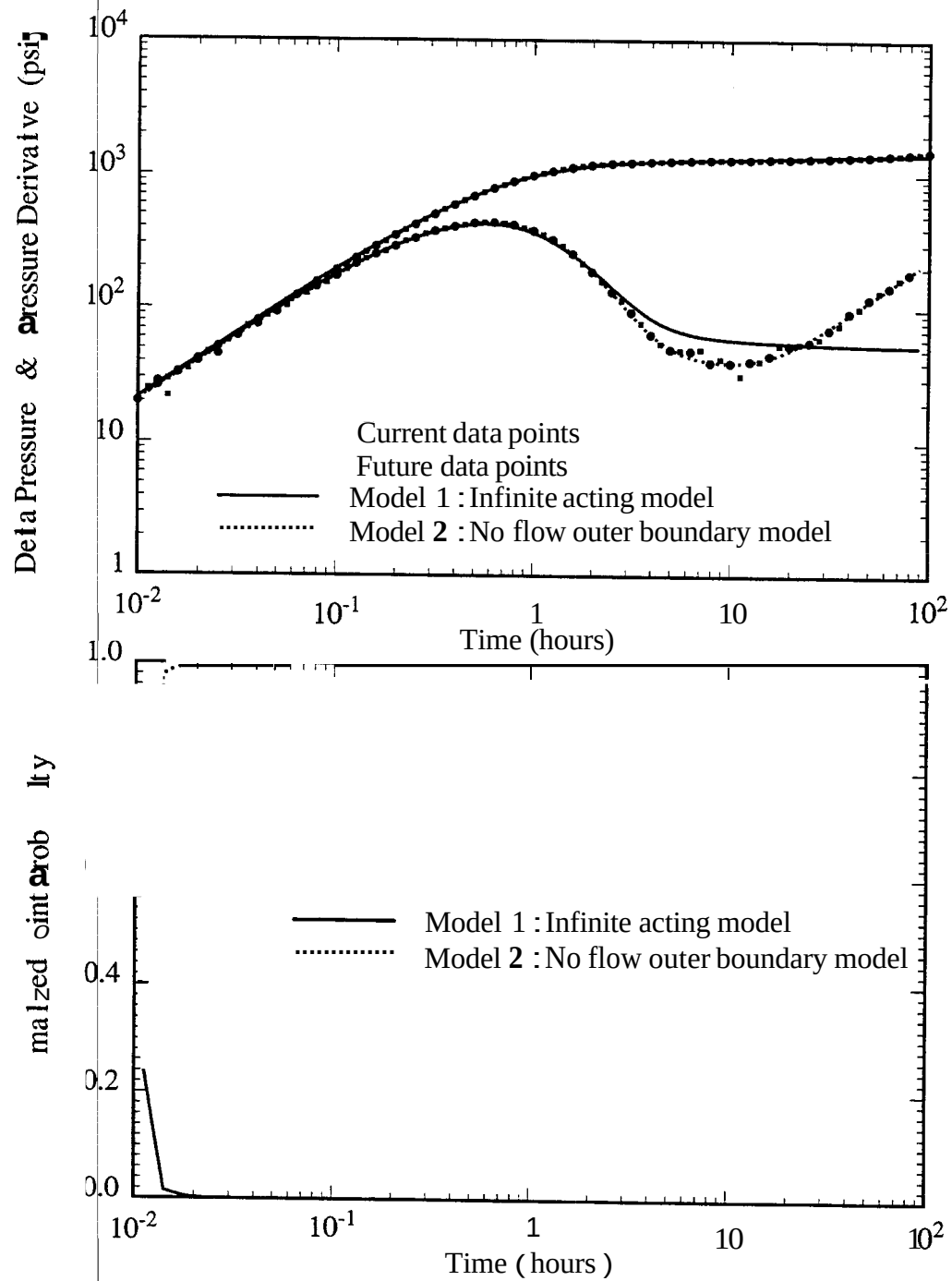


Figure 5.6: Alternating points selection: matches to the 81 data points (upper) and the corresponding normalized joint probability (lower)

with 31 data points and thereafter the data points are selected alternatively up to 8.91 hours. Fig. 5.6 shows the results of Case 2 obtained by the alternating points selection. The procedure starts at 0.0112 hour with 41 data points and thereafter the data points are selected alternatively up to 89.13 hours.

Fig. 5.2, Fig. 5.4, and Fig. 5.6 show that for cases where the boundary effect is significant all three methods select Model 2. While only a few steps are required to select Model 2 for the backward method and the alternating points method, the chronological method needs a larger number of steps. Hence, in cases where the boundary effect is significant, the chronological selection is less effective than the backward selection and the alternating points selection.

On the other hand, from Fig. 5.1, Fig. 5.3, and Fig. 5.5, in cases where the boundary effect is not significant, only the chronological method selects Model 1 while the other methods do not clearly select either Model 1 or Model 2. Hence, in the boundary effect is not significant, both the backward method and the points method produce inconclusive results.

to make the sequential predictive probability method reliable for both chronological selection is most effective. Hence, the chronological selection is a standard method of selecting the time steps.

Effect of the Starting Point

As indicated in Section 4.3.2, the starting point is selected just after the wellbore storage effect in the pressure derivative plot. This section examines how a change of starting point affects the results.

Two cases were considered. One is the case where the boundary effect is not significant (infinite acting state) and the other is the case where the boundary effect is significant (pseudosteady state).

In the case where the boundary effect is not significant, the total number of data points is 61 ($y_1, \dots, y_{61}$) and the time period is from 0.01 hour to 10 hours. This case is called Case 1. In the case where the boundary effect is significant, the total number of data points is 81 ($y_1, \dots, y_{81}$) and the time period is from 0.01 hour to 100 hours. This case is called Case 2.

In Case 1, the starting point was changed from 1 hour (y_{41}) to 8.9125 hours (y_{60}). The final point is always 10 hours (y_{61}). As the starting time becomes later, the number of sequential steps becomes smaller. Fig. 5.7 (upper) shows the final matches to the data, which were calculated using the final estimated values of the parameters, and Fig. 5.7 (lower) shows the effect of the starting point on the final normalized joint probability. Even though the starting point is changed from 1 hour (y_{41}) to 2.8184 hours (y_{50}), the final normalized joint probability does not change. This indicates that small changes in the starting point have small effect on the sequential predictive probability method. Thereafter, the effect of the starting point becomes more significant and the normalized joint probability associated with Model 1 approaches 0.5. This indicates that large changes in the starting point have large effect on the sequential predictive probability method. This is because there are not enough data available to support Model 1 as the starting point of the procedure approaches the final data point. As discussed in Section 4.3.8, the joint probability is used to confirm the adequacy of model fitting over all the time range. In the case where the starting point is 1 hour (y_{41}), 20 data points ($y_{42}, \dots, y_{61}$) are used to calculate the joint probability. On the other hand, in the case where the starting point is 8.9125 hours (y_{60}), only 1 data point (y_{61}) is used to calculate the joint probability. Therefore, in the case where the boundary effect is not significant, the results are insensitive to small changes in the starting point but are eventually sensitive to large changes.

In Case 2, the starting point was changed from 1 hour (y_{41}) to 89.125 hours (y_{80}). The final point is always 100 hours (y_{81}). Fig. 5.8 (upper) shows the final matches to the data, which were calculated using the final estimated values of the parameters, and Fig. 5.8 (lower) shows the effect of the starting point on the final normalized joint probabilities. In Case 2, changes of the starting point have little effect on the final normalized joint probability. The normalized joint probabilities associated with Model 2 are one except at the starting points of y_{79} and y_{80} . The normalized joint probabilities associated with Model 2 at the starting point of y_{79} and y_{80} are 0.99999 and 0.99966, respectively, and they are also close to one. This is because each data point at late time implies significant information about the boundary and, as a result, a small number of data points can confirm the adequacy of Model 2. Therefore, in

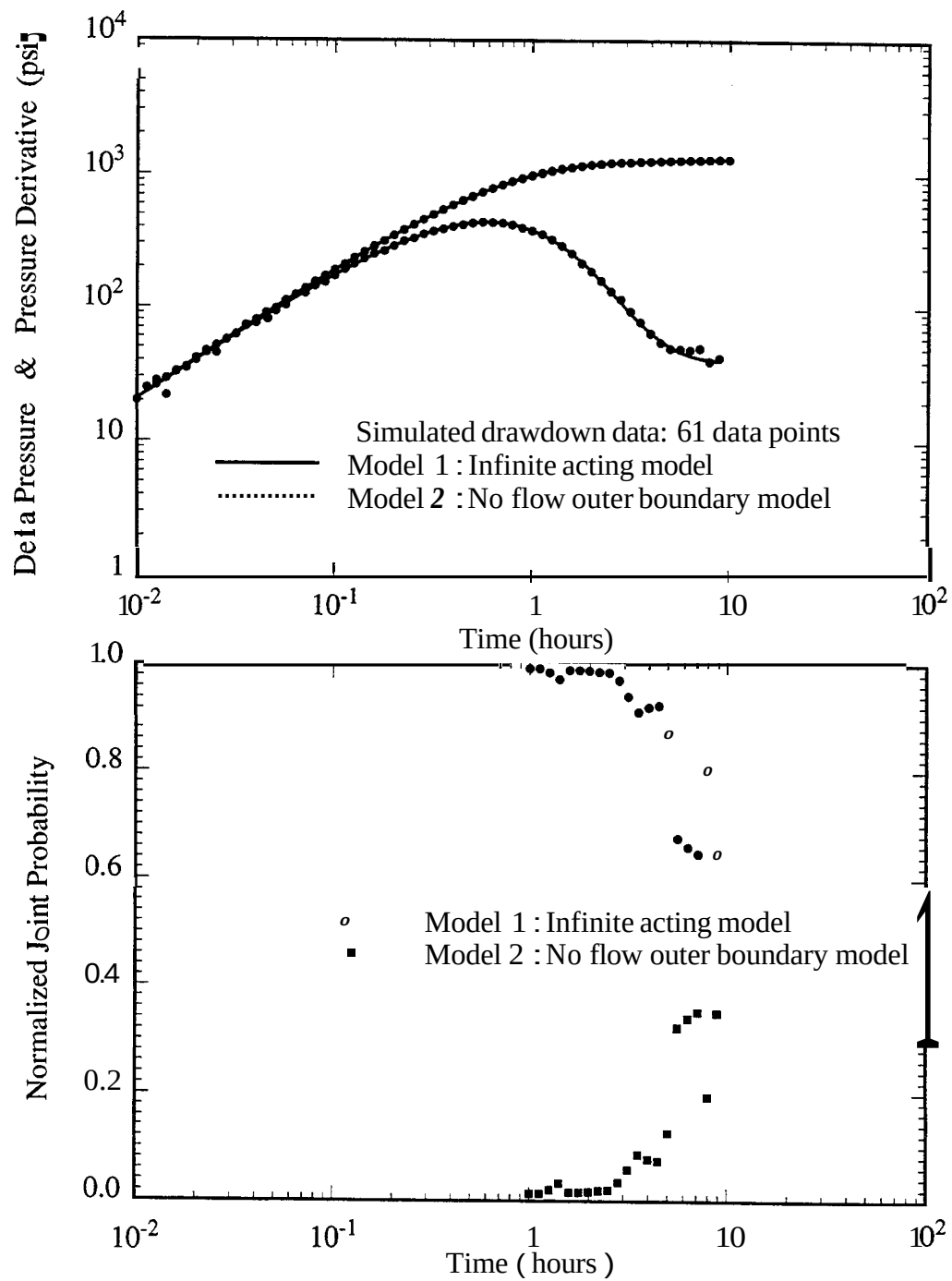


Figure 5.7: Effect of the starting point: matches to the 61 data points (upper) and the effect of the starting point on the normalized joint probability (lower)

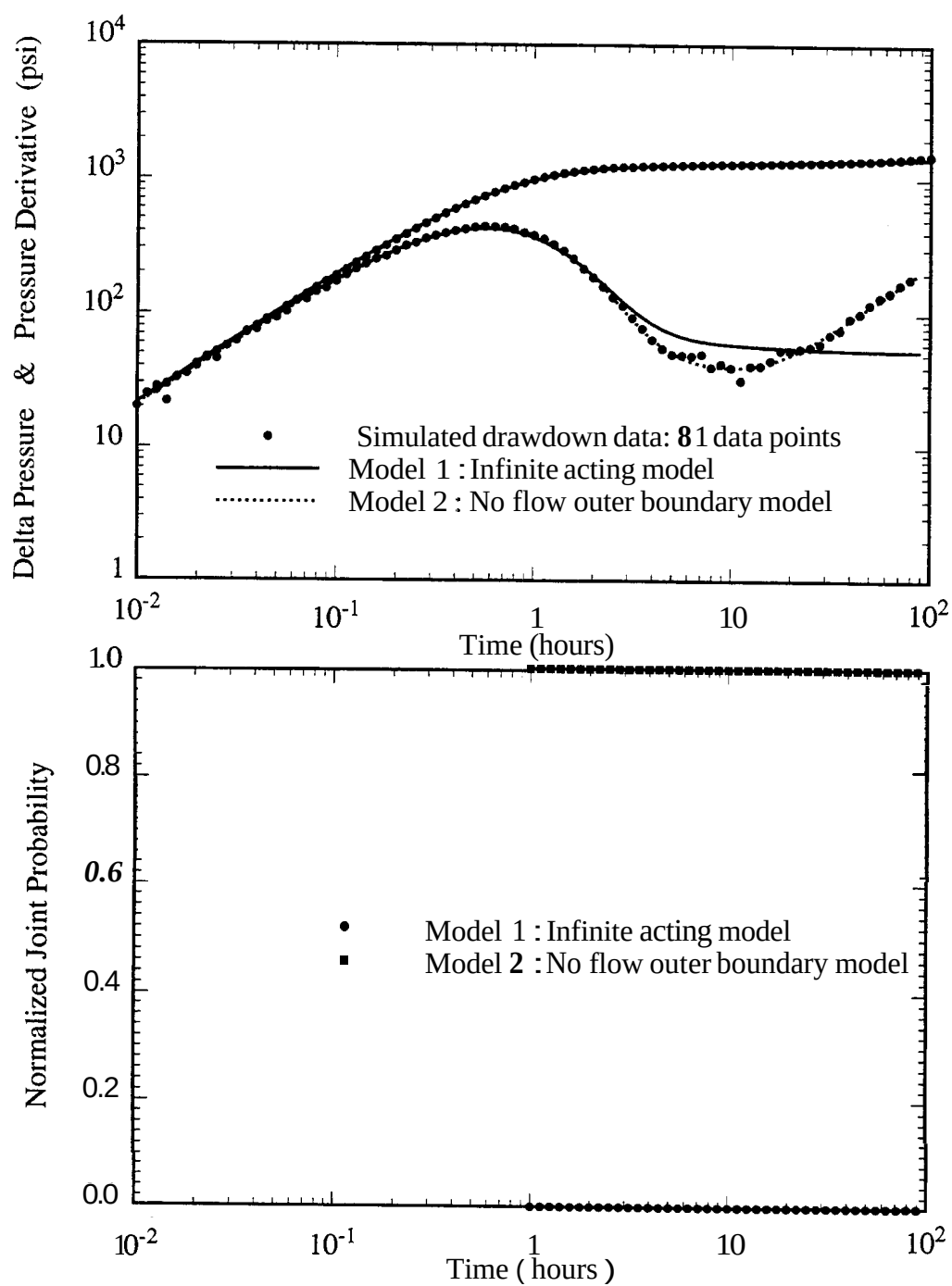


Figure 5.8: Effect of the starting point: matches to the 81 data points (upper) and the effect of the starting point on the final normalized joint probability (lower)

the case where the boundary effect is significant, the results are insensitive even to large changes to the starting point.

In order to make the sequential predictive probability method reliable for all cases, the starting point should be selected in the vicinity of the wellbore storage hump in the pressure derivative plot to retain enough data to confirm the adequacy of model fitting over the entire time range. However, it is not necessary to select the exact peak of the hump.

5.1.3 Effect of the Number of Data Points

This section examines the effect of the number of data points on the results obtained by the sequential predictive probability method. It is determined whether the sequential predictive probability method can discriminate effectively between possible reservoir models even in cases where the number of data points is small.

Three cases were considered. The case where the total number of data points is 21 is called Case 1. The case where the total number of data points is 41 is called Case 2. The case where the total number of data points is 81 is called Case 3. In all cases, the time period is from 0.01 hour to 100 hours.

Matches of the models to the data and the corresponding normalized joint probabilities are shown in Fig. 5.9, Fig. 5.10 and Fig. 5.11. Matches of the models to the data were calculated using the final estimated values of the parameters, which were evaluated using all of the data points. In all cases, the behavior of the normalized joint probabilities is similar. The sequential predictive probability method selects Model 1 before the boundary effect becomes significant and thereafter Model 2. All cases contain data points at late time when the boundary effect becomes significant. However, the final normalized joint probability associated with Model 2 is one in either Case 2 or Case 3 but is only 0.81 in Case 1, since in Case 1 only five data points are available in the period when the boundary effect becomes significant and these are insufficient to confirm the existence of the boundary completely. In addition, comparison of Case 2 and Case 3 shows that the time to select Model 2 in Case 2 is later than in Case 3. This is because Case 2 has a smaller number of data points in which the boundary effect is significant.

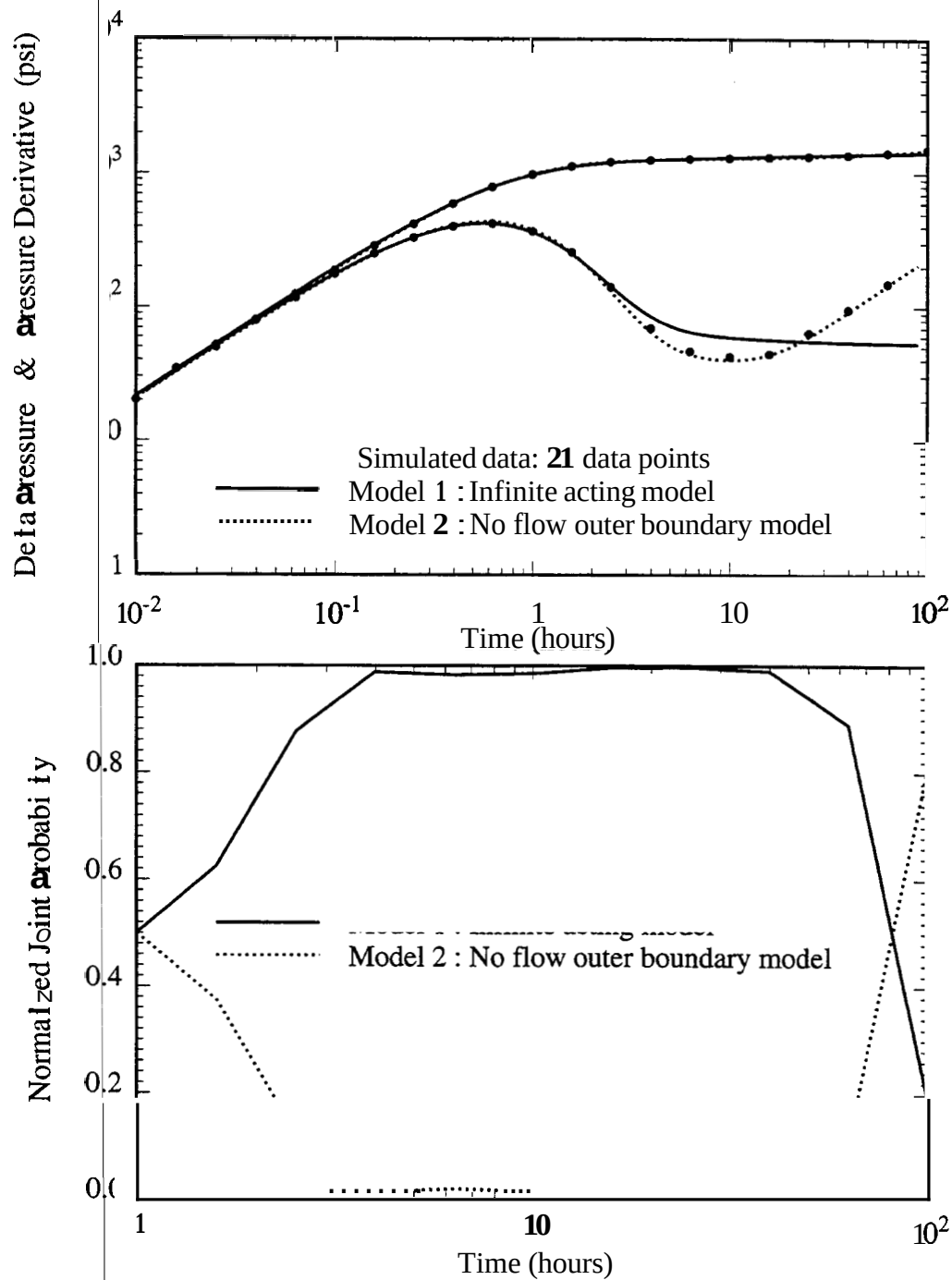


Figure 5.9 Pressure data of 21 data points: matches to the data (upper) and the corresponding normalized joint probability (lower)

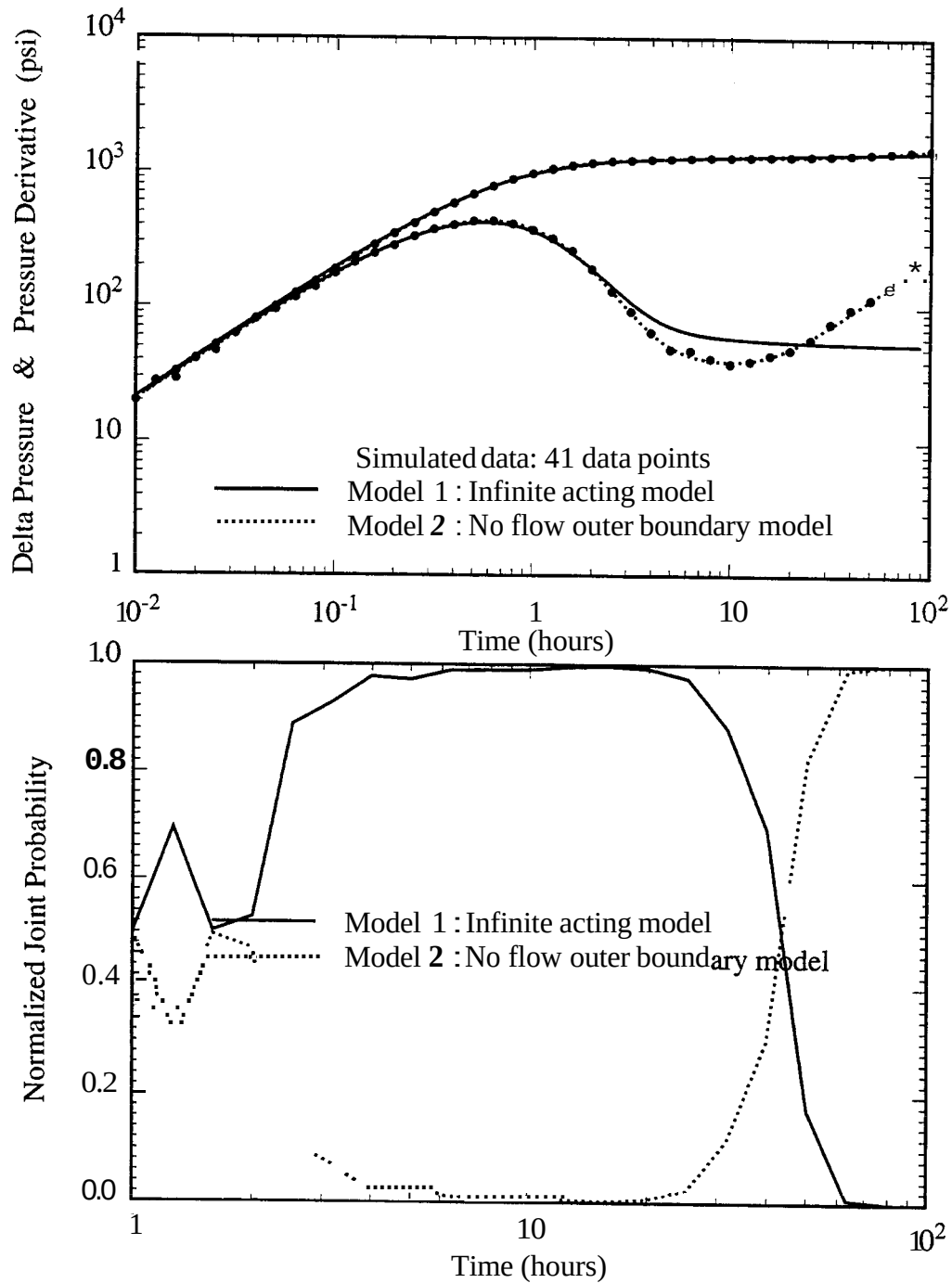


Figure 5.10: Pressure data of **41** data points: matches to the data (upper) and the corresponding normalized joint probability (lower)

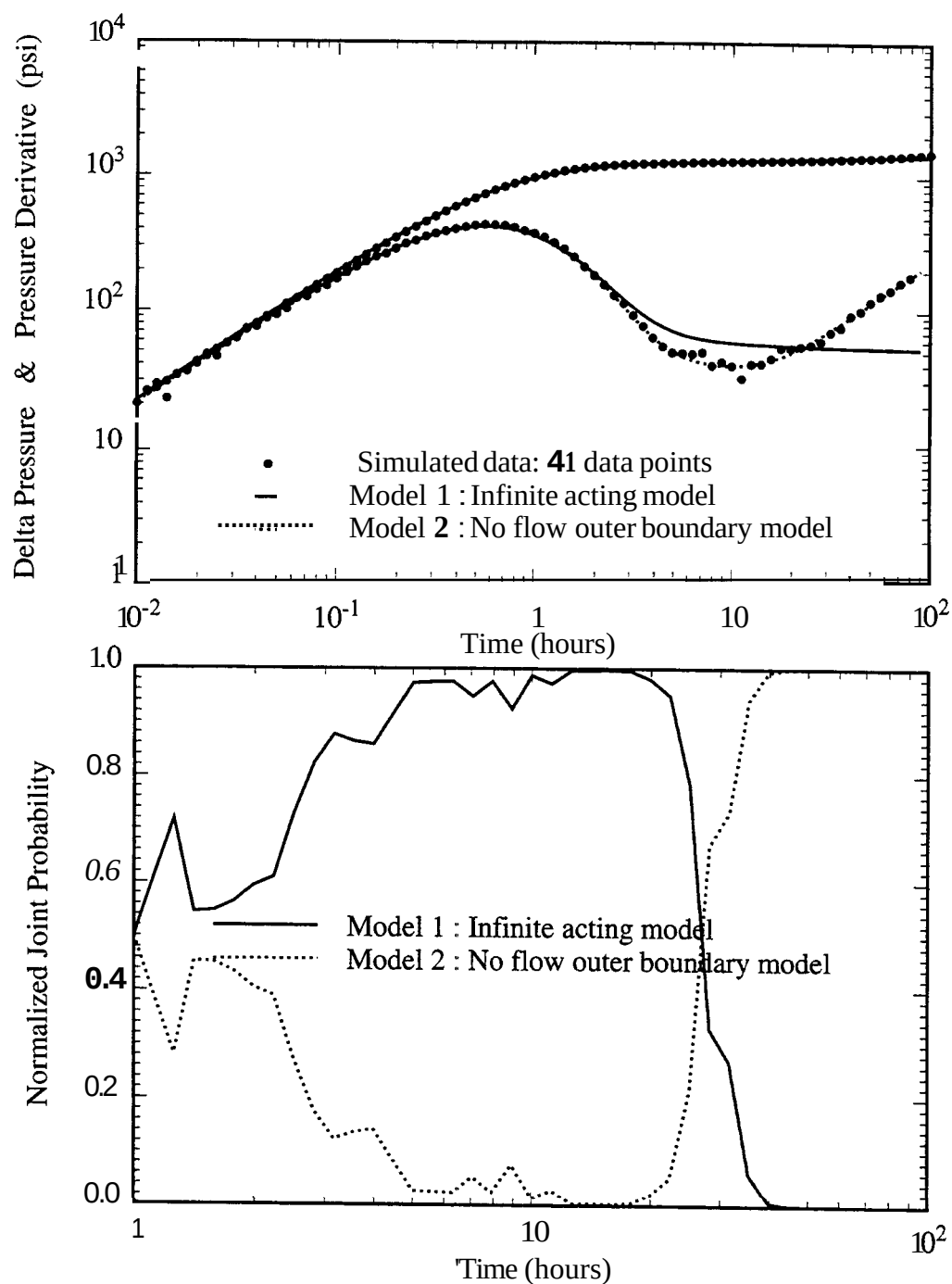


Figure 5.11: Pressure data of 81 data points: matches to the data (upper) and the corresponding normalized joint probability (lower)

Therefore, as long as the data contain data points at times when the boundary effect becomes significant, the sequential predictive probability method suggests the existence of the boundary. However, a sufficient number of data points is necessary to confirm the existence of the boundary.

5.1.4 Effect of the Magnitude of Errors

In this section, the effect of the magnitude of errors is investigated. It is determined whether the sequential predictive probability method can discriminate effectively between possible reservoir models even in cases where the data are noisy.

Three cases were considered. In each case, a random number generator was used to simulate random normal errors with zero mean. The difference is the variance. The case where the variance is 1.0 psi^2 is called Case 1. The case where the variance is 4.0 psi^2 is called Case 2. The case where the variance is 9.0 psi^2 is called Case 3.

The total number of data points is 81 and the time period is from 0.01 hour to 100 hours in each case. In each case, the corresponding random normal errors were added to the pressure data.

Matches of the models to the data and the corresponding normalized joint probabilities are shown in Fig. 5.12, Fig. 5.13, and Fig. 5.14. Matches of the models to the data were calculated using the final estimated values of the parameters, which were evaluated using all of the data points. In each case, the final normalized joint probability associated with Model 2 is one. Hence, the sequential predictive probability method selects Model 2 as the correct model at the final stage. However, the time to select Model 2 is different between the cases. In Case 1, the normalized joint probability associated with Model 2 reaches one after around 40 hours. In Case 2 the normalized joint probability reaches one after around 50 hours and in Case 3 after around 60 hours. The starting time of pseudosteady state is around 20 hours. Therefore, as the magnitude of errors increases, the time period between the time when the boundary effect becomes significant and the time when the sequential predictive probability method clearly indicates the existence of the boundary becomes longer. This is caused by the uncertainty in discriminating between the effects of errors in the data and the effects of reservoir boundaries. Thus, even in cases where the data

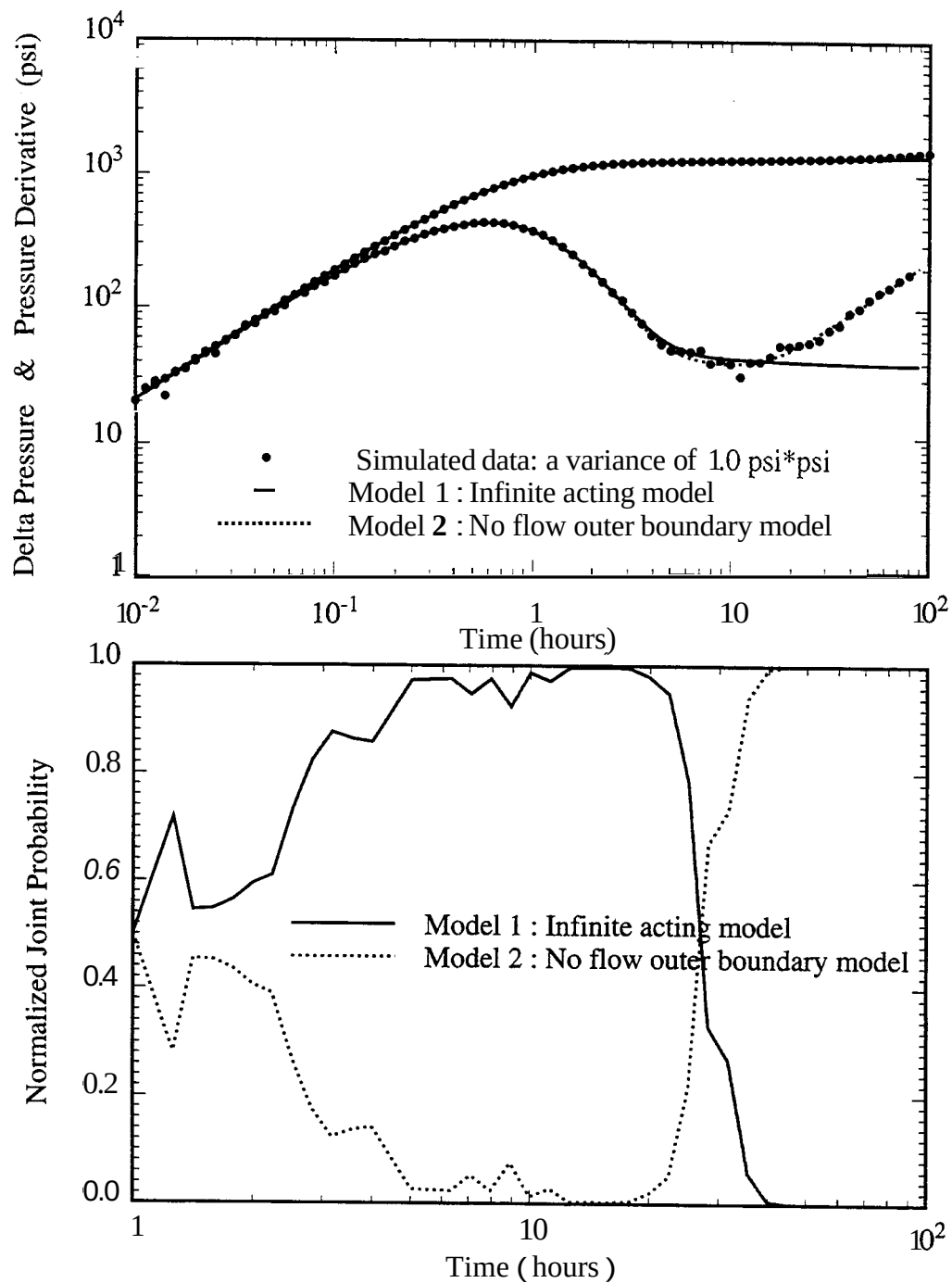


Figure 5.12: Pressure data with random normal errors with zero mean and a variance of 1.0 psi^2 : matches to the data (upper) and the corresponding normalized joint probability (lower)

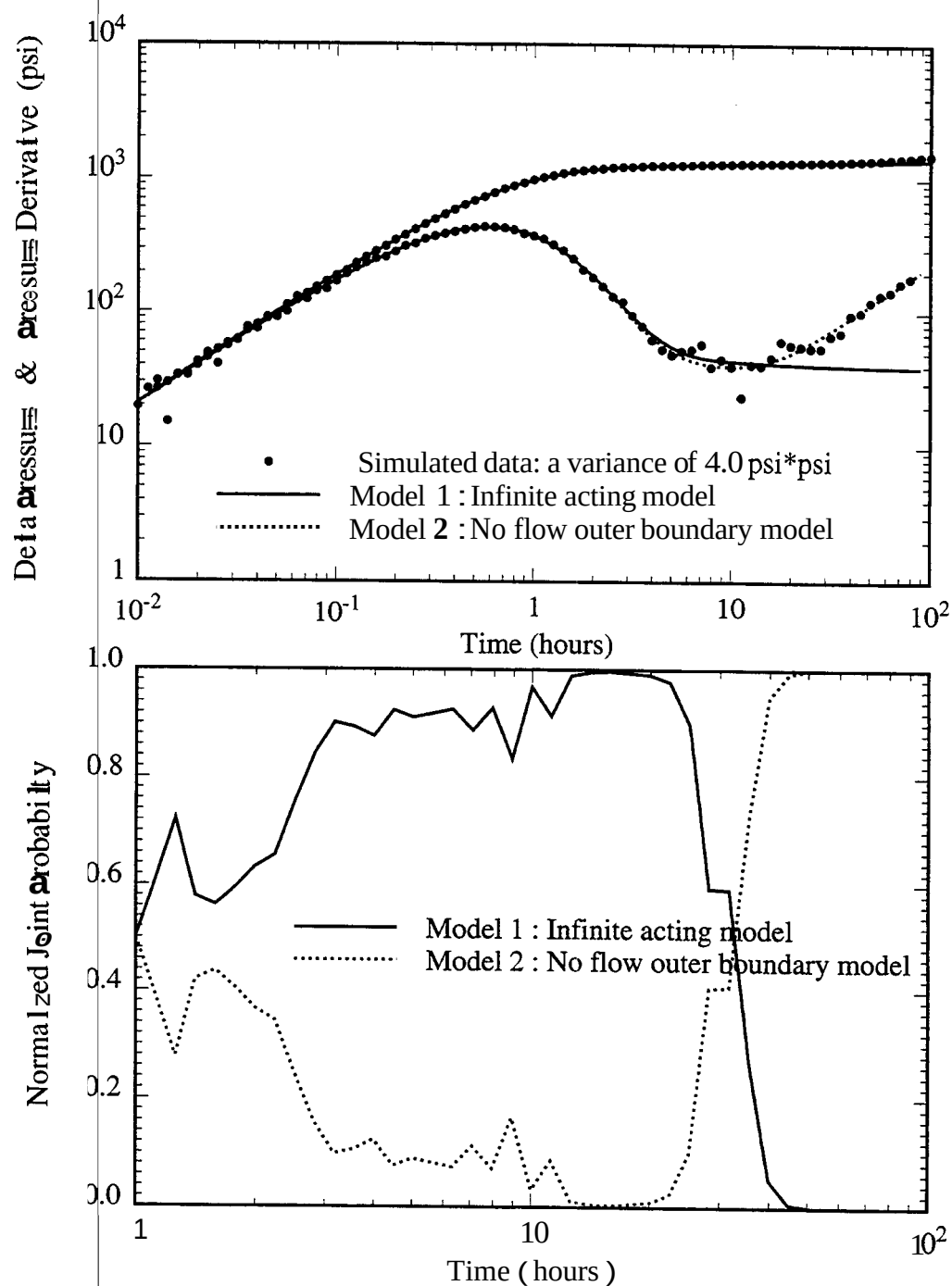


Figure 5.1 Pressure data with random normal errors with zero mean and a variance of 4.0 psi matches to the data (upper) and the corresponding normalized joint probability (lower)

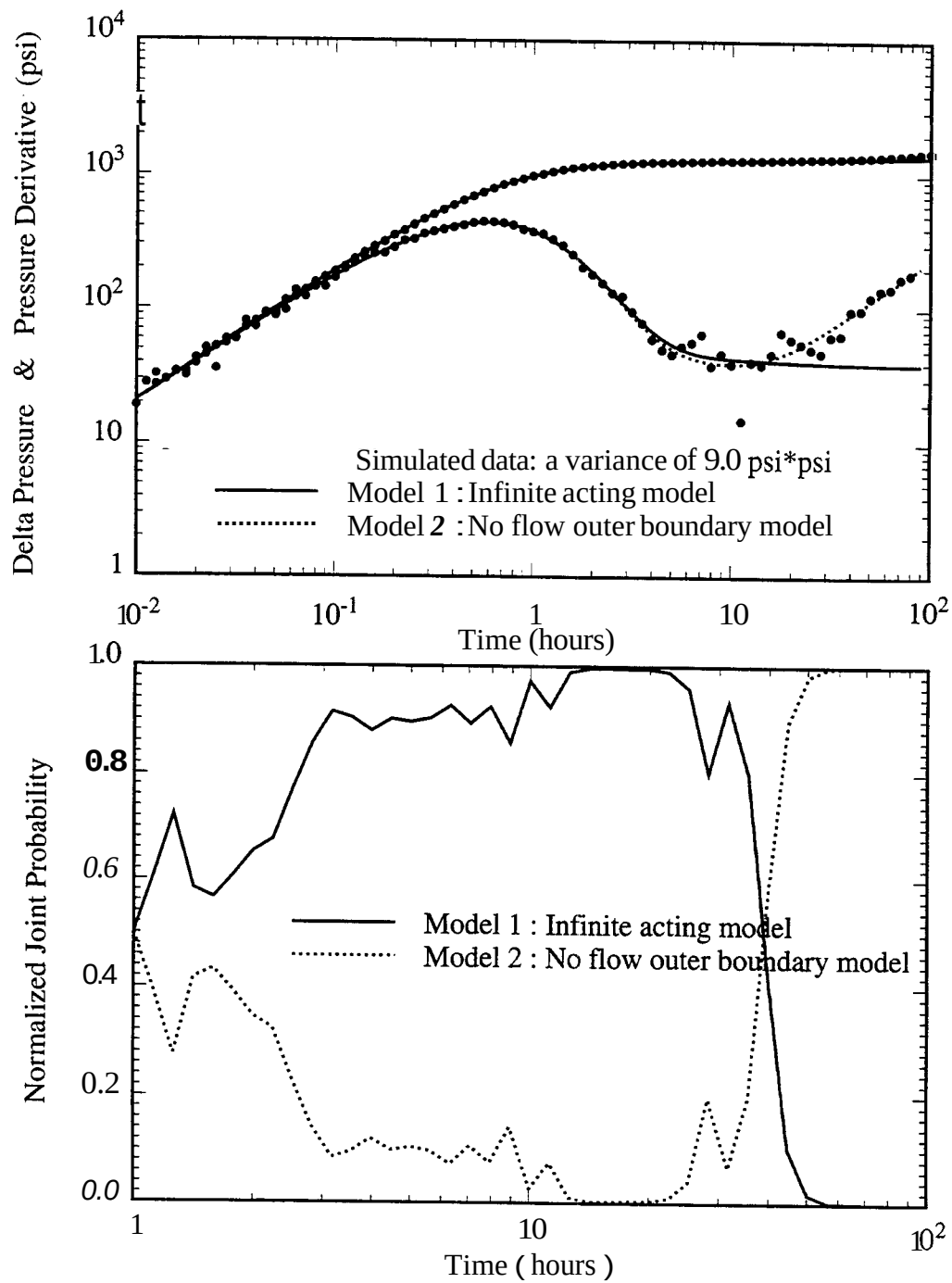


Figure 5.14: Pressure data with random normal errors with zero mean and a variance of 9.0 psi^2 : matches to the data (upper) and the corresponding normalized joint probability (lower)

are noisy, the sequential predictive probability method can discriminate between candidate reservoir models although the time period required to reach certainty becomes longer.

5.2 Advantages of the Sequential Predictive Probability Method Over Confidence Interval Analysis and Graphical Analysis

This section demonstrates that the utility of the sequential predictive probability method in comparison with confidence interval analysis and graphical analysis. Simulated draw down data were used as an example to illustrate a case in which the sequential predictive probability method works well for model discrimination while both confidence interval analysis and graphical analysis fail. In addition, it is demonstrated that the sequential predictive probability method can compare more than two models simultaneously.

Pressure data were calculated using a double porosity model with added random errors (Fig. 5.15). Reservoir and fluid data were the same as in Table 3.2. The true parameter values are $k = 50md$, $S = 10$, $C = 0.01STB/psi$, $w = 0.1$, and $\lambda = 5.0 \times 10^{-8}$. A random number generator was used to simulate random normd errors with zero mean and a variance of $4.0 psi^2$, which makes the data fairly noisy. The total number of data points is 81 and the time period is from 0.01 hour to 100 hours. The double porosity transition period is from 2 hours to 20 hours. In this example, the graphical analysis using the pressure derivative plot may suggest either a double porosity model, a sealing fault model or an infinite acting model. For simplicity, the infinite acting model is called Model 1, the sealing fault model Model 2 and the double porosity model Model 3. Model 1 has three parameters, which are h , S , and C . Model 2 has four parameters, which are k , S , C , and r_e . Model 3 has five parameters, which are h , S , C , w , and λ .

The data point at 1.0 hour is selected as the starting point since it lies just after the peak of the pressure derivative plot. The normalized joint probability associated

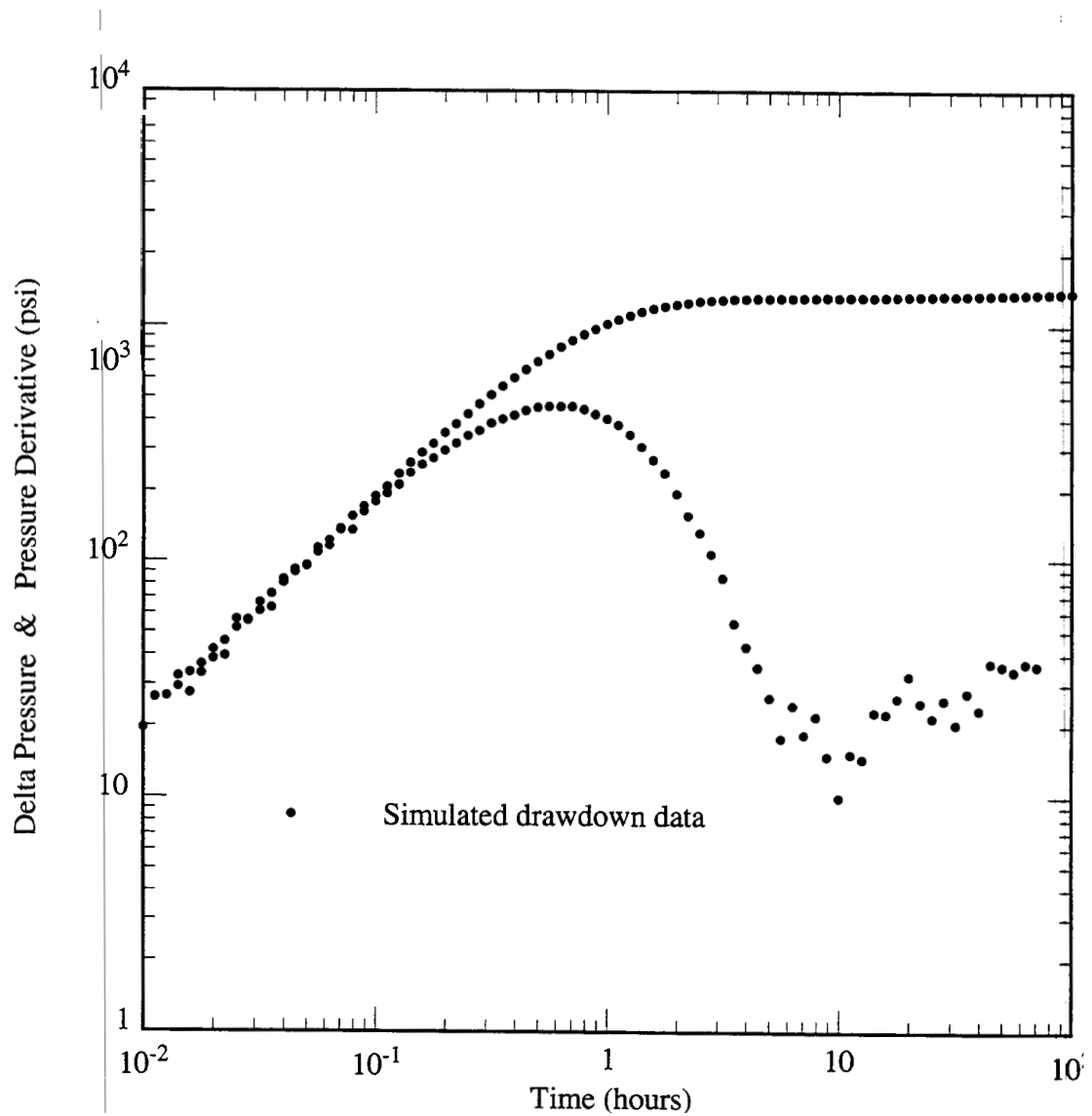


Figure 5.15: Simulated drawdown data using a double porosity model with normal random error

with each model at 1.0 hour is set to be equal to $1/3$, since three models are compared.

Final matches of the models to the data and the corresponding normalized joint probabilities are shown in Fig. 5.16. Final matches of the models to the data were calculated using the final estimated values of the parameters, which were evaluated using all of the data points.

Fig. 5.16 (upper) shows that any of the models seem to fit the data more or less equally. Therefore, graphical analysis does not provide useful results.

Fig. 5.16 (lower) shows that the normalized joint probability associated with Model 1 approaches one before 5 hours. Between 6 hours and 15 hours Model 1 and Model 3 have similar normalized joint probabilities, and thereafter the normalized joint probability associated with Model 3 reaches one. The normalized joint probability associated with Model 2 is zero after 10 hours.

Thus during the double porosity transition period, it is not possible to decide which model is better, Model 1 or Model 3, but as the double porosity transition time is close to the end the sequential predictive probability method clearly selects Model 3, which is the true model. The possibility of Model 2 is eliminated after 10 hours.

Fig. 5.17 (upper) shows the overall predictive variance in Model 1, Model 2, and Model 3, and Fig. 5.17 (lower) shows the pressure difference between the observed pressure response and the expected pressure response based on each model. The overall predictive variance in Model 3 is larger than that in Model 1 before 5 hours in Fig. 5.17 (upper), while the pressure differences are almost identical in Fig. 5.17 (lower). Therefore, the normalized joint probability associated with Model 1 is larger than that in Model 3. During the double porosity transition period the data contain valuable information about the double porosity behavior and the uncertainty about the parameters in Model 3 is greatly reduced. Hence, the overall predictive variance in Model 3 becomes small and approaches that in Model 1 between 6 hours and 15 hours in Fig. 5.17 (upper). As a result, Model 1 and Model 3 have similar normalized joint probabilities between 6 hours and 15 hours in Fig. 5.16 (lower). As the double porosity transition period is close to the end, the overall predictive variance in Model 3 becomes smaller than that in Model 1 in Fig. 5.17 (upper) and the normalized joint

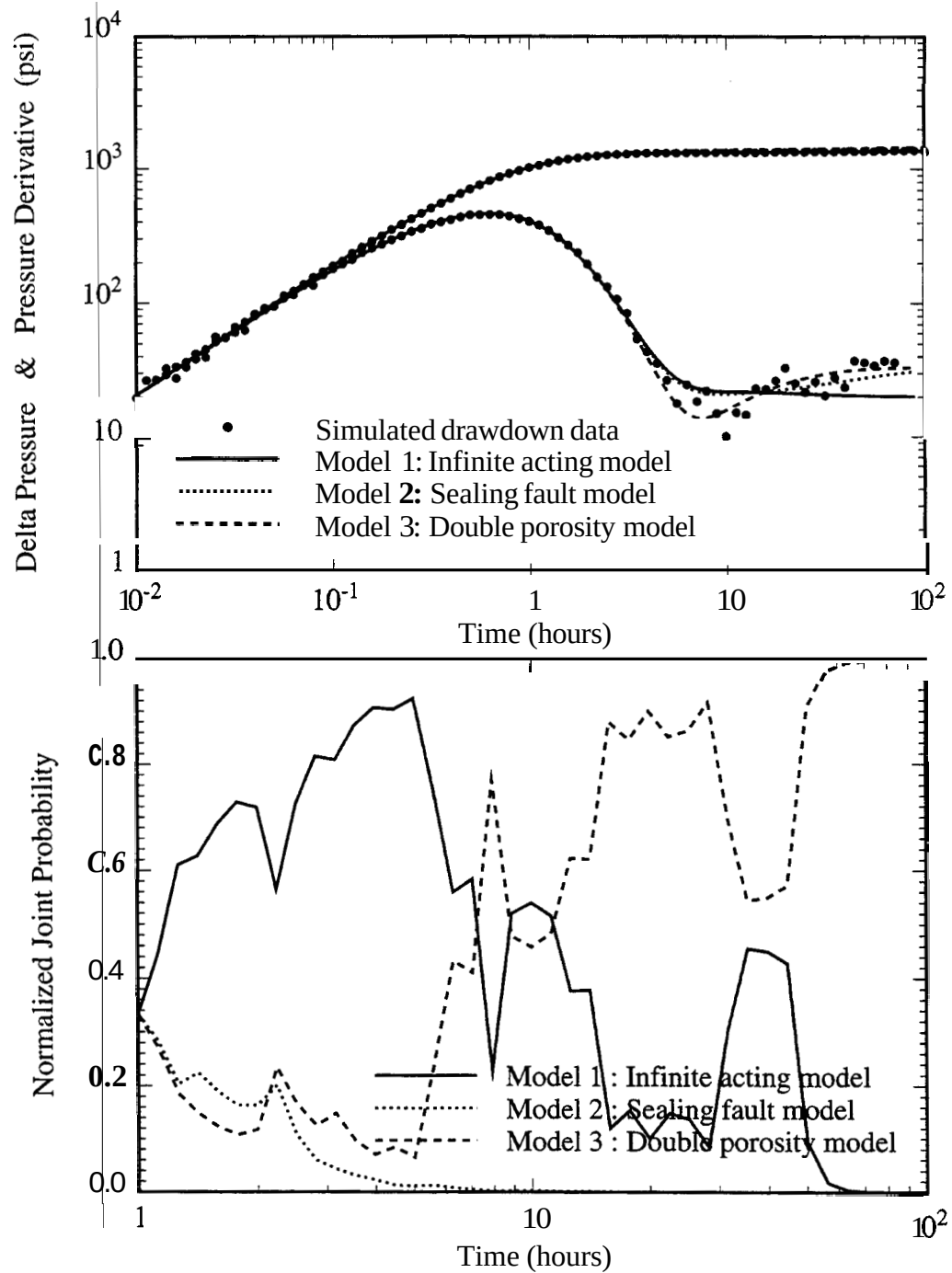


Figure 5.16: Final matches of Model 1, Model 2, and Model 3 to the data (upper) and normalized joint probability associated with each model (lower)

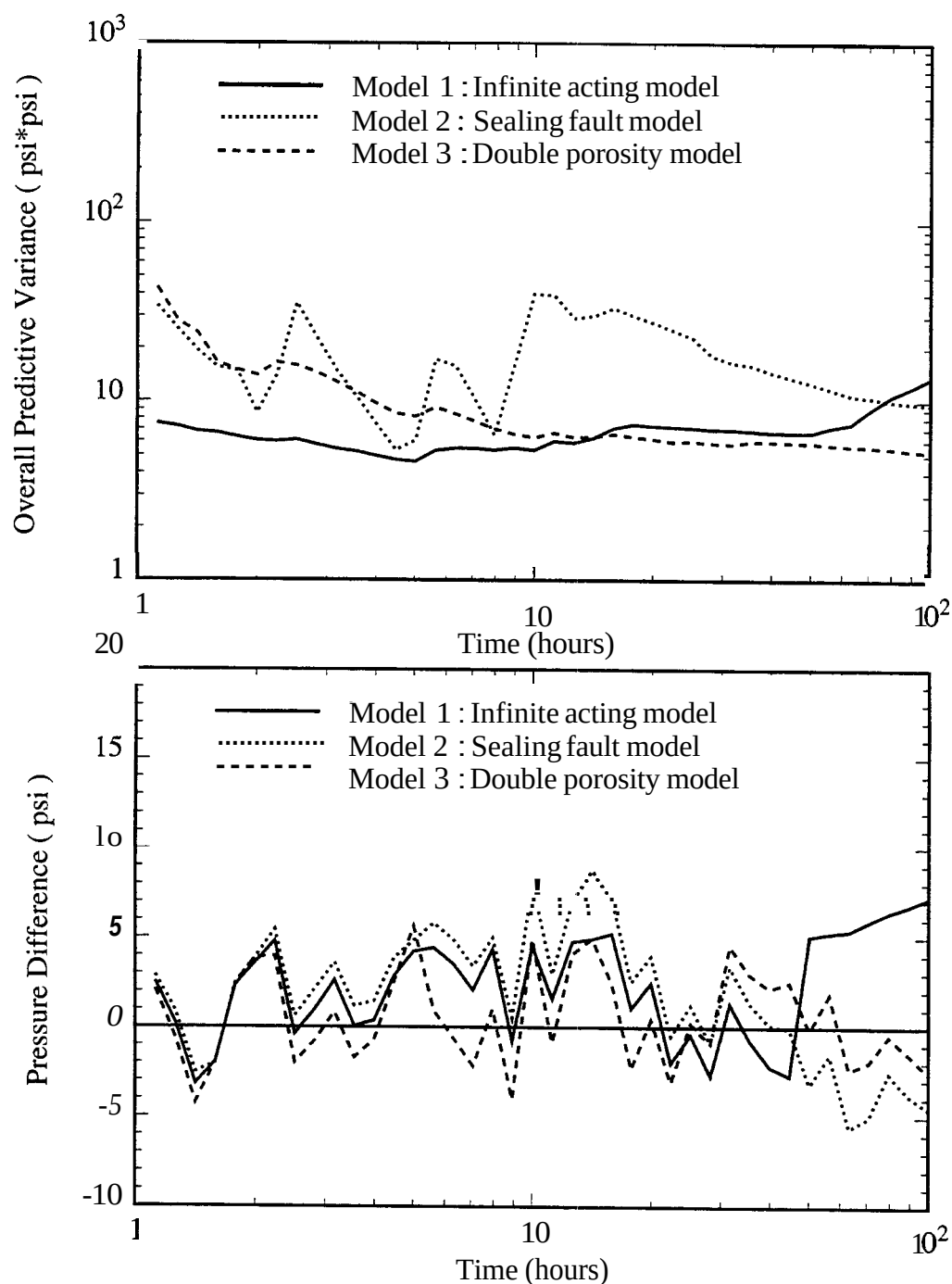


Figure 5.17: Overall predictive variance in Model 1, Model 2, and Model 3 (upper) and pressure difference between the observed pressure response and the expected pressure response based on each model (lower)

probability associated with Model 3 becomes larger than that in Model 1 in Fig. 5.16 (lower). On the other hand, the overall predictive variance in Model 2 is always large in Fig. 5.17 (upper), since the data have no information about the sealing fault. Therefore, the normalized joint probability associated with Model 2 is always small in Fig. 5.16 (lower).

Fig. 5.18 (upper) shows the estimated values of permeability in Model 1, Model 2, and Model 3. Fig. 5.18 (lower) shows the corresponding relative confidence interval of the permeability estimate in each model. From Fig. 5.18 (lower), the relative confidence interval of the permeability estimate may imply that Model 1 is better than Model 3, even after the double porosity transition period ends. Moreover, the relative confidence interval of the permeability estimate in Model 2 is narrower than that in Model 3. Therefore, according to the relative confidence interval of the permeability estimate, confidence interval analysis might suggest Model 1 as the true model. The reason why the relative confidence interval of the permeability estimate in the true model (Model 3) is wide is explained from the definition of confidence intervals. As discussed in Section 3.5.1, the confidence interval is directly proportional to the variance of the parameter, which is expressed as $\sigma_\theta^2 = \sigma^2 H^{-1}$. The variance of the parameter is also proportional to that of the errors. In this example, a large value of the variance of the errors of 4.0 psi^2 is used. In addition, the use of large number of parameters makes the diagonal elements of the inverse Hessian matrix large. Hence, in this example the confidence interval of the permeability estimate in the true model, which has larger number of parameters, turns out to be wider.

Fig. 5.18 indicates that in the case where Model 1 is selected as the true model, the estimated value of permeability is about 90 md , which is almost twice the true value (50 md). Therefore, correct selection of the true model is important not only for accurate reservoir description but also for accurate estimation of the reservoir properties.

The findings in this example are summarized as follows:

1. In cases where graphical analysis may indicate several models, the sequential predictive probability method can select the true model.

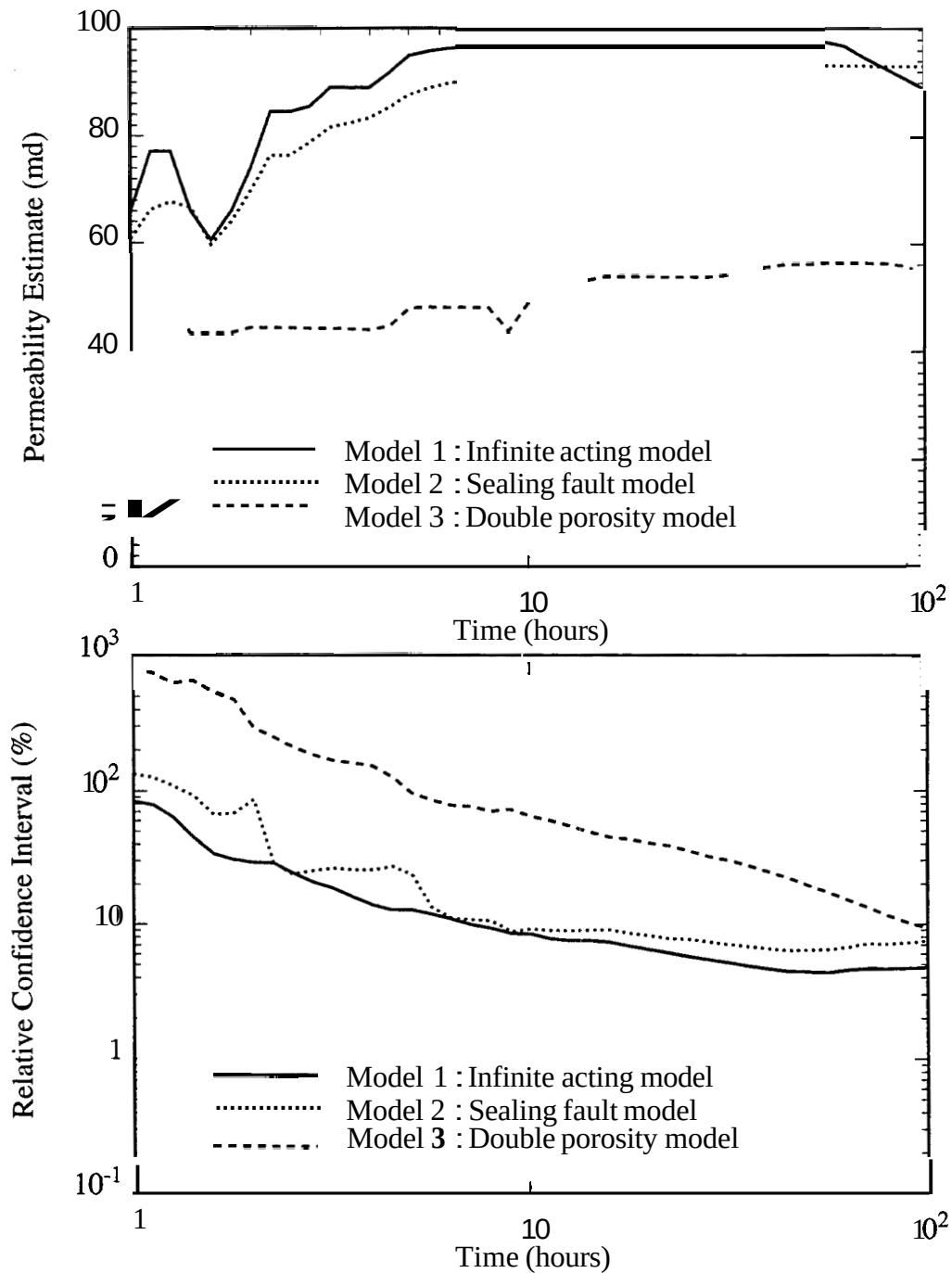


Figure 5.18: Permeability estimate for Model 1, Model 2, and Model 3 (upper) and relative confidence interval of permeability for each model (lower)

2. The use of confidence intervals alone may produce deceptive results.
3. The sequential predictive probability method can compare more than two models simultaneously.

5.3 Application to Simulated Well Test Data

In this section, application of the sequential predictive probability method to simulated well test data is discussed.

The purpose of this section is to demonstrate the utility of the sequential predictive probability method in several situations. Section 5.3.1 discusses situations which are commonly encountered in actual well test analysis. Section 5.3.2 discusses the applicability of the sequential predictive probability method to complex reservoir models.

5.3.1 Commonly Encountered Situations

This section discusses two situations which are commonly encountered in actual well test analysis. One is the situation where the selection between a sealing fault model and a no flow outer boundary model is necessary. The other is the situation where the selection between a constant pressure outer boundary model and a double porosity model is necessary. It was investigated whether the sequential predictive probability method can select the correct model in these situations.

Through this section, the same reservoir and fluid data were used as given in Table 3.2. A random number generator was used to simulate random normal errors with zero mean and a variance of 1.0 psi^2 . The total number of data points is 81 and the time period is from 0.01 hour to 100 hours. The data at 1.0 hour is selected as the starting point just after the peak of the pressure derivative plot.

First, the situation where discrimination between the sealing fault model and the no flow outer boundary model is necessary is discussed. As shown in Fig. 4.5 (upper), both the sealing fault model and the no flow outer boundary model show upward

trends at late time on the pressure derivative plot. Therefore, in cases where upward trends may be recognized on the pressure derivative plot, it is necessary to decide whether the sealing fault model, the no flow outer boundary model, or the infinite acting model should be selected as the true model. For simplicity, the infinite acting model, the sealing fault model, and the no flow outer boundary model are named Model 1, Model 2, and Model 3, respectively. Model 1 has three parameters, which are k , S , and C , while both Model 2 and Model 3 have four parameters, which are k , S , C , and r_e . Drawdown pressure data were calculated using Model 2 and Model 3, and random errors were added. The true parameter values are $k = 50md$, $S = 10$, $C = 0.01STB/psi$, and $r_e = 2000ft$. The starting time when the boundary effects become significant is around 20 hours. The normalized joint probability associated with each model at 1.0 hour is set to be equal to $1/3$, since three models are compared.

Fig. 5.19 shows the simulated drawdown data using the sealing fault model (Model 2). Final matches of the models to the data and the corresponding normalized joint probabilities are shown in Fig. 5.20. Final matches of the models to the data were calculated using the final estimated values of the parameters, which were evaluated using all of the data points. Fig. 5.21 shows the simulated drawdown data using the no flow outer boundary model (Model 3). Final matches of the models to the data and the corresponding normalized joint probabilities are shown in Fig. 5.22. Final matches of the models to the data were calculated using the final estimated values of the parameters, which were evaluated using all of the data points. In either case, the sequential predictive probability method can select the true model at the final stage. However, there are significant differences between these two cases.

Fig. 5.20 (lower) shows that the normalized joint probability associated with Model 1 is close to one before around 50 hours. Between 50 hours and 80 hours the normalized joint probability associated with Model 1 decreases rapidly to zero, while that with Model 2 increases rapidly to one. The normalized joint probability associated with Model 2 reaches one after 80 hours. The normalized joint probability associated with Model 3 is almost zero after 5 hours. Therefore, in cases where the sealing fault model is the correct model, the possibility of selecting the no flow outer boundary model is low.

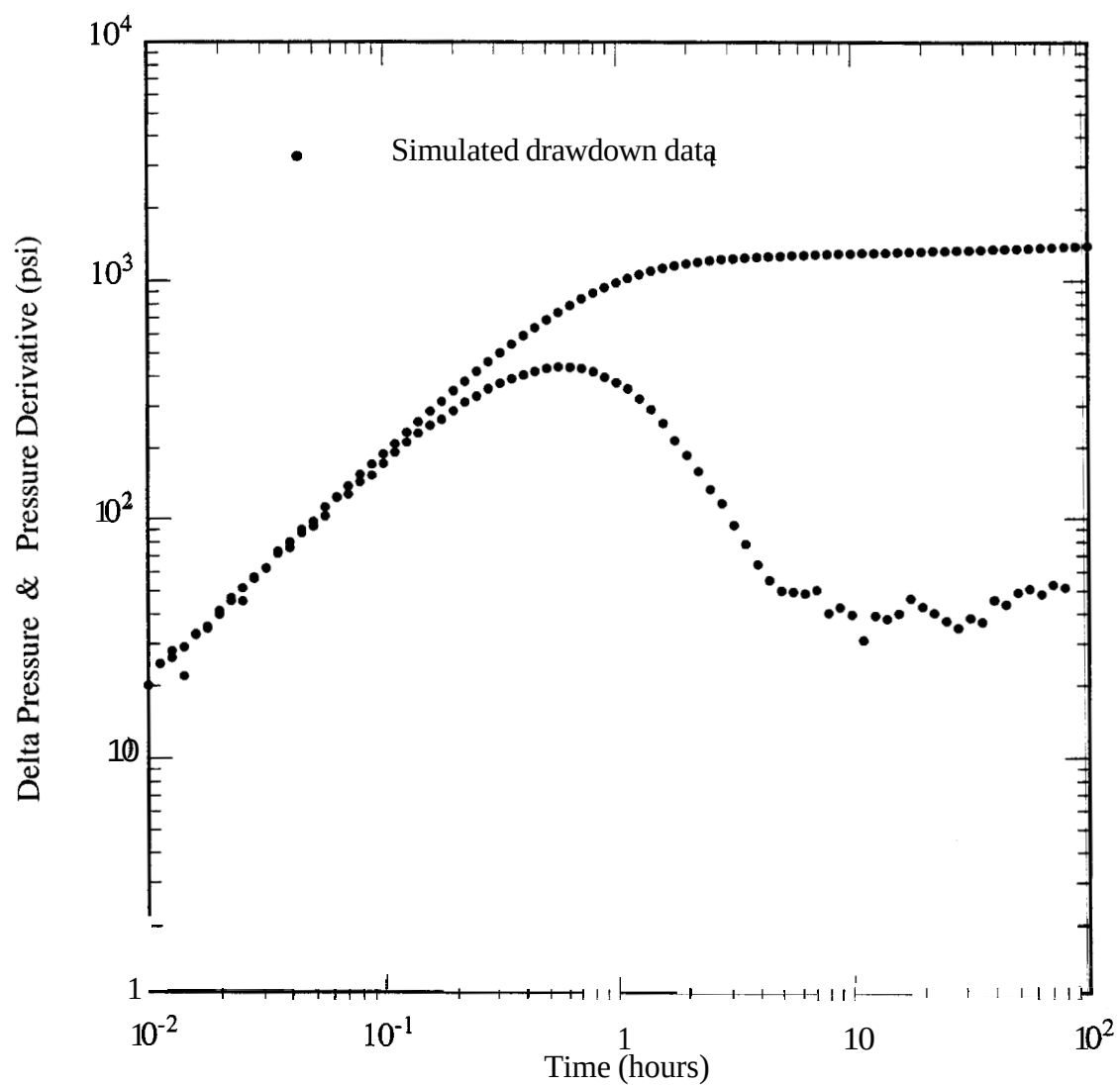


Figure 5.19: Simulated drawdown data using a sealing fault model (Model 2) with normal random errors

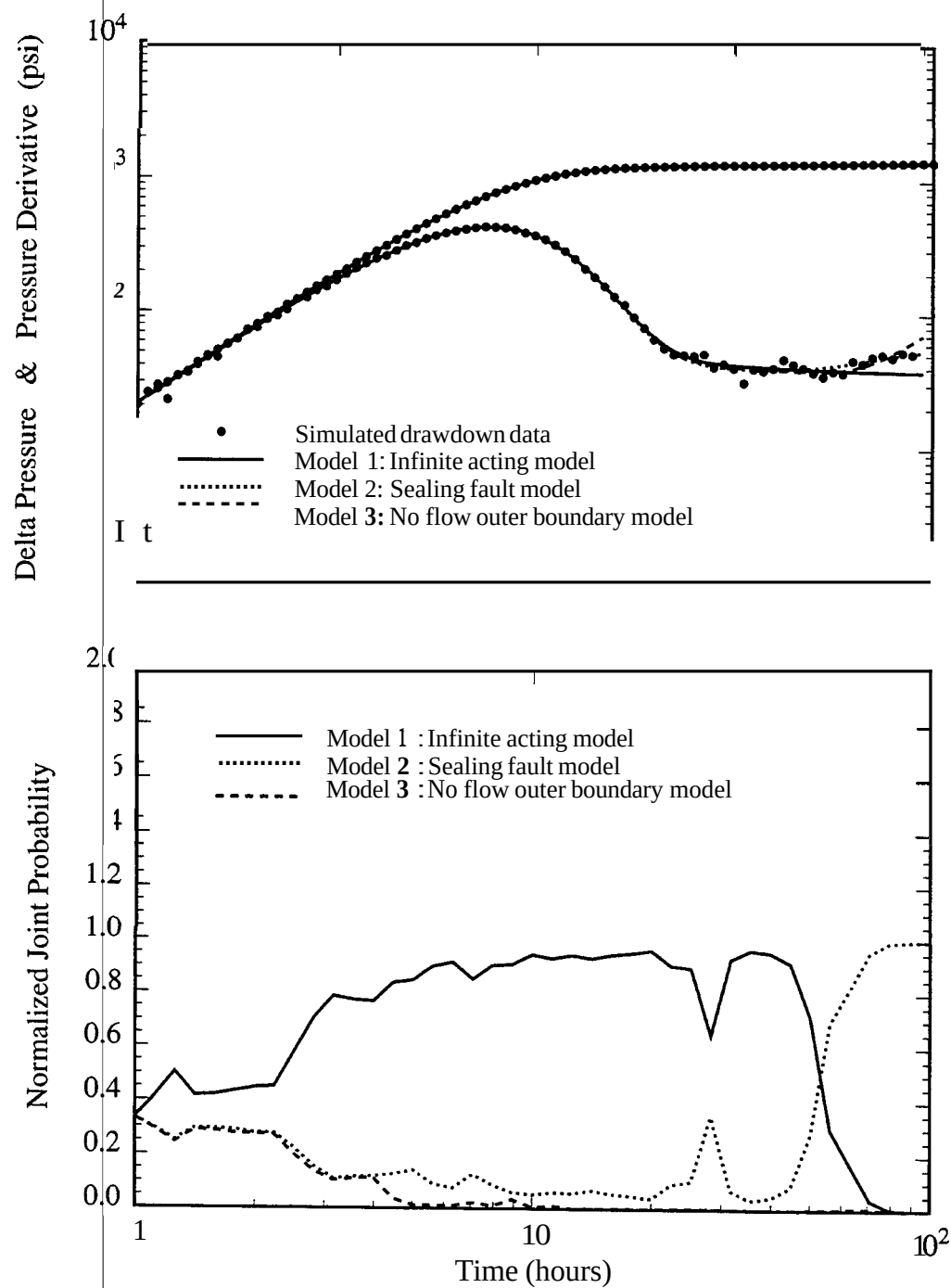


Figure 5.20: Final matches of Model 1, Model 2, and Model 3 to the data (upper) and normalized joint probability associated with each model (lower)

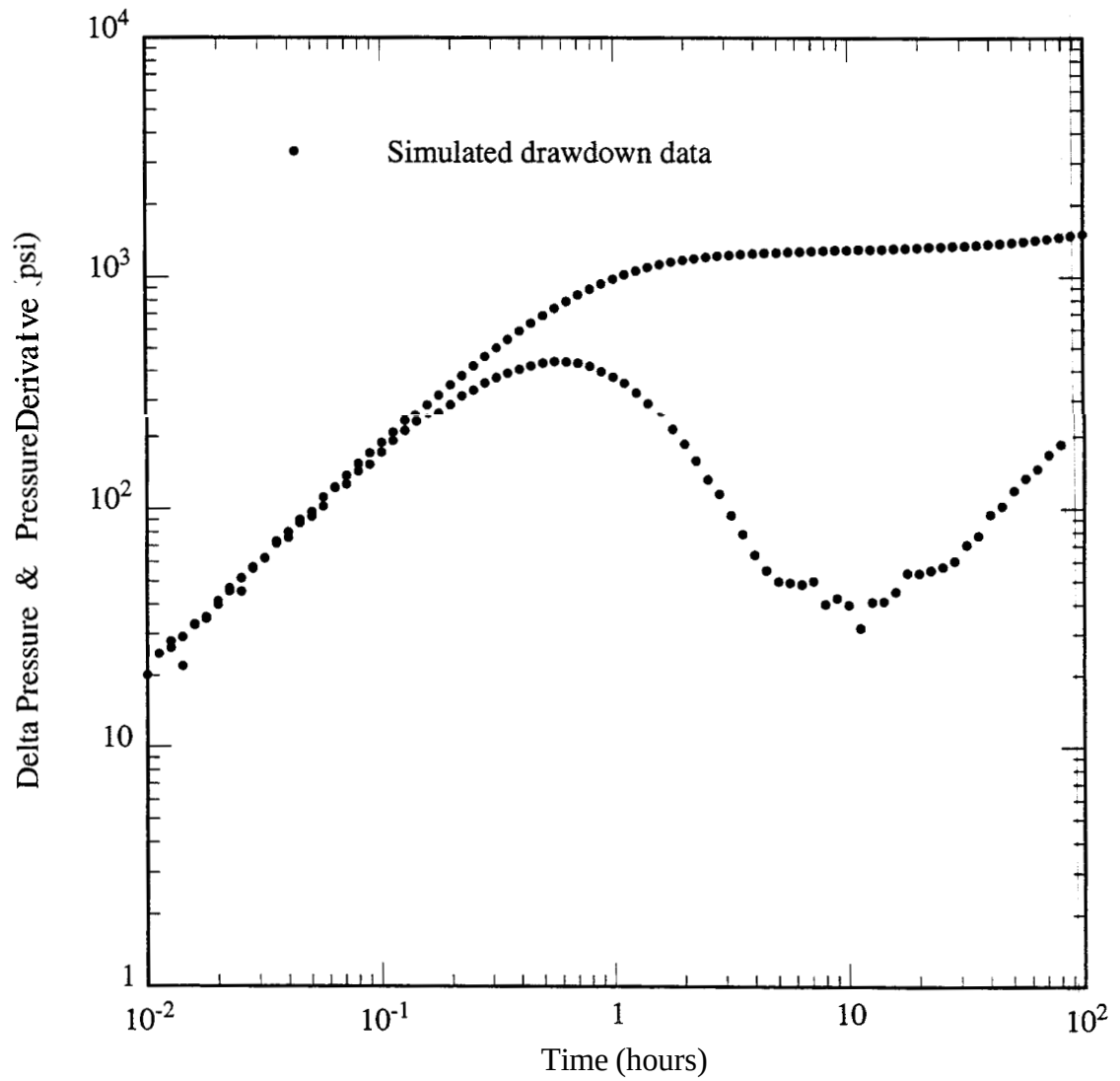


Figure 5.21: Simulated drawdown data using a no flow outer boundary model (Model 3) with normal random errors

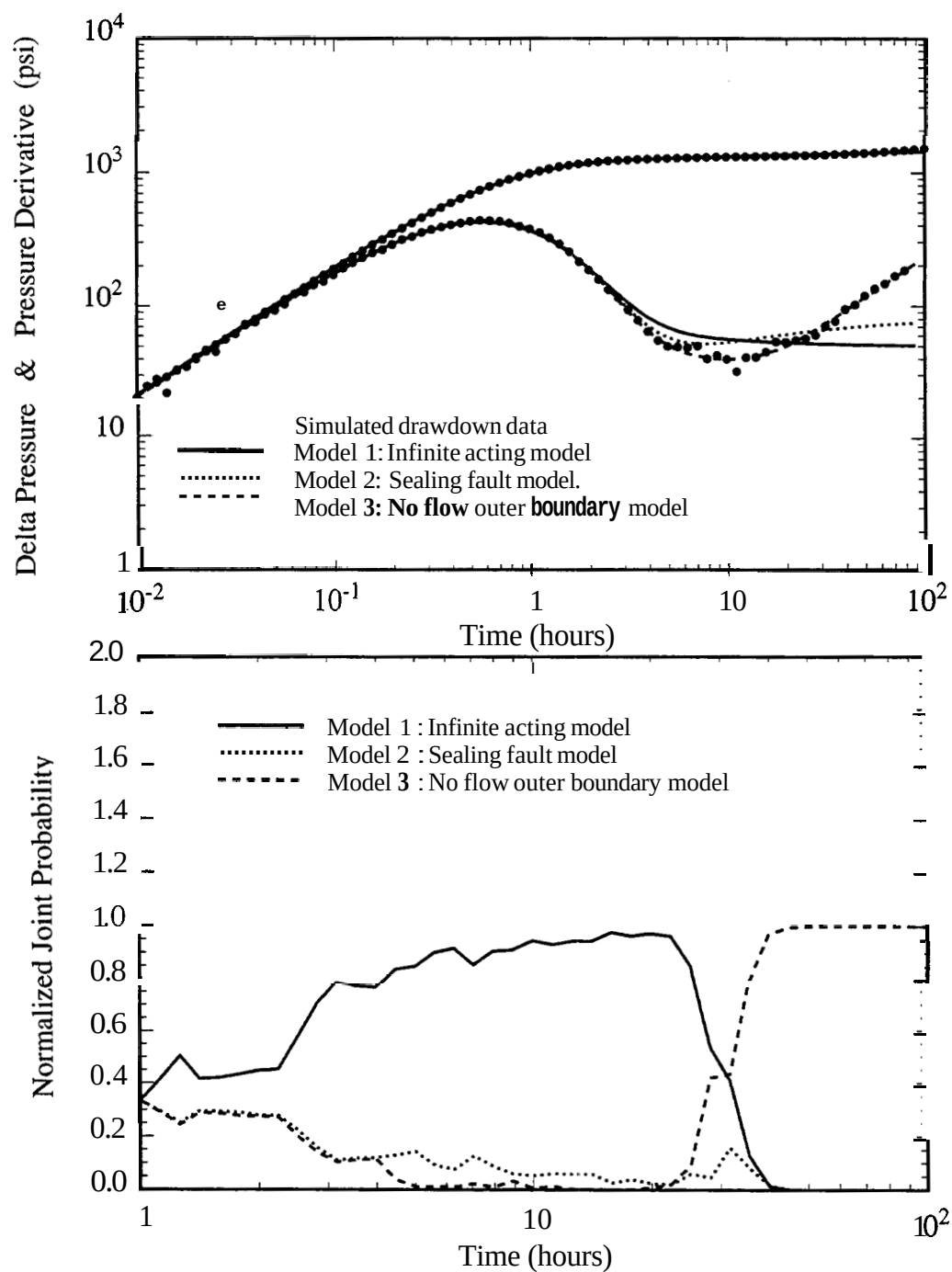


Figure 5.22: Final matches of Model 1, Model 2, and Model 3 to the data (upper) and normalized joint probability associated with each model (lower)

On the other hand, Fig. 5.22 (lower) shows that the normalized joint probability associated with Model 1 is close to one before around 20 hours. Between 20 hours and 40 hours the normalized joint probability associated with Model 1 decreases rapidly to zero. The normalized joint probability associated with Model 3 increases rapidly and reaches one after 40 hours. The normalized joint probability associated with Model 2 increases slightly around 20 hours, when the boundary effects due to the no flow outer boundary become significant, but it decreases rapidly to zero. Therefore, in cases where the no flow outer boundary model is the correct model, the possibility of selecting the sealing fault model is low. However, it is interesting to compare Fig. 5.20 (lower) with Fig. 5.22 (lower). The normalized joint probability associated with the sealing fault model reaches one after 80 hours in Fig. 5.20 (lower), while that with the no flow outer boundary model reaches one after 40 hours in Fig. 5.22 (lower). The starting time when the boundary effects become significant is around 20 hours in both cases. Hence, the detection of a sealing fault requires more data points than the detection of a no flow outer boundary.

Next, the situation where discrimination between the constant pressure outer boundary model and the double porosity model is necessary is discussed. As shown in Fig. 4.5 (upper) and Fig. 4.6 (upper), the constant pressure outer boundary model shows a downward trend at late time and the double porosity model also shows a downward trend at intermediate time on the pressure derivative plot. Therefore, in cases where downward trends may be recognized on the pressure derivative plot, it is necessary to decide whether the constant pressure outer boundary model, the double porosity model, or the infinite acting model should be selected as the true model. For simplicity, the infinite acting model, the constant pressure outer boundary model, and the double porosity model are named Model 1, Model 2, and Model 3, respectively. Model 1 has three parameters, which are k , S , and C , Model 2 has four parameters, which are k , S , C , and r_e , and Model 3 has five parameters, which are k , S , C , w , and λ . Drawdown pressure data were calculated using Model 2 and Model 3, and random errors were added. The true parameter values for Model 2 are $k = 50md$, $S = 10$, $C = 0.015TB/psi$, and $r_e = 2000ft$, while the true parameter values for Model 3 are

$k = 50md$, $S = 10$, $C = 0.01STB/psi$, $\omega = 0.1$, and $\lambda = 1.0 \times 10^{-8}$. The starting time when the boundary effects become significant for Model 2 is around 20 hours. The normalized joint probability associated with each model at 1.0 hour is set to be equal to $1/3$, since three models are compared.

Fig. 5.23 shows the simulated drawdown data using the constant pressure outer boundary model (Model 2). Final matches of the models to the data and the corresponding normalized joint probabilities are shown in Fig. 5.24. Final matches of the models to the data were calculated using the final estimated values of the parameters, which were evaluated using all of the data points. Fig. 5.25 shows the simulated drawdown data using the double porosity model (Model 3). Final matches of the models to the data and the corresponding normalized joint probabilities are shown in Fig. 5.26. Final matches of the models to the data were calculated using the final estimated values of the parameters, which were evaluated using all of the data points. In either case, the sequential predictive probability method can select the true model at the final stage. However, there are significant differences between these two cases.

Fig. 5.24 (lower) shows that the normalized joint probability associated with Model 1 is close to one before around 25 hours. Between 25 hours and 50 hours the normalized joint probability associated with Model 1 decreases rapidly to zero, while that with Model 2 increases rapidly to one. The normalized joint probability associated with Model 2 reaches one after 50 hours. The normalized joint probability associated with Model 3 is almost zero after 3 hours. Therefore, in cases where the constant pressure outer boundary model is the correct model, the possibility of selecting the double porosity model falsely is low.

On the other hand, Fig. 5.26 (lower) shows that the normalized joint probability associated with Model 1 is close to one before around 15 hours. Between 15 hours and 30 hours the normalized joint probability associated with Model 1 decreases rapidly to zero. The normalized joint probability associated with Model 2 increases to about 0.9 between 30 hours and 50 hours. Thereafter the normalized joint probability associated with Model 2 decreases to zero, while that with Model 3 reaches one after 70 hours. Therefore, in cases where the double porosity model is the correct model, discrimination between the constant pressure outer boundary model and the double

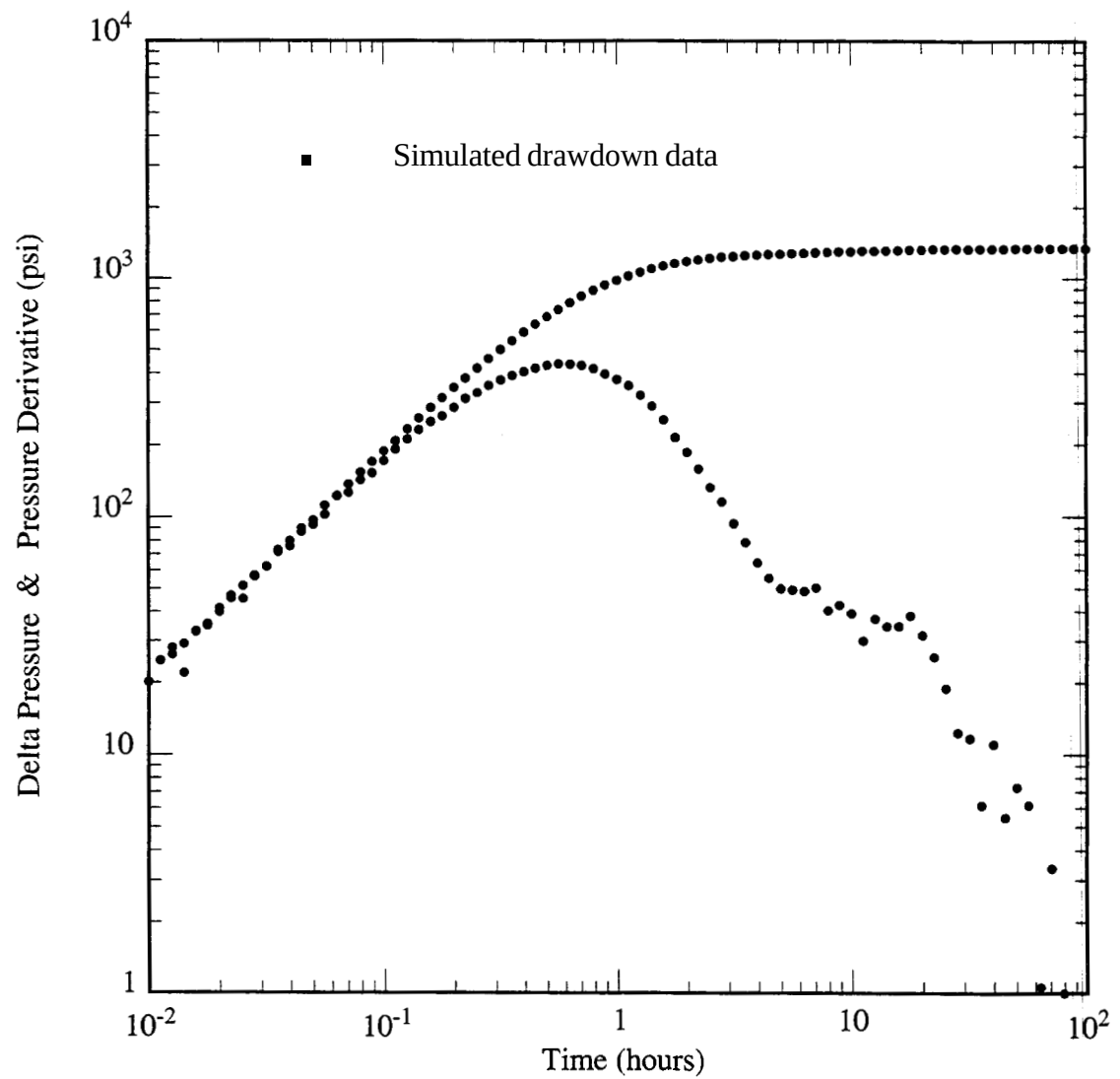


Figure 5.23: Simulated drawdown data using a constant pressure outer boundary model (Model 2) with normal random errors

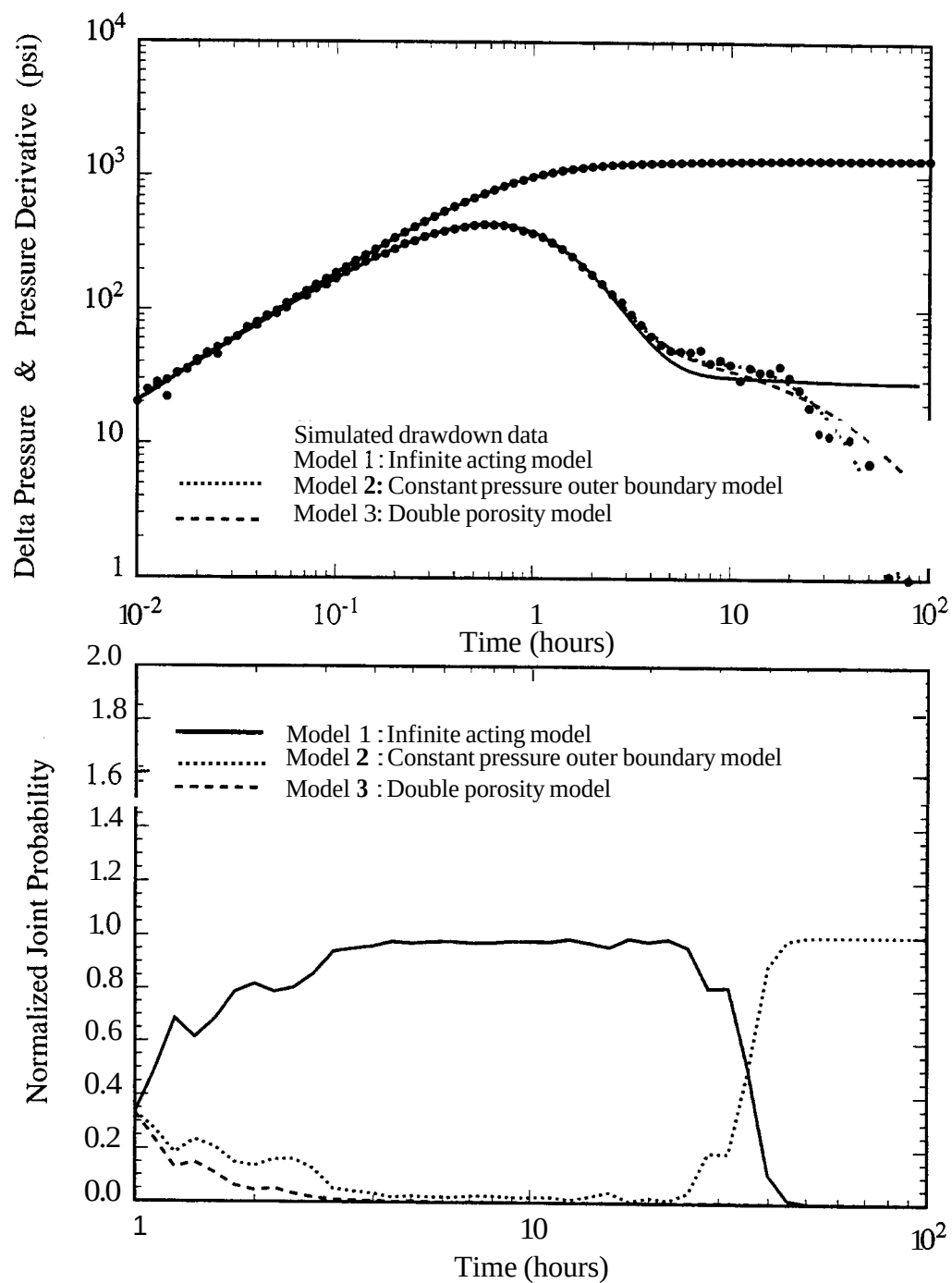


Figure 5.24: Final matches of Model 1, Model 2, and Model 3 to the data (upper) and normalized joint probability associated with each model (lower)

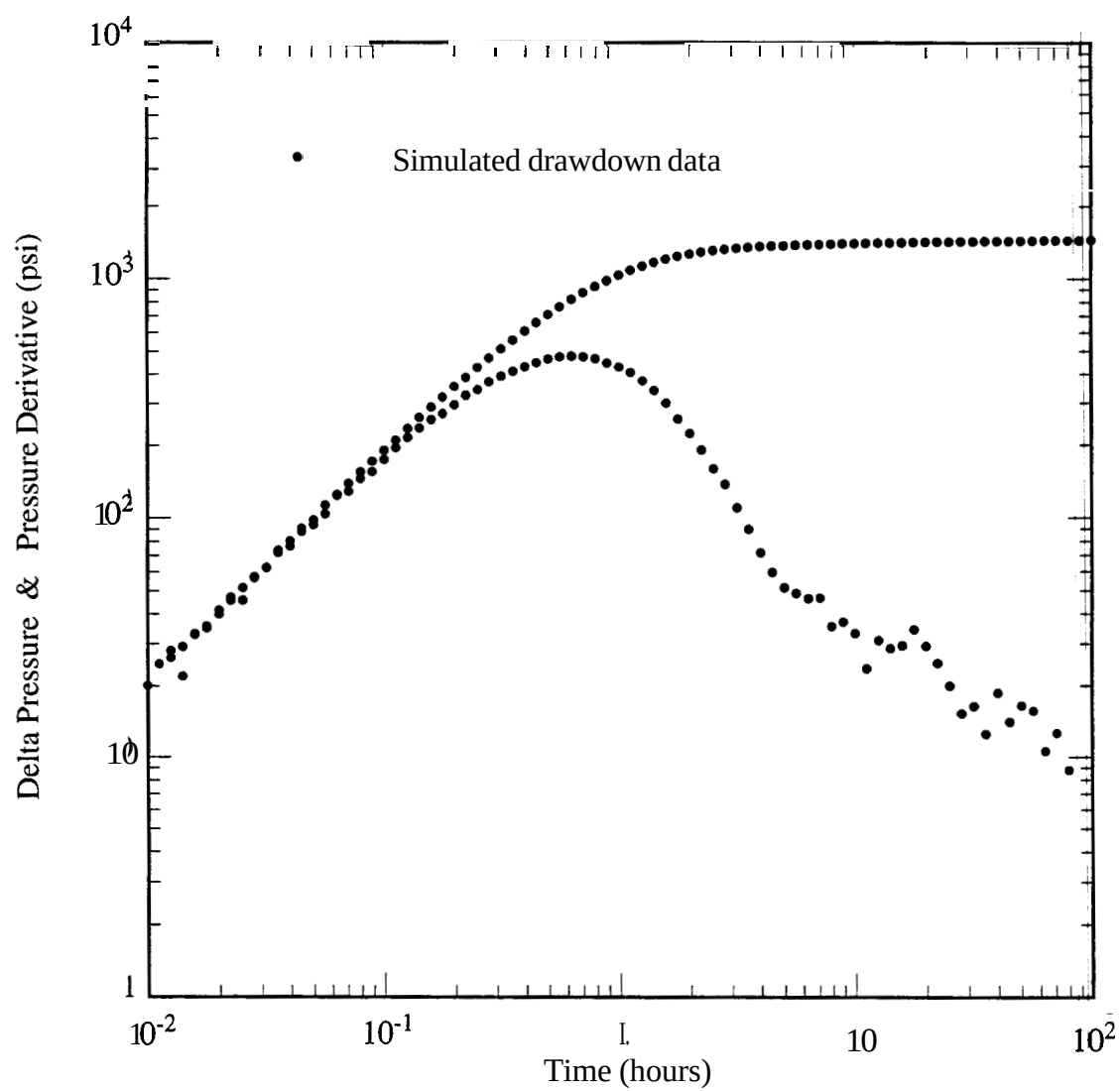


Figure 5.25: Simulated drawdown data using a double porosity model (Model 3) with normal random errors

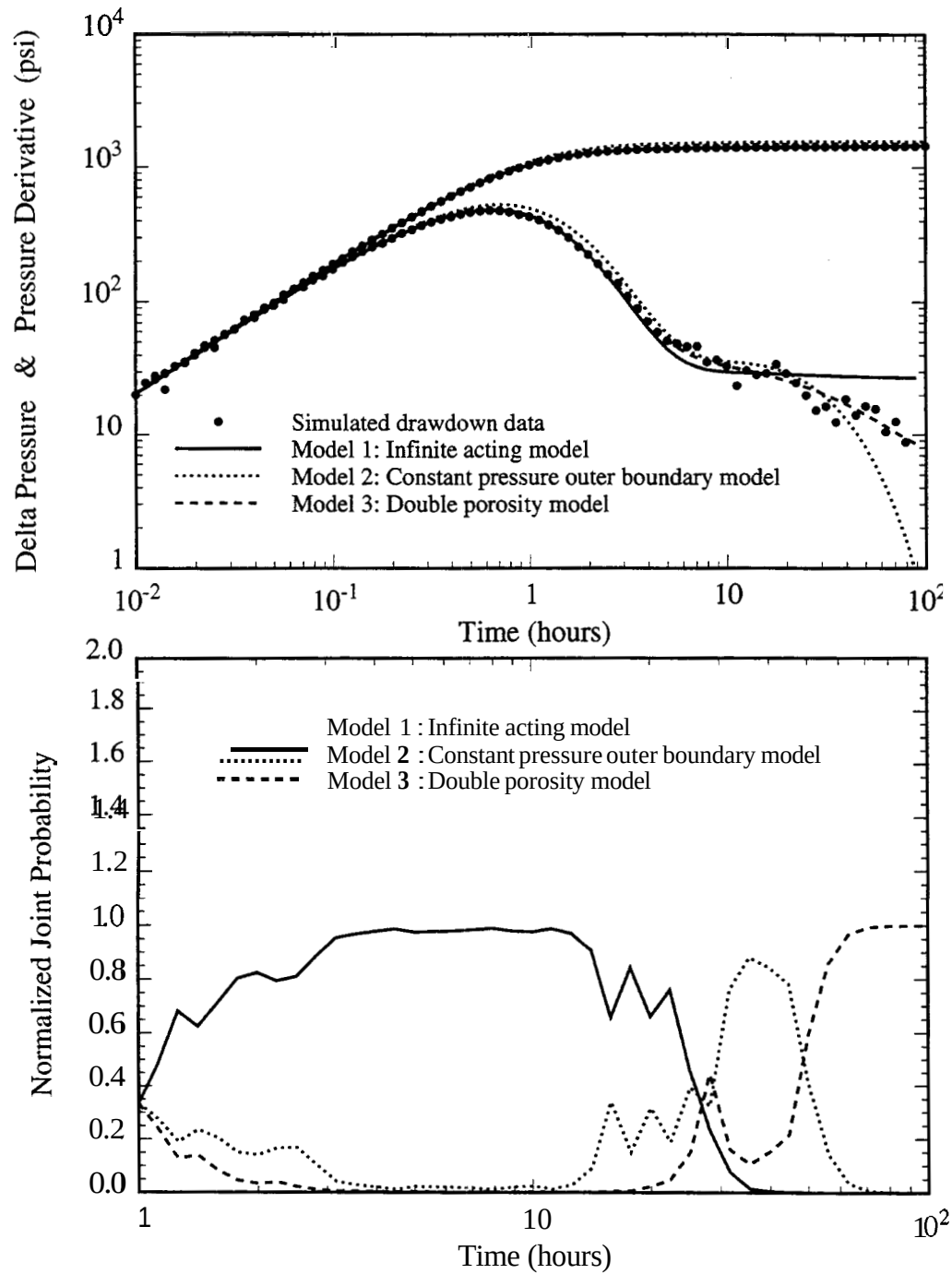


Figure 5.26: Final matches of Model 1, Model 2, and Model 3 to the data (upper) and normalized joint probability associated with each model (lower)

porosity model is important at the beginning of the double porosity transition period. In this particular case, if the data are truncated before 40 hours, there is a risk of selecting the constant pressure outer boundary model as the true model.

In addition to the two situations discussed above, it is sometimes necessary to discriminate between the double porosity model, the sealing fault model, and the no flow outer boundary model, since the double porosity behavior has an upward trend during the later half of the double porosity transition period. Discrimination between the double porosity model and the sealing fault model was discussed in Section 5.2, Discrimination between the double porosity model and the no flow outer boundary model will be discussed in Section 5.4.2 using actual field well test data.

5.3.2 Complex Reservoir Models

This section discusses situations where complex reservoir models are used. As shown in Fig. 4.5 (lower), Fig. 4.6 (lower), and Fig. 4.7 (lower), the magnitude of $\mathbf{g}^T \mathbf{H}^{-1} \mathbf{g}$ generally increases according to the complexity of the model. This implies that uncertainty associated with a complex model is higher than that with a simple model, since uncertainty due to the model is expressed mathematically as $\mathbf{g}^T \mathbf{H}^{-1} \mathbf{g} \cdot \sigma^2$. As discussed in Section 3.1, it is straightforward to extend the utility of the sequential predictive probability method to other reservoir models than the eight basic models which are described in this study, since theoretically the sequential predictive probability method can discriminate between any number of reservoir models as long as the reservoir models can be expressed approximately as a linear form with respect to the reservoir parameters. Hence, it is a necessary step to investigate whether the sequential predictive probability method can select a complex model as the true model for a limited number of data.

In this work, the double porosity and sealing fault model, the double porosity and no flow outer boundary model, and the double porosity and constant pressure outer boundary model are regarded as complex models, since these models have six reservoir parameters (k , S , C , ω , λ , and r_e) and express both the double porosity effects and the boundary effects.

Through this section, the same reservoir and fluid data were used as given in Table 3.2. A random number generator was used to simulate random normal errors with zero mean and a variance of 1.0 psi^2 . The total number of data points is 101 and the time period is from 0.01 hour to 1000 hours. The data at 1.0 hour is selected as the starting point just after the peak of the pressure derivative plot. The true parameter values are $k = 50 \text{ md}$, $S = 10$, $C = 0.01 \text{ STB/psi}$, $w = 0.1$, $\lambda = 1.0 \times 10^{-8}$, and $r_e = 4000 \text{ ft}$. The boundary effects do not become significant until after the double porosity transition period.

First, the double porosity and sealing fault model was employed as the true model, Pressure data are shown in Fig. 5.27. During intermediate time the double porosity behavior is present and at late time the boundary effects due to the sealing fault boundary appear on the pressure derivative plot. In this example, six models, which were the infinite acting model (Model 1), the sealing fault model (Model 2), the no flow outer boundary model (Model 3), the double porosity model (Model 4), the double porosity and sealing fault model (Model 5), and the double porosity and no flow outer boundary model (Model 6), were employed as candidate reservoir models, Model 1 has three parameters, which are k , S , and C . Model 2 and Model 3 have four parameters, which are k , S , C , and r_e . Model 4 has five parameters, which are k , S , C , w , and λ . Model 5 and Model 6 have six parameters, which are k , S , C , w , λ , and r_e . The normalized joint probability associated with each model at 1.0 hour is set to be equal to $1/6$. Final matches of the models to the data and the corresponding normalized joint probabilities are shown in Fig. 5.28. Final matches of the models to the data were calculated using the final estimated values of the parameters, which were evaluated using all of the data points.

From Fig. 5.28 (lower), the normalized joint probabilities indicate that Model 1, Model 4, and Model 5 become possible models during the sequential procedure. First, the normalized joint probability associated with Model 1 is close to one before 10 hours, since neither the double porosity effects nor the boundary effects have become significant. Next, the normalized joint probability associated with Model 4 is close to one between 20 hours and 200 hours, since the double porosity effects dominate

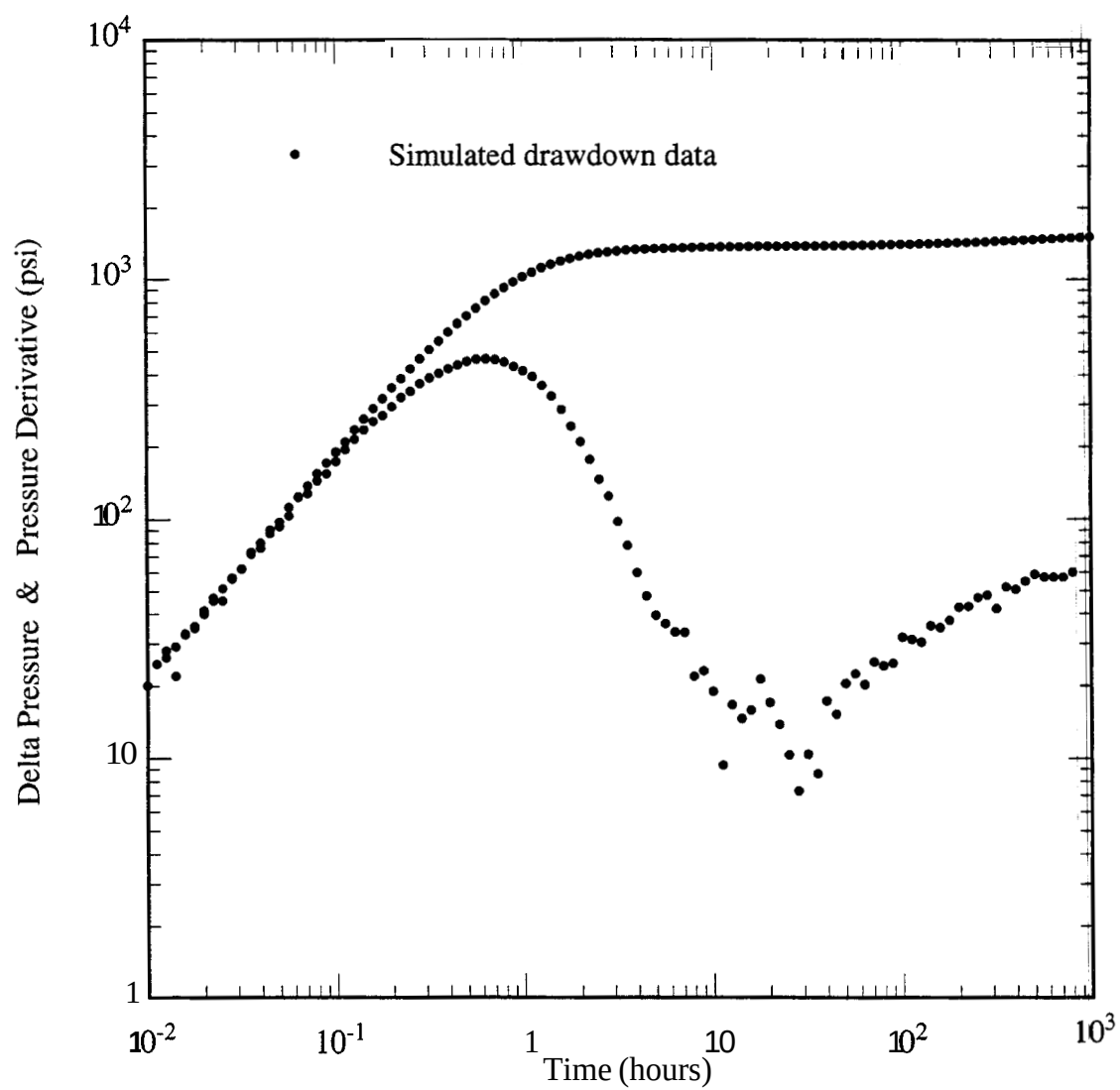


Figure 5.27: Simulated drawdown data using a double porosity and sealing fault model (Model 5) with normal random errors

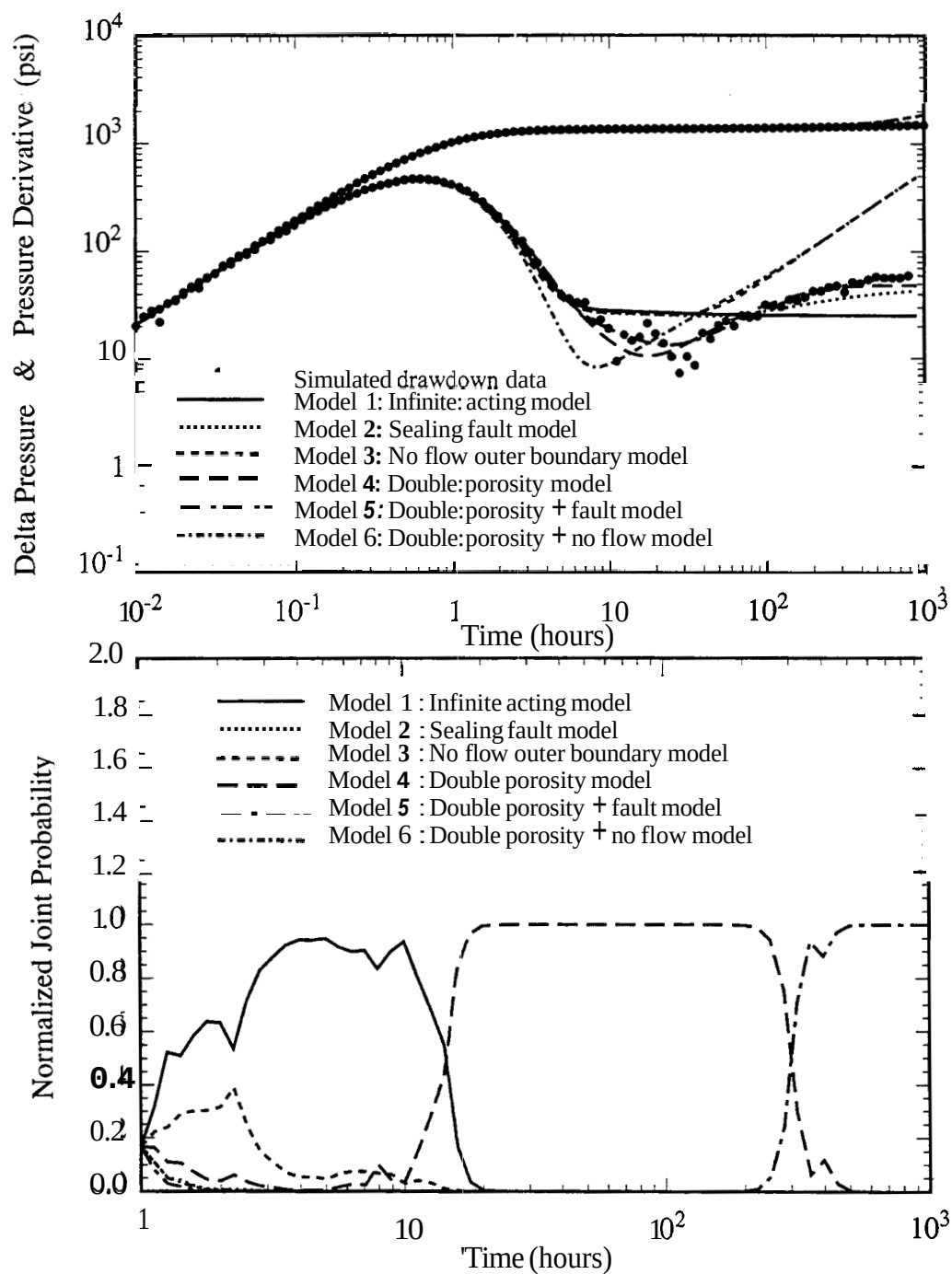


Figure 5.28: Final matches of Model 1, Model 2, Model 3, Model 4, Model 5, and Model 6 to the data (upper) and normalized joint probability associated with each model (lower)

the pressure responses but the boundary effects are not yet significant during this period. Finally, the normalized joint probability associated with Model 5 is close to one after 500 hours, since the boundary effects due to the sealing fault boundary become significant. At the final stage, the sequential predictive probability method clearly selects Model 5, which is the true model. The possibilities of other models which are Model 2, Model 3, and Model 6 are always low. Therefore, the sequential predictive probability method produces results consistent with pressure behavior, in cases where both the double porosity effects and the boundary effects due to the sealing fault boundary exist.

Next, the double porosity and no flow outer boundary model was employed as the true model. Pressure data are shown in Fig. 5.29. During intermediate time the double porosity behavior is present and at late time the boundary effects due to the no flow outer boundary appear on the pressure derivative plot. In this example, six models, which were the infinite acting model (Model 1), the sealing fault model (Model 2), the no flow outer boundary model (Model 3), the double porosity model (Model 4), the double porosity and sealing fault model (Model 5), and the double porosity and no flow outer boundary model (Model 6), were employed as candidate reservoir models. Model 1 has three parameters, which are k , S , and C . Model 2 and Model 3 have four parameters, which are k , S , C , and r_e . Model 4 has five parameters, which are k , S , C , w , and A . Model 5 and Model 6 have six parameters, which are k , S , C , w , A , and r_e . The inormalized joint probability associated with each model at 1.0 hour is set to be equal to $1/6$. Final matches of the models to the data and the corresponding normalized joint probabilities are shown in Fig. 5.30. Final matches of the models to the data were calculated using the final estimated values of the parameters, which were evaluated using all of the data points.

From Fig. 5.30 (lower), the normalized joint probabilities indicate that Model 1, Model 4, and Model 6 become possible models during the sequential procedure. First, the normalized joint probability associated with Model 1 is close to one before 15 hours, since neither the double porosity effects nor the boundary effects become significant. Next, the normalized joint probability associated with Model 4 is close

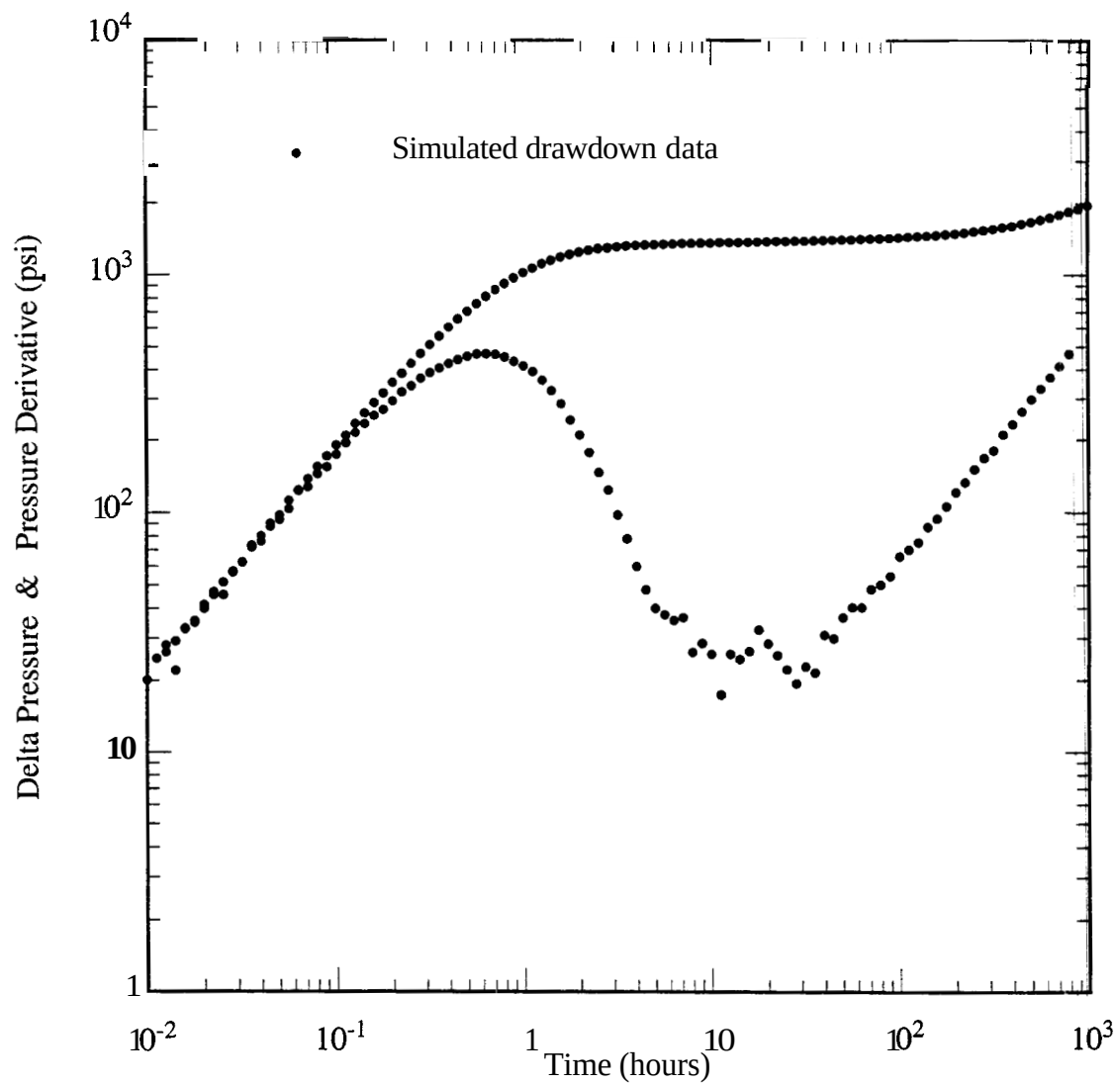


Figure 5.29: Simulated drawdown data, using a double porosity and no flow outer boundary model (Model 6) with normal random errors

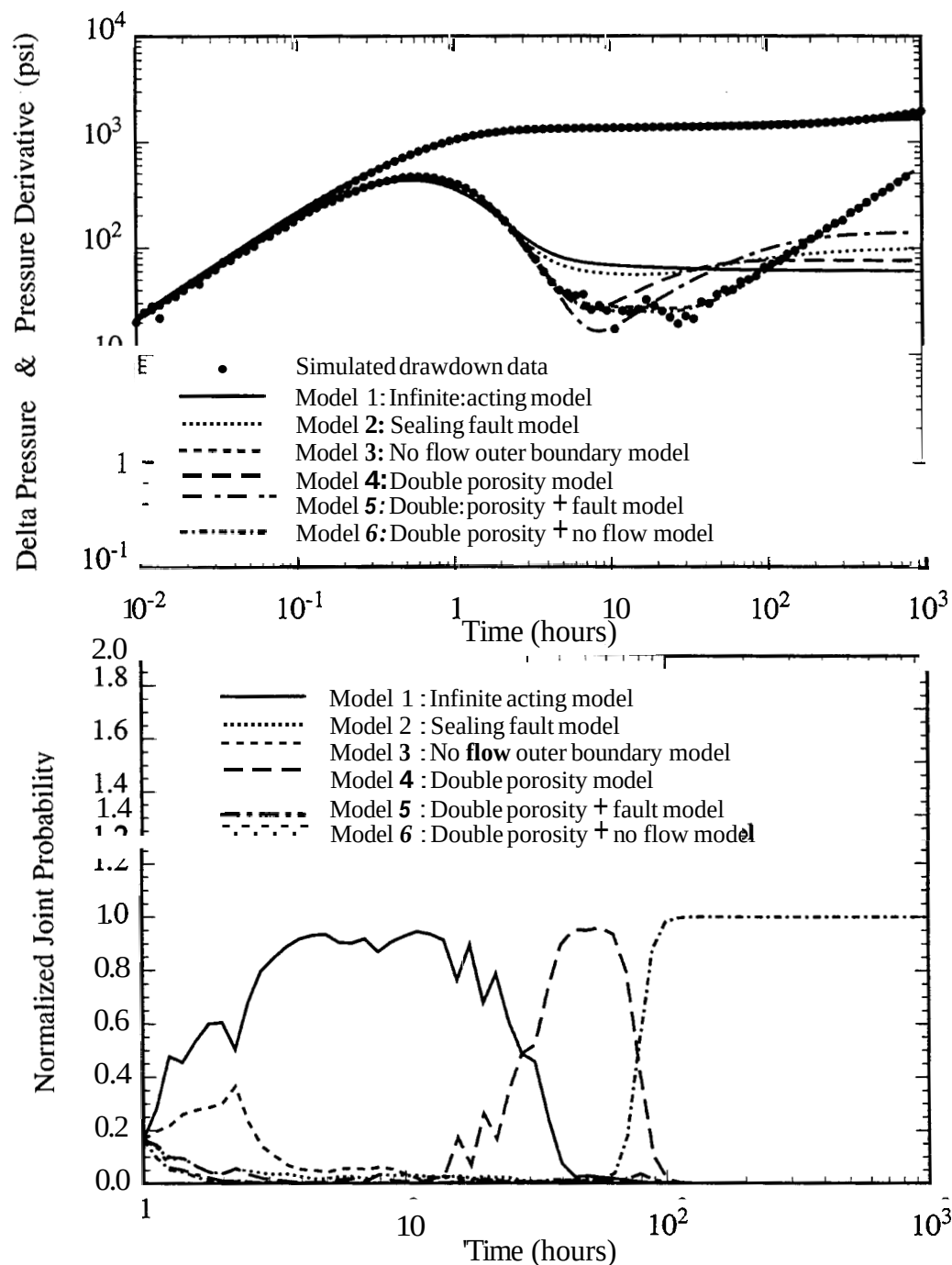


Figure 5.30: Final matches of Model 1, Model 2, Model 3, Model 4, Model 5, and Model 6 to the data (upper) and normalized joint probability associated with each model (lower)

to one between 40 hours and 60 hours, since the double porosity effects dominate the pressure responses but the boundary effects are not yet significant during this period. Finally, the normalized joint probability associated with Model 6 is close to one after 100 hours, since the boundary effects due to the no flow outer boundary become significant. At the final stage, the sequential predictive probability method clearly selects Model 6, which is the true model. The possibilities of other models which are Model 2, Model 3, and Model 5 are always low. Therefore, the sequential predictive probability method produces results consistent with pressure behavior, in cases where both the double porosity effects and the boundary effects due to the no flow outer boundary exist.

Now, it is interesting to compare Fig. 5.28 (lower) with Fig. 5.30 (lower). The normalized joint probability associated with the double porosity and sealing fault model reaches one after 500 hours in Fig. 5.28 (lower), while that with the double porosity and no flow outer boundary model reaches one after 100 hours in Fig. 5.30 (lower). Hence, in conjunction with the results shown in Fig. 5.20 (lower) and Fig. 5.22 (lower), the detection of the sealing fault requires more data points which contain the sealing fault effects than in the case of the no flow outer boundary, whether the double porosity behavior exists or not.

Finally, the double porosity and constant pressure outer boundary model was employed as the true model. Pressure data are shown in Fig. 5.31. During intermediate time the double porosity behavior is present and at late time the boundary effects due to the constant pressure outer boundary appear on the pressure derivative plot. In this example, four models, which were the infinite acting model (Model 1), the constant pressure outer boundary model (Model 2), the double porosity model (Model 3), and the double porosity and constant pressure outer boundary model (Model 4), were employed as candidate reservoir models. Model 1 has three parameters, which are k , S , and C . Model 2 has four parameters, which are k , S , C , and r_e . Model 3 has five parameters, which are ℓ , S , C , ω , and λ . Model 4 has six parameters, which are ℓ , S , C , ω , λ , and r_e . The normalized joint probability associated with each model at 1.0 hour is set to be equal to $1/4$. Final matches of the models to the data

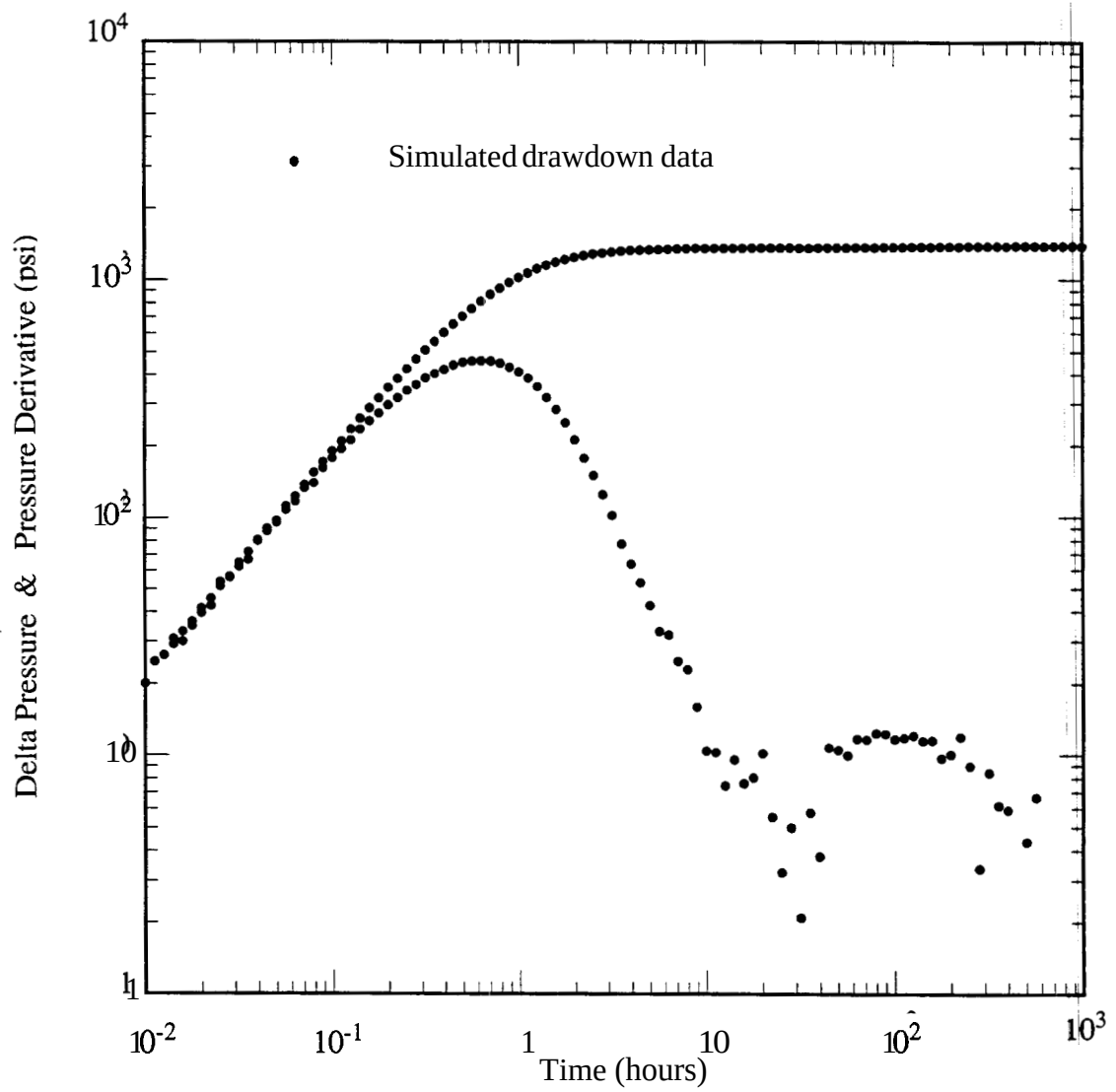


Figure 5.31: Simulated drawdown data using a double porosity and constant pressure outer boundary model (Model **4**) with normal random errors

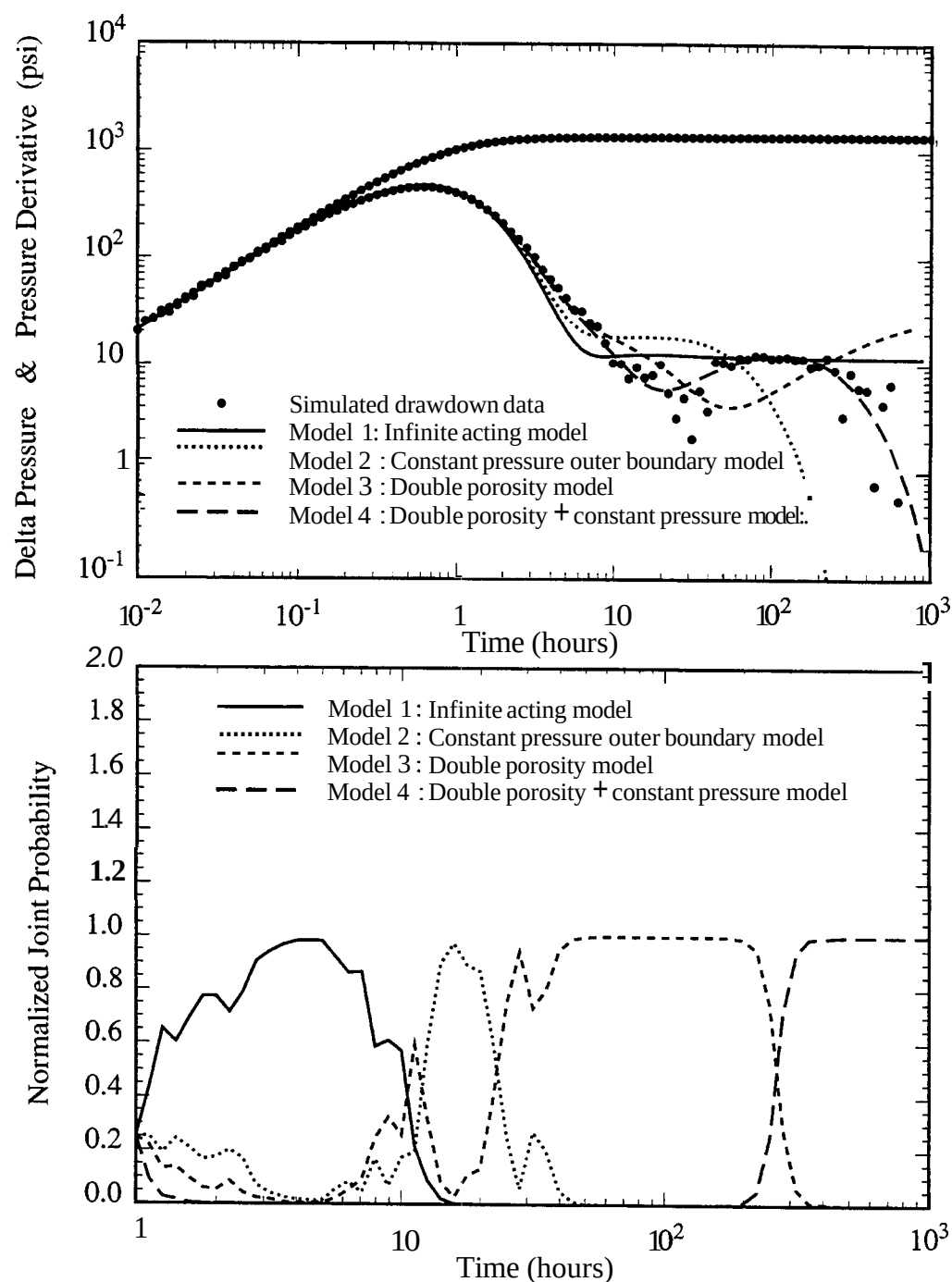


Figure 5.32: Final matches of Model 1, Model 2, Model 3, and Model 4 to the data (upper) and normalized joint probability associated with each model (lower)

and the corresponding normalized joint probabilities are shown in Fig. 5.32. Final matches of the models to the data were calculated using the final estimated values of the parameters, which were evaluated using all of the data points.

From Fig. 5.32 (lower), the normalized joint probabilities indicate that all models become possible models during the sequential procedure. Firstly, the normalized joint probability associated with Model 1 is close to one before 7 hours, since neither the double porosity effects nor the boundary effects have become significant. Secondly, the normalized joint probability associated with Model 2 is close to one between 15 hours and 20 hours. This behavior is explained by the same reasons discussed in Fig. 5.26 (lower). Thirdly, the normalized joint probability associated with Model 3 is close to one between 40 hours and 200 hours, since the double porosity effects dominate the pressure responses but the boundary effects are not yet significant during this period. Finally, the normalized joint probability associated with Model 4 is close to one after 300 hours, since the boundary effects due to the constant pressure outer boundary become significant. At the final stage, the sequential predictive probability method clearly selects Model 4, which is the true model. Therefore, the sequential predictive probability method produces results consistent with pressure behavior, in cases where both the double porosity effects and the boundary effects due to the constant pressure outer boundary exist.

As discussed above, the sequential predictive probability method can discriminate between possible reservoir models even in cases where the true models are complex. This provides the basis for future work to extend the utility of the sequential predictive probability method to reservoir models other than the eight basic models which were employed in this study. Moreover, during the procedure the sequential predictive probability method provides quantitative verification for model recognition. For instance, Fig. 5.30 (lower) indicates that the flow regime before 15 hours is dominated by the infinite acting behavior, the flow regime between 40 hours and 60 hours is dominated by the double porosity behavior, and the flow regime after 100 hours is dominated by the no flow outer boundary effects.

5.4 Application to Actual Field Well Test Data.

In this section, application of the sequential predictive probability method to actual field well test data is discussed. Three actual field well test data were examined. Each serves a special purpose to demonstrate different features of the sequential predictive probability method.

5.4.1 Case 1: Multirate Pressure Data

These field multirate pressure data were obtained from Bourdet *et al.* (1983b). The test was a multirate test in which the well was flowed at three different rates prior to shut-in. Bourdet *et al.* (1983b) reported that, the most likely reservoir model is the double porosity model. Allain and Horne (1990) applied an AI approach using a rule-based expert system to the data and reported that the sealing fault model and the double porosity model are possible reservoir models. Therefore, two models, the sealing fault model (Model 1) and the double porosity model (Model 2), were compared using the sequential predictive probability method. Theoretically, the sequential predictive probability method can discriminate between reservoir models irrespective of the flow history. However, only drawdown pressure data were employed in Section 5.3. Therefore, a second purpose in considering this case was to investigate the application of the sequential predictive probability method to multirate data.

The data are plotted in Fig. 5.33. The total number of data points is 179 and the time period is from 0.0024883 hour to 17.6059 hours. The data at 0.100618 hour just after the peak of the pressure derivative plot was selected as the starting point for the sequential predictive probability procedure. The number of data points between 0.0024883 hour and 0.100618 hour is 70. Model 1 has four parameters, which are k , S , C , and r_e , while Model 2 has five parameters, which are k , S , C , w , and λ . The normalized joint probability associated with each model at 0.100618 hour is set to be 1/2. Pressure responses were calculated exactly using the superposition in time, instead of using the approximation of the Agarwal equivalent time. Final matches of the models to the data and the corresponding, normalized joint probabilities are shown in Fig. 5.34. Final matches of the models to the data were calculated using

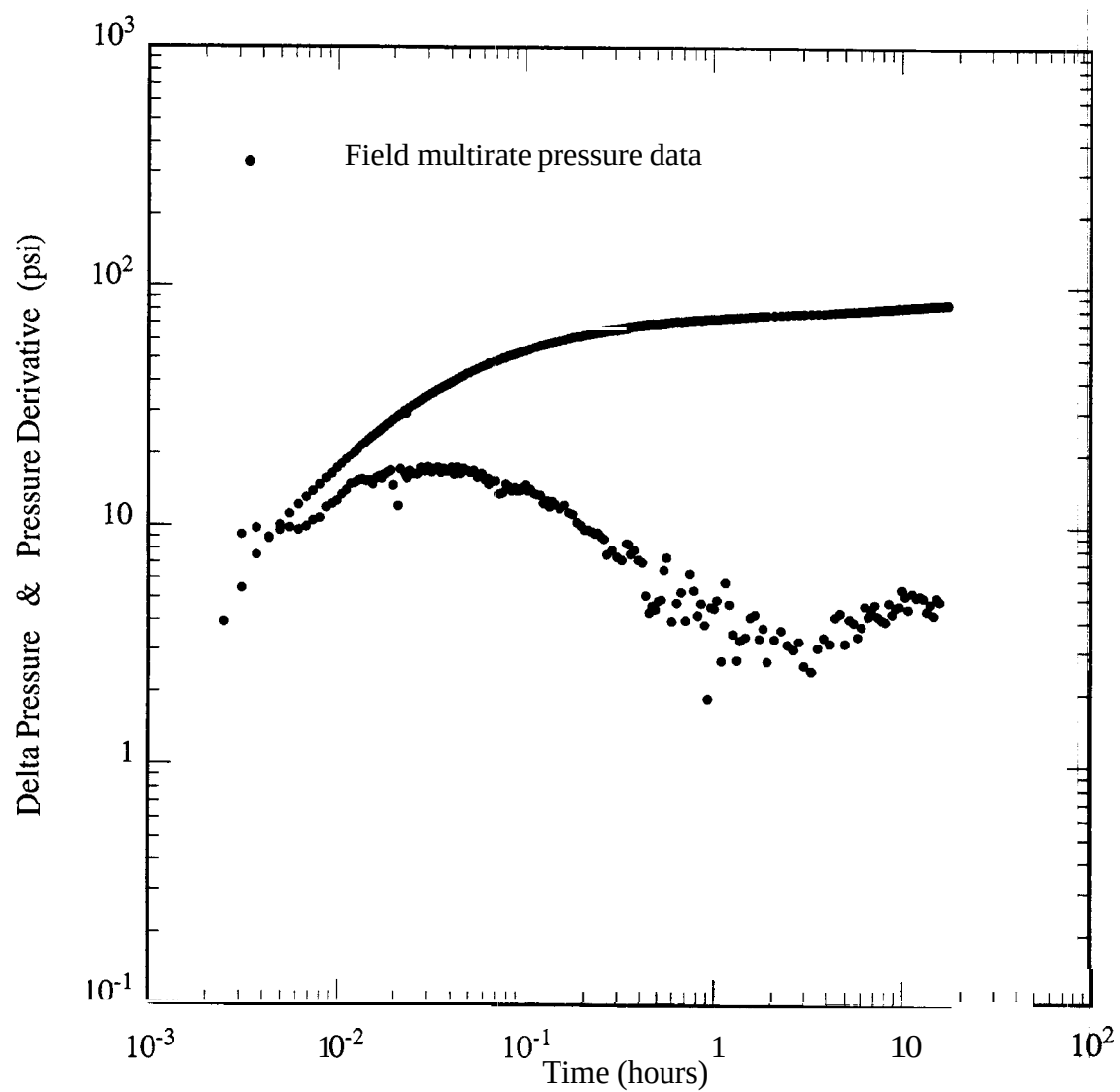


Figure 5.33: Field multirate pressure data from Bourdet *et al.* (1983b)

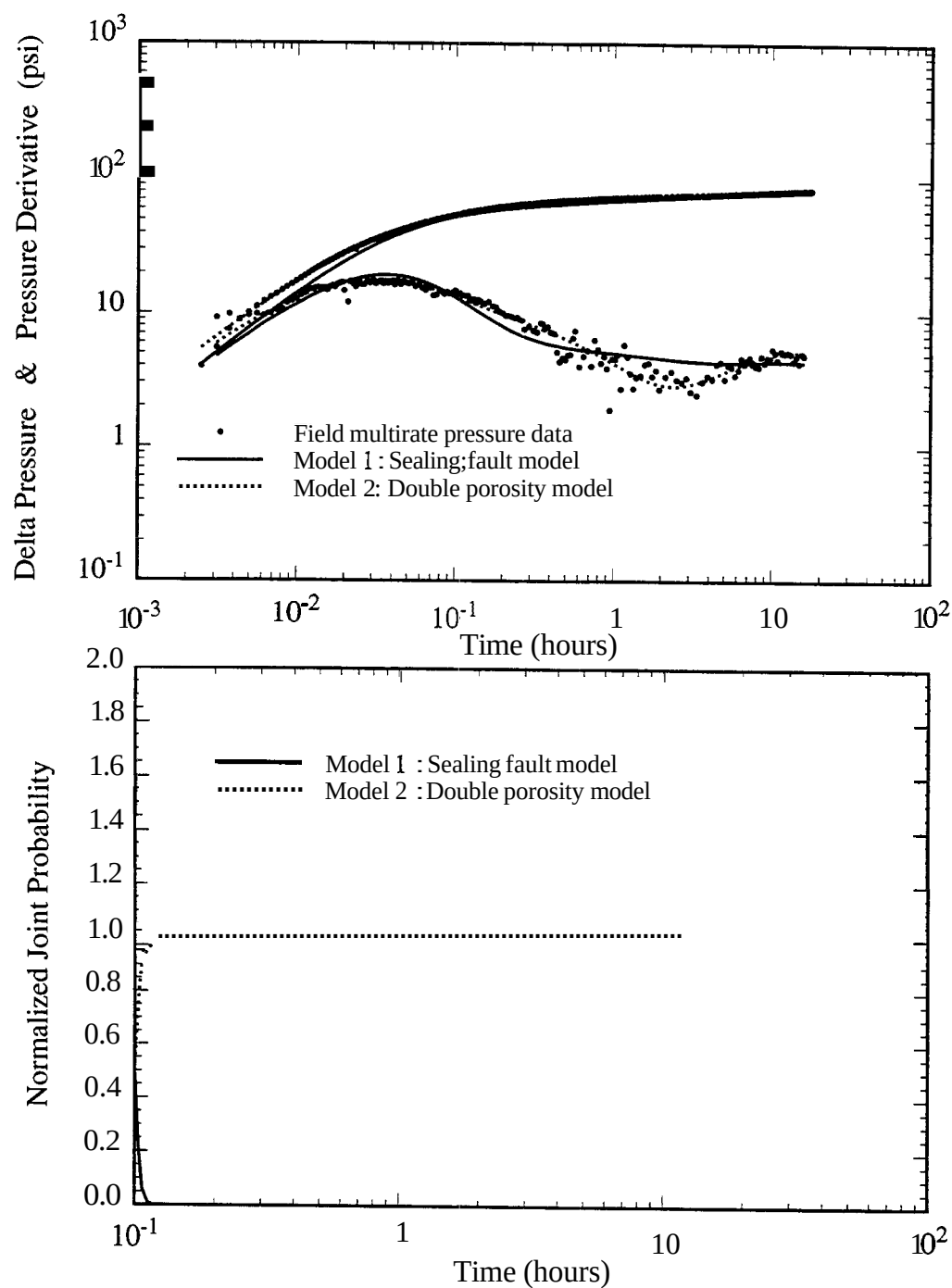


Figure 5.34: Final matches of Model 1 and Model 2 to the data (upper) and normalized joint probability associated with each model (lower)

the final estimated values of the parameters, which were evaluated using all of the data points.

Fig. 5.34 (upper) shows that Model 2 fits the data better than Model 1 on the pressure derivative plot. Fig. 5.34 (lower) shows that the normalized joint probability associated with Model 2 reaches one rapidly during the procedure. Hence, the sequential predictive probability method clearly selects Model 2, which was believed to be the true model. This result is consistent with graphical analysis.

Therefore, the sequential predictive probability method can select the true model in cases where multirate data are employed.

5.4.2 Case 2: Drawdown Pressure Data

These field drawdown pressure data were obtained from Da Prat (1990). Da Prat (1990) reported that the most likely reservoir model is the double porosity model. As discussed in Section 5.3.1, it is sometimes necessary to discriminate between the double porosity model and the no flow outer boundary model. Therefore, one purpose in considering this case was to discriminate between the double porosity model and the no flow outer boundary model in the case where the double porosity model is believed to be a correct model. A second purpose was to examine whether boundary effects appear after the double porosity transition period. Therefore, four models were compared: the no flow outer boundary model (Model 1), the double porosity model (Model 2), the double porosity and sealing fault model (Model 3), and the double porosity and no flow outer boundary model (Model 4).

The data are plotted in Fig. 5.35. The total number of data points is 63 and the time period is from 0.25 hour to 178 hours. The data at 10 hours is selected as the starting point for the sequential predictive probability procedure. The number of data points between 0.25 hour and 10 hours is 20. As discussed in Section 5.1.2, the starting time should be selected around the wellbore storage hump in the pressure derivative plot to retain enough data to confirm the adequacy of the model fits over all time range. However, in this case only 5 data points are available before the wellbore storage hump and these data points are not sufficient to obtain reliable parameter estimates at the start of the procedure. Therefore, the starting time was selected at

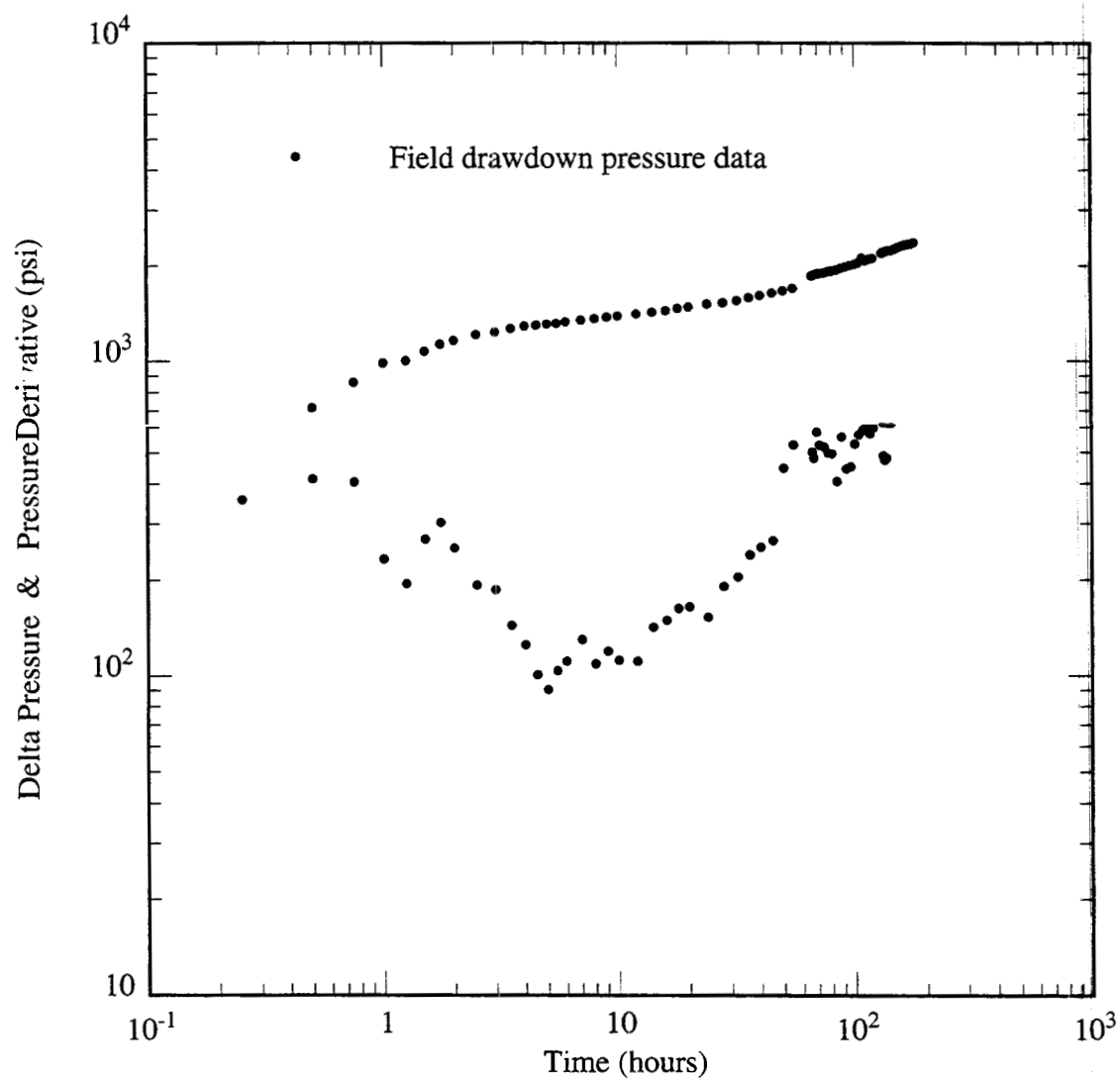


Figure 5.35: Field drawdown pressure data from Da Prat (1990)

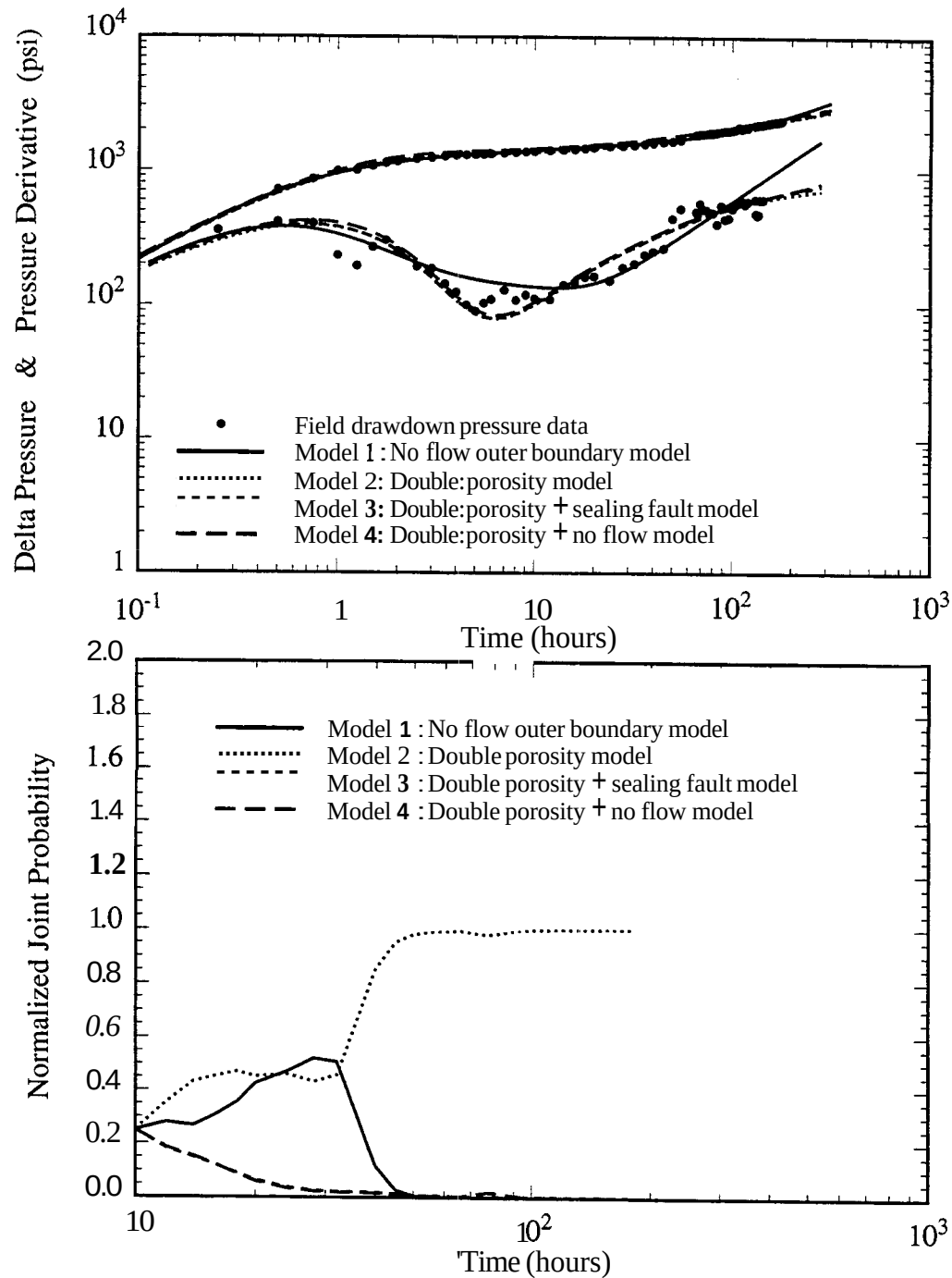


Figure 5.36: Final matches of Model 1, Model 2, Model 3, and Model 4 to the data (upper) and normalized joint probability associated with each model (lower)

10 hours to include enough data points. Model 1 has four parameters, which are k , S , C , and r_e . Model 2 has five parameters, which are k , S , C , ω , and λ . Model 3 and Model 4 have six parameters, which are k , S , C , w , λ , and r_e . The normalized joint probability associated with each model at 10 hours is set to be 1/4. Final matches of the models to the data and the corresponding normalized joint probabilities are shown in Fig. 5.36. Final matches of the models to the data were calculated using the final estimated values of the parameters, which were evaluated using all of the data points.

Fig. 5.36 (upper) shows that Model 2, Model 3, and Model 4 seem to fit the data more or less equally on the pressure derivative plot. In Fig. 5.36 (lower), Model 1 and Model 2 have similar normalized joint probabilities up to 35 hours, and thereafter the normalized joint probability associated with Model 2 increases up to one. The normalized joint probabilities associated with Model 3 and Model 4 are always low. Hence, the sequential predictive probability method clearly selects Model 2, which was believed to be the correct model, and indicates that no boundary effects are apparent.

From Fig. 5.36 (lower), it is not possible to decide which model is better between 10 hours and 35 hours, Model 1 or Model 2. Therefore, in cases where the double porosity model is the correct model, discrimination between the no flow outer boundary model and the double porosity model is important during the later half of the double porosity transition period.

5.4.3 Case 3: Buildup Pressure Data

These field buildup pressure data were obtained from Vieira and Rosa (1993). The data are plotted in Fig. 5.37. In Fig. 5.37, a downward trend is recognized at the end of the pressure derivative plot. Even though Vieira and Rosa (1993) reported that an adequate reservoir model is the infinite acting model from the match using the Agarwal equivalent time, the Agarwal equivalent time inherently assumes infinite acting behavior. Therefore, the sequential predictive probability method was used to examine whether the downward trend is caused by the buildup superposition effects or the boundary effects. Two models, the infinite acting model (Model 1) and the

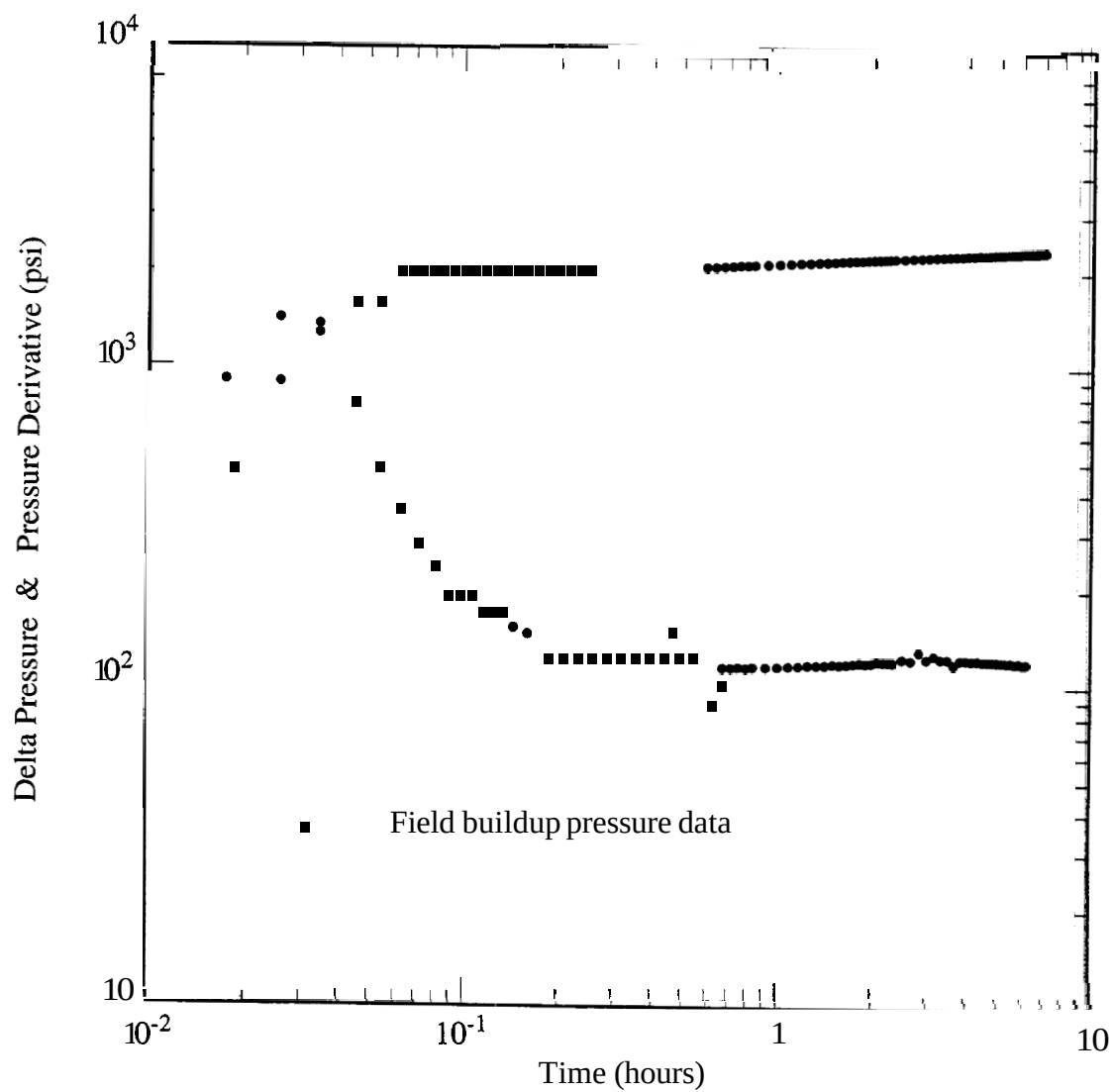


Figure 5.37: Field buildup pressure data from Vieira and Rosa (1993)

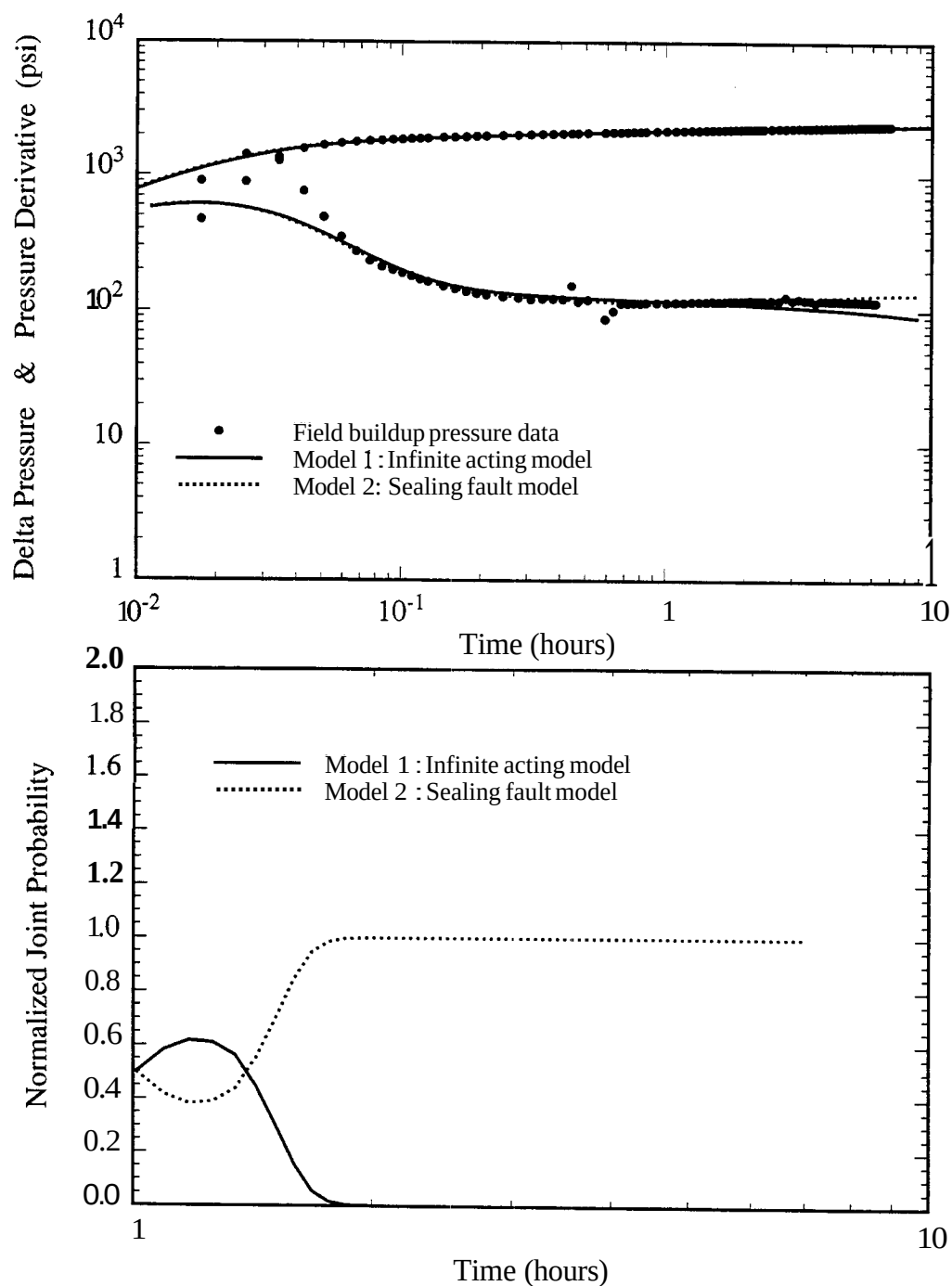


Figure 5.38: Final matches of Model 1 and Model 2 to the data (upper) and normalized joint probability associated with each model (lower)

sealing fault model (Model 2), were compared.

The total number of data points is 73 and the time period is from 0.0842 hour to 7.0092 hours. The data at 1.0092 hours was selected as the starting point. The number of data points between 0.0842 hour and 1.0092 hours is 29. In this case only 3 data points are available before the wellbore storage hump. Therefore, for the same reasons discussed in Case 2, the starting time was selected at 1.0092 hours to include sufficient data points for reliable parameter estimates at the start of the procedure. Model 1 has three parameters, which are k , S , and C , while Model 2 has four parameters, which are k , S , C , and r_e . The normalized joint probability associated with each model at 1.0092 hours is set to be $1/2$. Pressure responses were calculated exactly using the superposition in time, instead of using the approximation of the Agarwal equivalent time. Final matches of the models to the data and the corresponding normalized joint probabilities are shown in Fig. 5.38. Final matches of the models to the data were calculated using the final estimated values of the parameters, which were evaluated using all of the data points.

Fig. 5.38 (lower) shows that Model 2 seems to fit the data but Model 1 deviates from the data after 2 hours. Therefore, graphical analysis implies the existence of a sealing fault. Fig. 5.38 (lower) shows that the normalized joint probability associated with Model 2 reaches one after 2 hours. Hence, the sequential predictive probability method clearly selects Model 2. This result indicates that the sealing fault may exist and the downward trend on the pressure derivative plot in Fig. 5.37 may be caused by the sealing fault effects. This result confirms the results implied by graphical analysis.

This case shows the potential of the sequential predictive probability method to correctly interpret actual field well test data. In cases where graphical analysis implies the existence of a boundary, the sequential predictive probability method provides quantitative verification.

Chapter 6

Conclusions and Recommendations

discriminate between candidate reservoir models. The method employed in this study

1. Bayesian inference makes it possible to express uncertainty involved in the estimated reservoir parameters in terms of probability.

2. Confidence intervals were derived in the framework of Bayesian inference and the problems inherently involved in their application to model discrimination were discussed. Models with different numbers of parameters have probability distributions for the parameters with different dimensions. Therefore, different models are inherently incomparable in the parameter space, and confidence interval analysis alone is not sufficient to compare probability distributions with different dimensions.
3. The sequential predictive probability method was derived and implemented for effective use in model discrimination in well test analysis. Eight fundamental models, which are the infinite acting model, the sealing fault model, the no flow outer boundary model, the constant pressure outer boundary model, the double porosity model, the double porosity and sealing fault model, the double porosity and no flow outer boundary model. and the double porosity and constant pressure outer boundary model, were employed in this study and the sequential predictive probability method was shown to be able to discriminate effectively between these models.
4. The sequential predictive probability method provides a unified measure of model discrimination, which is the normalized joint probability, regardless of the number of the parameters in reservoir models, and can compare any number of reservoir models simultaneously.
5. The sequential predictive probability method can be used to identify the flow regimes. Even in cases where the results obtained by the sequential predictive probability method are not conclusive, the behavior of the normalized joint probability provides some information about the model.
6. The modified sequential predictive probability method was introduced as a way to reduce the computation cost. Although the modified sequential predictive probability method and the sequential predictive probability method produce similar results, the modified sequential predictive probability method does not reveal the behavior of the normalized joint probability during the procedure.

Therefore, in cases where the results are not conclusive, the sequential predictive probability method is preferable.

7. For practical application of the sequential predictive probability method for model discrimination, the following two situations need to be scrutinized carefully: one is the discrimination between the constant pressure outer boundary model and the double porosity model at the first half of the double porosity transition period and the other is the discrimination between the no flow outer boundary model and the double porosity model at the later half of the double porosity transition period.

Throughout this study, it has been found that the sequential predictive probability method is a promising technique to discriminate between possible reservoir models in well test analysis. The following future work is recommended:

1. It is straightforward to extend the utility of the sequential predictive probability method to other reservoir models than the eight fundamental models employed in this study. Theoretically, the sequential predictive probability method can discriminate between any number of reservoir models as long as the reservoir models can be expressed approximately as a linear form with respect to the reservoir parameters. As discussed in Section 3.1, the composite model and the multilayered model could be candidate models for future research.
2. In this work, the Gauss-Marquardt method with penalty function and interpolation and extrapolation technique was employed as the nonlinear regression algorithm. The inverse Hessian matrix and the gradient are key components for the sequential predictive probability method. Therefore, any other type of Newton's method could be implemented for the sequential predictive probability method.

Nomenclature

B_o	=	Formation volume factor, bbl/STB
C	=	Wellbore storage constant, STB/psi
C_D	=	Dimensionless wellbore storage constant
c_t	=	Total system compressibility, psi^{-1}
E	=	Objective function
E^*	=	Approximate objective function
f	=	Function F in Laplace space
F	=	Reservoir model function
$F_{1-\alpha}$	=	Tabulated F value at $1 - \frac{\alpha}{2}$ confidence level
$\mathbf{g}$	=	Gradient vector
h	=	Formation thickness, ft
H	=	Hessian matrix
H^{-1}	=	Inverse of Hessian matrix
J	=	Jacobian
k	=	Permeability, md
L	=	Likelihood function
L^{-1}	=	Inverse Laplace operator
m	=	Number of parameters
M	=	Stehfest sampling number
M	=	Mobility ratio defined by Eq. 3.2
n	=	Number of data
N	=	Stehfest sampling number
n_m	=	Number of models

p	=	Pressure, <i>psi</i>
Δp	=	Delta pressure, <i>psi</i>
$\delta p'$	=	Pressure derivative
p_D	=	Dimensionless pressure
p_i	=	Initial pressure, <i>psi</i>
P	=	Probability
<i>Prob</i>	=	Probability
q	=	Flow rate, <i>STB/D</i>
r	=	Distance from the wellbore, <i>ft</i>
r_e	=	Distance to the outer boundary, <i>ft</i>
r_w	=	Wellbore radius, <i>ft</i>
$\mathbf{R}$	=	Residual vector
S	=	Skin factor, dimensionless
S	=	Entropy defined by Eq. 4.21
S_f	=	Skin effect at the discontinuity
s^2	=	Error mean square, <i>psi</i> ²
<i>SSR</i>	=	Sum of squared residuals
t	=	Time, <i>hour</i>
Δt	=	Delta time, <i>hour</i>
t_D	=	Dimensionless time
t_{DA}	=	Dimensionless time based upon reservoir area
$t_{1-\alpha}$	=	Tabulated student t value at $1 - \frac{\alpha}{2}$ confidence level
V	=	Covariance matrix
x_f	=	Fracture length, <i>ft</i>
x	=	Observed dependent variable
y	=	Observed independent variable
$\hat{y}$	=	Estimated value of y
y^*	=	True value of y
z	=	Argument of a function in Laplace space

GREEK LETTERS

α	=	Confidence level
α	=	Interporosity flow shape factor defined by Eq. 3.5
Γ	=	Gamma function
ϵ_i	=	Error, <i>psi</i>
θ	=	Parameter vector
θ	=	Unknown parameter
θ	=	Estimated value of parameter
θ^*	=	True value of parameter
λ	=	Transmissivity ratio, dimensionless
μ	=	Viscosity, <i>cp</i>
ν	=	Degrees of freedom
Π	=	Joint probability
ρ	=	Correlation coefficient
σ^2	=	Variance of error, <i>psi</i> ²
σ_p^2	=	Predictive variance, <i>psi</i> ²
σ_θ^2	=	variance of parameter estimate, <i>psi</i> ²
ϕ	=	Porosity, dimensionless
ω	=	Storativity ratio, dimensionless

SUBSCRIPTS

D	=	Dimensionless
f	=	Fracture defined by Eq. 3.4
i, j, k	=	Integer indices
m	=	Matrix defined by Eq. 3.4

SUPERSCRIPTS

0	=	Initial guess
-1	=	Inverse
(n)	=	Number of data
T	=	Transpose

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Appendix A

Derivatives With Respect To Parameters

This appendix presents the derivatives of the reservoir models with respect to the reservoir parameters. Eight fundamental reservoir models, which are the infinite acting model, the sealing fault model, the no flow outer boundary model, the constant pressure outer boundary model, the double porosity model, the double porosity and sealing fault model, the double porosity and no flow outer boundary model, and the double porosity and constant pressure outer boundary model, are employed. Field units are used.

A.1 Dimensionless Variables

Dimensionless variables are defined as follows:

The dimensionless pressure p_D is defined in field units as:

$$p_D = \frac{kh}{141.2qB\mu} (p_i - p_{wf}) \quad (\text{A.1})$$

where

- k = permeability, *md*
 h = thickness, *ft*
 p_i = initial reservoir pressure, *psi*
 p_{wf} = well flowing pressure, *psi*
 q = production rate, *STB/D*
 B = formation volume factor, *res vol/std vol*
 μ = viscosity, *cp*

In consistent unit sets, p_D is defined as:

$$p_D = \frac{2\pi kh}{qB\mu} (p_i - p_{wf}) \quad (\text{A.2})$$

The dimensionless time t_D is defined in field units as:

$$t_D = \frac{0.000264kt}{\phi\mu c_t r_w^2} \quad (\text{A.3})$$

where

- t = time, *hours*
 ϕ = porosity, *pore volume/bulk volume*
 c_t = total system compressibility, *psi⁻¹*
 r_w = wellbore radius, *ft*

In consistent unit sets, t_D is defined as:

$$t_D = \frac{kt}{\phi\mu c_t r_w^2} \quad (\text{A.4})$$

The dimensionless time based on reservoir size, t_{DA} is defined as:

$$t_{DA} = \frac{0.000264kt}{\phi\mu c_t A} \quad (\text{A.5})$$

where

- A = reservoir area, *ft²*
 r_e = reservoir radius, *ft*

There is a direct relationship between t_D and t_{DA} :

$$t_D = t_{DA} \frac{A}{r_w^2} = t_{DA} \pi \frac{r_e^2}{r_w^2} \quad (\text{A.6})$$

The dimensionless radius r_D is defined as:

$$r_D = \frac{r}{r_w} \quad (\text{A.7})$$

This definition is independent of any particular set of units.

The dimensionless radius r_{eD} is defined as:

$$\bar{r}_{eD} = \frac{r_e}{r_w} \quad (\text{A.8})$$

The dimensionless wellbore storage constant C_D is defined in field units as:

$$C_D = \frac{5.615C}{2\pi\phi c_t h r_w^2} \quad (\text{A.9})$$

where C is in STB/psi .

A.2 Reservoir Models

A.2.1 Infinite Acting Model

The infinite acting model has three parameters, which are k , S , and C .

The dimensionless wellbore pressure for the infinite acting model is given in Laplace space by:

$$\bar{p}_D = \frac{K_0(\sqrt{z}) + S\sqrt{z}K_1(\sqrt{z})}{z[\sqrt{z}K_1(\sqrt{z}) + C_D z[K_0(\sqrt{z}) + S\sqrt{z}K_1(\sqrt{z})]]} \quad (\text{A.10})$$

In nonlinear regression, the derivatives with respect to actual values of parameters need to be calculated, and the equations are expressed in field units.

To convert the dimensionless forms into the forms in field units, the following two theorems of Laplace transformation are used:

$$a \cdot L\{F(t)\} = a \cdot f(z) \quad (\text{A.11})$$

$$L\{F(at)\} = \frac{1}{a} f\left(\frac{z}{a}\right) \quad (\text{A.12})$$

For simplicity, the following symbols are used:

$$\Delta p = p_i - p_{wj} \quad (\text{A.13})$$

$$\eta = \frac{k}{\phi \mu c_t} \quad (\text{A.14})$$

$$u = \frac{r_w^2}{0.000264 \eta} z \quad (\text{A.15})$$

$$c_1 = \frac{h}{141.2 \mu} \quad (\text{A.16})$$

Then, the wellbore pressure for the infinite acting model in field units is given in Laplace space by:

$$\overline{\Delta p} = \frac{qB}{z} \cdot \frac{A_1}{A_2} \quad (\text{A.17})$$

where

$$A_1 = K_0(\sqrt{u}) + S\sqrt{u}K_1(\sqrt{u}) \quad (\text{A.18})$$

$$A_2 = c_1 k \sqrt{u} K_1(\sqrt{u}) + 24Cz \cdot Ai \quad (\text{A.19})$$

The derivative of the wellbore pressure with respect to permeability is given by:

$$\frac{\partial \overline{\Delta p}}{\partial k} = \frac{qB}{z} \cdot \frac{A_3 \cdot A_2 - A_1 \cdot A_4}{A_2 \cdot A_2} \quad (\text{A.20})$$

where

$$\begin{aligned} A_3 &= \frac{\partial(A_1)}{\partial k} \\ &= \frac{\sqrt{u}}{2k} \cdot [K_1(\sqrt{u}) + S\sqrt{u}K_0(\sqrt{u})] \end{aligned} \quad (\text{A.21})$$

$$\begin{aligned} A_4 &= \frac{\partial(A_2)}{\partial k} \\ &= c_1 \sqrt{u} K_1(\sqrt{u}) + c_1 \frac{\sqrt{u}}{2} \sqrt{u} K_0(\sqrt{u}) + 24Cz \cdot A_3 \end{aligned} \quad (\text{A.22})$$

The following relationships are used:

$$\frac{\partial f(g)}{\partial \theta} = \frac{\partial (f(g))}{\partial g} \cdot \frac{\partial g}{\partial \theta} \quad (\text{A.23})$$

$$\frac{\partial \sqrt{g}}{\partial \theta} = \frac{1}{2\sqrt{g}} \cdot \frac{\partial g}{\partial \theta} \quad (\text{A.24})$$

$$\frac{\partial (f \cdot g)}{\partial \theta} = f' \cdot g + f \cdot g' \quad (\text{A.25})$$

$$\frac{\partial}{\partial \theta} \left(\frac{g}{f} \right) = \frac{g' \cdot f - g \cdot f'}{f \cdot f} \quad (\text{A.26})$$

The derivatives of the Modified Bessel functions are given by:

$$\frac{dK_0(\theta)}{d\theta} = -K_1(\theta) \quad (\text{A.27})$$

$$\frac{dK_1(\theta)}{d\theta} = -\left[\frac{1}{\theta} K_1(\theta) + K_0(\theta) \right] \quad (\text{A.28})$$

$$\frac{dI_0(\theta)}{d\theta} = I_1(\theta) \quad (\text{A.29})$$

$$\frac{dI_1(\theta)}{d\theta} = \left[\frac{1}{\theta} I_1(\theta) - I_0(\theta) \right] \quad (\text{A.30})$$

The following relationships are important:

$$\frac{\partial u}{\partial k} = -\frac{u}{k} \quad (\text{A.31})$$

$$\frac{\partial \sqrt{u}}{\partial k} = -\frac{\sqrt{u}}{2k} \quad (\text{A.32})$$

$$\frac{\partial}{\partial k} (K_0(\sqrt{u})) = \frac{\sqrt{u}}{2k} K_1(\sqrt{u}) \quad (\text{A.33})$$

$$\frac{\partial}{\partial k} (K_1(\sqrt{u})) = \frac{\sqrt{u}}{2k} \left[\frac{1}{\sqrt{u}} K_1(\sqrt{u}) + K_0(\sqrt{u}) \right] \quad (\text{A.34})$$

$$\frac{\partial}{\partial k} (\sqrt{u} K_1(\sqrt{u})) = \frac{\sqrt{u}}{2k} \sqrt{u} K_0(\sqrt{u}) \quad (\text{A.35})$$

The derivative of the wellbore pressure with respect to skin is given by:

$$\frac{\partial \overline{\Delta p}}{\partial S} = \frac{qB}{z} \cdot \frac{A_5 \cdot A_2 - A_1 \cdot A_6}{A_2 \cdot A_2} \quad (\text{A.36})$$

where

$$\begin{aligned} A_5 &= \frac{\partial(A_1)}{\partial S} \\ &= \sqrt{u} K_1(\sqrt{u}) \end{aligned} \quad (\text{A.37})$$

$$\begin{aligned} A_6 &= \frac{\partial(A_2)}{\partial S} \\ &= 24Cz \cdot A_5 \end{aligned} \quad (\text{A.38})$$

The derivative of the wellbore pressure with respect to wellbore storage constant is given by:

$$\frac{\partial \overline{\Delta p}}{\partial C} = \frac{qB}{z} \cdot \frac{-A_1 \cdot A_7}{A_2 \cdot A_2} \quad (\text{A.39})$$

where

$$\begin{aligned} \frac{\partial(A_1)}{\partial C} &= 0 \\ A_7 &= \frac{\partial(A_2)}{\partial C} \\ &= 24z \cdot A_1 \end{aligned} \quad (\text{A.40})$$

In cases where skin is negative, the concept of the effective wellbore radius is used to avoid numerical instabilities.

The effective wellbore radius is defined by:

$$r_{weff} = r_w e^{-S} \quad (\text{A.41})$$

Then, u is redefined by:

$$u = \frac{r_{weff}^2}{0.000264\eta} z \quad (\text{A.42})$$

The following relationships hold:

$$\frac{\partial r_{weff}}{\partial S} = -r_{weff} \quad (\text{A.43})$$

$$\frac{\partial u}{\partial S} = -2u \quad (\text{A.44})$$

$$\frac{\partial \sqrt{u}}{\partial S} = -\sqrt{u} \quad (\text{A.45})$$

$$\frac{\partial}{\partial S} (K_0(\sqrt{u})) = \sqrt{u} K_1(\sqrt{u}) \quad (\text{A.46})$$

$$\frac{\partial}{\partial S} (K_1(\sqrt{u})) = K_1(\sqrt{u}) + \sqrt{u} K_0(\sqrt{u}) \quad (\text{A.47})$$

$$\frac{\partial}{\partial S} (\sqrt{u} K_1(\sqrt{u})) = u K_0(\sqrt{u}) \quad (\text{A.48})$$

The following expressions need to be modified, when the effective wellbore radius is used:

$$A_1 = K_0(\sqrt{u}) \quad (\text{A.49})$$

$$A_3 = \frac{\sqrt{u}}{2k} K_1(\sqrt{u}) \quad (\text{A.50})$$

$$A_6 = c_1 k u K_0(\sqrt{u}) + 24 C z \cdot A_5 \quad (\text{A.51})$$

The sealing fault model has four parameters, which are k , S , C and r_e .

The sealing fault model can be handled through the concept of superposition, using the method of images.

The dimensionless wellbore pressure for the sealing fault model is given in Laplace space by:

$$\overline{p_D} = \frac{K_0(\sqrt{z}) + S\sqrt{z}K_1(\sqrt{z}) + K_0(2r_{eD}\sqrt{z})}{z[\sqrt{z}K_1(\sqrt{z}) + C_D z[K_0(\sqrt{z}) + S\sqrt{z}K_1(\sqrt{z})]]} \quad (\text{A.52})$$

Defining the expressions by:

$$u = \frac{r_w^2}{0.000264\eta} z \quad (\text{A.53})$$

$$w = \frac{4r_e^2}{0.000264\eta} z \quad (\text{A.54})$$

The wellbore pressure for the sealing fault model in field units is given in Laplace space by:

$$\overline{\Delta p} = \frac{qB}{z} \cdot \frac{A_1}{A_2} \quad (\text{A.55})$$

where

$$A_1 = A_{11} + A_{12} \quad (\text{A.56})$$

$$A_{11} = K_0 \left(\frac{2S}{\sqrt{u}} K_1(\sqrt{u}) \right) \quad (\text{A.57})$$

$$A_{12} = K_0(\sqrt{w}) \quad (\text{A.58})$$

$$A_2 = c_1 k \sqrt{u} K_1(\sqrt{u}) + 24Cz \cdot A_{11} \quad (\text{A.59})$$

The derivative of the wellbore pressure with respect to permeability is given by:

$$\frac{\partial \overline{\Delta p}}{\partial k} = \frac{qB}{z} \cdot \frac{A_3 \cdot A_2 - A_1 \cdot A_4}{A_2 \cdot A_2} \quad (\text{A.60})$$

where

$$\begin{aligned} A_3 &= \frac{\partial(A_1)}{\partial k} \\ &= A_{31} + A_{32} \end{aligned} \quad (\text{A.61})$$

$$\begin{aligned} A_{31} &= \frac{\partial(A_{11})}{\partial k} \\ &= \frac{\sqrt{u}}{2k} \cdot [K_1(\sqrt{u}) + S\sqrt{u}K_0(\sqrt{u})] \end{aligned} \quad (\text{A.62})$$

$$\begin{aligned} A_{32} &= \frac{\partial(A_{12})}{\partial k} \\ &= \frac{\sqrt{w}}{2k} K_1(\sqrt{w}) \end{aligned} \quad (\text{A.63})$$

$$\begin{aligned} A_4 &= \frac{\partial(A_2)}{\partial k} \\ &= c_1 \sqrt{u} K_1(\sqrt{u}) + c_1 \frac{\sqrt{u}}{2} \sqrt{u} K_0(\sqrt{u}) + 24Cz \cdot A_{31} \end{aligned} \quad (\text{A.64})$$

The derivative of the wellbore pressure with respect to skin is given by:

$$\frac{\partial \overline{\Delta p}}{\partial S} = \frac{qB}{z} \cdot \frac{A_5 \cdot A_2 - A_1 \cdot A_6}{A_2 \cdot A_2} \quad (\text{A.65})$$

where

$$\begin{aligned} A_5 &= \frac{\partial(A_1)}{\partial S} \\ &= \sqrt{u} K_1(\sqrt{u}) \end{aligned} \quad (\text{A.66})$$

$$\begin{aligned} A_6 &= \frac{\partial(A_2)}{\partial S} \\ &= 24Cz \cdot A_5 \end{aligned} \quad (\text{A.67})$$

The derivative of the wellbore pressure with respect to wellbore storage constant, is given by:

$$\frac{\partial \overline{\Delta p}}{\partial C} = \frac{qB}{z} \cdot \frac{-A_1 \cdot A_7}{A_2 \cdot A_2} \quad (\text{A.68})$$

where

$$\begin{aligned} A_7 &= \frac{\partial(A_2)}{\partial C} \\ &= 24z \cdot A_{11} \end{aligned} \quad (\text{A.69})$$

The derivative of the wellbore pressure with respect to the distance to the bound; ary is given by:

$$\frac{\partial \overline{\Delta p}}{\partial r_e} = \frac{qB}{z} \cdot \frac{A_8}{A_2} \quad (\text{A.70})$$

where

$$\begin{aligned} A_8 &= \frac{\partial(A_1)}{\partial r_e} \\ &= -\frac{\sqrt{w}}{r_e} K_1(\sqrt{w}) \end{aligned} \quad (\text{A.71})$$

The following relationships are useful:

$$\frac{\partial w}{\partial r_e} = \frac{2w}{r_e} \quad (\text{A.72})$$

$$\frac{\partial \sqrt{w}}{\partial r_e} = \frac{\sqrt{w}}{r_e} \quad (\text{A.73})$$

$$\frac{\partial}{\partial r_e} (K_0(\sqrt{w})) = -\frac{\sqrt{w}}{r_e} K_1(\sqrt{w}) \quad (\text{A.74})$$

In cases where skin is negative, the following expressions need to be modified:

$$A_{11} = K_0(\sqrt{u}) \quad (\text{A.75})$$

$$A_{31} = \frac{\sqrt{u}}{2k} K_1(\sqrt{u}) \quad (\text{A.76})$$

$$A_6 = c_1 k u K_0(\sqrt{u}) + 24Cz \cdot A_5 \quad (\text{A.77})$$

A.2.3 No Flow Outer Boundary Model

The no flow outer boundary model has four parameters, which are k , S , C and r_e .

The dimensionless wellbore pressure for the no flow outer boundary model is given in Laplace space by:

$$\overline{p_D} = \frac{A_{01} + S\sqrt{z} \cdot A_{02}}{z[\sqrt{z} \cdot A_{02} + C_D z[A_{01} + S\sqrt{z} \cdot A_{02}]]} \quad (\text{A.78})$$

where

$$A_{01} = I_1(r_{eD}\sqrt{z}) K_0(\sqrt{z}) + K_1(r_{eD}\sqrt{z}) I_0(\sqrt{z}) \quad (\text{A.79})$$

$$A_{02} = I_1(r_{eD}\sqrt{z}) K_1(\sqrt{z}) - K_1(r_{eD}\sqrt{z}) I_1(\sqrt{z}) \quad (\text{A.80})$$

Defining the expressions by:

$$u = \frac{r_w^2}{0.000264\eta} z \quad (\text{A.81})$$

$$w = \frac{r_e^2}{0.000264\eta} z \quad (\text{A.82})$$

The wellbore pressure for the no flow outer boundary model in field units is given in Laplace space by:

$$\overline{\Delta p} = \frac{qB}{z} \cdot \frac{A_1}{A_2} \quad (\text{A.83})$$

where

$$A_1 = B_1 + S\sqrt{u} \cdot B_2 \quad (\text{A.84})$$

$$B_1 = I_1(\sqrt{w}) K_0(\sqrt{u}) + K_1(\sqrt{w}) I_0(\sqrt{u}) \quad (\text{A.85})$$

$$B_2 = I_1(\sqrt{w}) K_1(\sqrt{u}) - K_1(\sqrt{w}) I_1(\sqrt{u}) \quad (\text{A.86})$$

$$B_3 = I_0(\sqrt{w}) K_0(\sqrt{u}) - K_0(\sqrt{w}) I_0(\sqrt{u}) \quad (\text{A.87})$$

$$B_4 = I_0(\sqrt{w}) K_1(\sqrt{u}) + K_0(\sqrt{w}) I_1(\sqrt{u}) \quad (\text{A.88})$$

$$A_2 = c_1 k \sqrt{u} \cdot B_2 + 24Cz \cdot A_1 \quad (\text{A.89})$$

The derivative of the wellbore pressure with respect to permeability is given by:

$$\frac{\partial \overline{\Delta p}}{\partial k} = \frac{qB}{z} \cdot \frac{A_3 \cdot A_2 - A_1 \cdot A_4}{A_2 \cdot A_2} \quad (\text{A.90})$$

where

$$\begin{aligned} A_3 &= \frac{\partial(A_1)}{\partial k} \\ &= A_{31} + S \cdot A_{32} \end{aligned} \quad (\text{A.91})$$

$$\begin{aligned} A_{31} &= \frac{\partial(B_1)}{\partial k} \\ &= \frac{\sqrt{u}}{2k} \left[\frac{1}{\sqrt{u}} \cdot B_1 + B_2 - \frac{\sqrt{w}}{\sqrt{u}} \cdot B_3 \right] \end{aligned} \quad (\text{A.92})$$

$$\begin{aligned} A_{32} &= \frac{\partial(\sqrt{u} \cdot B_2)}{\partial k} \\ &= \frac{\sqrt{u}}{2k} [B_2 + \sqrt{u} \cdot B_1 - \sqrt{w} \cdot B_4] \end{aligned} \quad (\text{A.93})$$

$$\begin{aligned} A_4 &= \frac{\partial(A_2)}{\partial k} \\ &= c_1 \sqrt{u} \cdot B_2 + c_1 k \cdot A_{32} + 24Cz \cdot A_3 \end{aligned} \quad (\text{A.94})$$

The derivative of the wellbore pressure with respect to skin is given by:

$$\frac{\partial \overline{\Delta p}}{\partial S} = \frac{qB}{z} \cdot \frac{A_5 \cdot A_2 - A_1 \cdot A_6}{A_2 \cdot A_2} \quad (\text{A.95})$$

where

$$A_5 = \frac{\partial(A_1)}{\partial S}$$

$$= \sqrt{u} \cdot B_2 \quad (\text{A.96})$$

$$\begin{aligned} A_6 &= \frac{\partial(A_2)}{\partial S} \\ &= 24Cz \cdot A_5 \end{aligned} \quad (\text{A.97})$$

The derivative of the wellbore pressure with respect to wellbore storage constant is given by:

$$\frac{\partial \overline{\Delta p}}{\partial C} = \frac{qB}{z} \cdot \frac{-A_1 \cdot A_7}{A_2 \cdot A_2} \quad (\text{A.98})$$

where

$$\begin{aligned} A_7 &= \frac{\partial(A_2)}{\partial C} \\ &= 24z \cdot A_1 \end{aligned} \quad (\text{A.99})$$

The derivative of the wellbore pressure with respect to the distance to the boundary is given by:

$$\frac{\partial \overline{\Delta p}}{\partial r_e} = \frac{qB}{z} \cdot \frac{A_8 \cdot A_2 - A_1 \cdot A_9}{A_2 \cdot A_2} \quad (\text{A.100})$$

where

$$\begin{aligned} A_8 &= \frac{\partial(A_1)}{\partial r_e} \\ &= A_{81} + S \cdot A_{82} \end{aligned} \quad (\text{A.101})$$

$$\begin{aligned} A_{81} &= \frac{\partial(B_1)}{\partial r_e} \\ &= -\frac{\sqrt{w}}{r_e} \left[\frac{1}{\sqrt{w}} \cdot B_1 - B_3 \right] \end{aligned} \quad (\text{A.102})$$

$$\begin{aligned} A_{82} &= \frac{\partial(\sqrt{u} \cdot B_2)}{\partial r_e} \\ &= -\sqrt{u} \frac{\sqrt{w}}{r_e} \left[\frac{1}{\sqrt{w}} \cdot B_2 - B_4 \right] \end{aligned} \quad (\text{A.103})$$

$$\begin{aligned} A_9 &= \frac{\partial(A_2)}{\partial r_e} \\ &= c_1 k \cdot A_{82} + 24Cz \cdot A_8 \end{aligned} \quad (\text{A.104})$$

The following relationships are useful:

$$\frac{\partial w}{\partial r_e} = \frac{2w}{r_e} \quad (\text{A.105})$$

$$\frac{\partial \sqrt{w}}{\partial r_e} = \frac{\sqrt{w}}{r_e} \quad (\text{A.106})$$

$$\frac{\partial}{\partial r_e} (K_1(\sqrt{w})) = -\frac{\sqrt{w}}{r_e} \left[\frac{1}{\sqrt{w}} K_1(\sqrt{w}) + K_0(\sqrt{w}) \right] \quad (\text{A.107})$$

$$\frac{\partial}{\partial r_e} (I_1(\sqrt{w})) = -\frac{\sqrt{w}}{r_e} \left[\frac{1}{\sqrt{w}} I_1(\sqrt{w}) - I_0(\sqrt{w}) \right] \quad (\text{A.108})$$

In cases where skin is negative, the following expressions need to be modified:

$$A_1 = B_1 \quad (\text{A.109})$$

$$A_3 = A_{31} \quad (\text{A.110})$$

$$A_6 = c_1 k u \cdot B_1 + 24 C z \cdot A_5 \quad (\text{A.111})$$

$$A_8 = A_{81} \quad (\text{A.112})$$

A.2.4 Constant Pressure Outer Boundary Model

The constant pressure outer boundary model has four parameters, which are k , S , C and r_e .

The dimensionless wellbore pressure for the constant pressure outer boundary model is given in Laplace space by:

$$\overline{p_D} = \frac{A_{03} + S\sqrt{z} \cdot A_{04}}{z [\sqrt{z} \cdot A_{04} + C_D z [A_{03} + S\sqrt{z} \cdot A_{04}]]} \quad (\text{A.113})$$

where

$$A_{03} = I_0(r_{eD}\sqrt{z}) K_0(\sqrt{z}) - K_0(r_{eD}\sqrt{z}) I_0(\sqrt{z}) \quad (\text{A.114})$$

$$A_{04} = K_0(r_{eD}\sqrt{z}) I_1(\sqrt{z}) + I_0(r_{eD}\sqrt{z}) K_1(\sqrt{z}) \quad (\text{A.115})$$

Defining the expressions by:

$$u = \frac{r_w^2}{0.000264\eta} z \quad (\text{A.116})$$

$$w = \frac{r_e^2}{0.000264\eta} z \quad (\text{A.117})$$

The wellbore pressure for the constant pressure boundary boundary model in field units is given in Laplace space by:

$$\overline{\Delta p} = \frac{qB}{z} \cdot \frac{A_1}{A_2} \quad (\text{A.118})$$

where

$$A_1 = B_3 + S\sqrt{u} \cdot B_4 \quad (\text{A.119})$$

$$B_1 = I_0(\sqrt{u}) + K_1(\sqrt{w}) I_0(\sqrt{u}) \quad (\text{A.120})$$

$$B_2 = I_1(\sqrt{u}) - K_1(\sqrt{w}) I_1(\sqrt{u}) \quad (\text{A.121})$$

$$B_3 = I_0(\sqrt{u}) - K_0(\sqrt{w}) I_0(\sqrt{u}) \quad (\text{A.122})$$

$$B_4 = I_1(\sqrt{u}) + K_0(\sqrt{w}) I_1(\sqrt{u}) \quad (\text{A.123})$$

$$A_2 = c_1 k \sqrt{u} \cdot B_4 + 24Cz \cdot A_1 \quad (\text{A.124})$$

The derivative of the wellbore pressure with respect to permeability is given by:

$$\frac{\partial \overline{\Delta p}}{\partial k} = \frac{qB}{z} \cdot \frac{A_3 \cdot A_2 - A_1 \cdot A_4}{A_2 \cdot A_2} \quad (\text{A.125})$$

where

$$\begin{aligned} A_3 &= \frac{\partial(A_1)}{\partial k} \\ &= A_{31} + S \cdot A_{32} \end{aligned} \quad (\text{A.126})$$

$$\begin{aligned} A_{31} &= \frac{\partial(B_3)}{\partial k} \\ &= \frac{\sqrt{u}}{2k} \left[B_4 - \frac{\sqrt{w}}{\sqrt{u}} \cdot B_1 \right] \end{aligned} \quad (\text{A.127})$$

$$\begin{aligned} A_{32} &= \frac{\partial(\sqrt{u} \cdot B_4)}{dk} \\ &= \frac{\sqrt{u}}{2k} [\sqrt{u} \cdot B_3 + \sqrt{w} \cdot B_2] \end{aligned} \quad (\text{A.128})$$

$$\begin{aligned}
A_4 &= \frac{\partial(A_2)}{\partial k} \\
&= c_1 \sqrt{u} \cdot B_4 + c_1 k \cdot A_{32} + 24Cz \cdot A_3
\end{aligned} \tag{A.129}$$

The derivative of the wellbore pressure with respect to skin is given by:

$$\frac{\partial \overline{\Delta p}}{\partial S} = \frac{qB}{z} \cdot \frac{A_5 \cdot A_2 - A_1 \cdot A_6}{A_2 \cdot A_2} \tag{A.130}$$

where

$$\begin{aligned}
A_5 &= \frac{\partial(A_1)}{\partial S} \\
&= \sqrt{u} \cdot B_4
\end{aligned} \tag{A.131}$$

$$\begin{aligned}
A_6 &= \frac{\partial(A_2)}{\partial S} \\
&= 24Cz \cdot A_5
\end{aligned} \tag{A.132}$$

The derivative of the wellbore pressure with respect to wellbore storage constant is given by:

$$\frac{\partial \overline{\Delta p}}{\partial C} = \frac{qB}{z} \cdot \frac{-A_1 \cdot A_7}{A_2 \cdot A_2} \tag{A.133}$$

where

$$\begin{aligned}
A_7 &= \frac{\partial(A_2)}{\partial C} \\
&= 24z \cdot A_1
\end{aligned} \tag{A.134}$$

The derivative of the wellbore pressure with respect to the distance to the boundary is given by:

$$\frac{\partial \overline{\Delta p}}{\partial r_e} = \frac{qB}{z} \cdot \frac{A_8 \cdot A_2 - A_1 \cdot A_9}{A_2 \cdot A_2} \tag{A.135}$$

where

$$\begin{aligned}
A_8 &= \frac{\partial(A_1)}{\partial r_e} \\
&= A_{81} + S \cdot A_{82} \\
A_{81} &= \frac{\partial(B_3)}{\partial r_e}
\end{aligned} \tag{A.136}$$

$$= \frac{\sqrt{w}}{r_e} \cdot B_1 \quad (\text{A.137})$$

$$\begin{aligned} A_{82} &= \frac{\partial(\sqrt{u} \cdot B_4)}{\partial r_e} \\ &= -\sqrt{u} \frac{\sqrt{w}}{r_e} \cdot B_2 \end{aligned} \quad (\text{A.138})$$

$$\begin{aligned} A_9 &= \frac{\partial(A_2)}{\partial r_e} \\ &= c_1 k \cdot A_{82} + 24Cz \cdot A_8 \end{aligned} \quad (\text{A.139})$$

The following relationships are useful:

$$\frac{\partial w}{\partial r_e} = \frac{2w}{r_e} \quad (\text{A.140})$$

$$\frac{\partial \sqrt{w}}{\partial r_e} = \frac{\sqrt{w}}{r_e} \quad (\text{A.141})$$

$$\frac{\partial}{\partial r_e} (K_0(\sqrt{w})) = -\frac{\sqrt{w}}{r_e} K_1(\sqrt{w}) \quad (\text{A.142})$$

$$\frac{\partial}{\partial r_e} (I_0(\sqrt{w})) = \frac{\sqrt{w}}{r_e} I_1(\sqrt{w}) \quad (\text{A.143})$$

In cases where skin is negative, the following expressions need to be modified:

$$A_1 = B_3 \quad (\text{A.144})$$

$$A_3 = A_{31} \quad (\text{A.145})$$

$$A_6 = c_1 k u \cdot B_3 + 24Cz \cdot A_5 \quad (\text{A.146})$$

$$A_8 = A_{81} \quad (\text{A.147})$$

A.2.5 Double Porosity Model

The double porosity model has five parameters, which are k , S , C , w , and λ .

In the double porosity model, both the dimensionless pressure and the dimensionless time are modified slightly. In field units, they are defined by:

$$p_D = \frac{E h}{141.2 q B \mu} (p_i - p_{wf}) \quad (\text{A.148})$$

$$t_D = \frac{0.000264 k_f t}{(\phi_f c_{tf} + \phi_m c_{tm}) \mu r_w^2} \quad (\text{A.149})$$

The double porosity effects are expressed in terms of two parameters that relate primary and secondary properties.

The first of the two parameters is the storativity ratio, ω , which relates the secondary storativity to that of the entire system:

$$\omega = \frac{\phi_f c_{tf}}{\phi_f c_{tf} + \phi_m c_{tm}} \quad (\text{A.150})$$

The second parameter is the transmissivity ratio, λ , which relates the interporosity flow between the matrix and the fracture:

$$\lambda = \alpha \frac{k_m}{k_f} r_w^2 \quad (\text{A.151})$$

where α is a factor that depends on the geometry.

The dimensionless wellbore pressure for the double porosity model is given in Laplace space by:

$$\bar{p}_D = \frac{K_0 \left(\sqrt{zf(z)} \right) + S \sqrt{zf(z)} K_1 \left(\sqrt{zf(z)} \right)}{z \left[\sqrt{zf(z)} K_1 \left(\sqrt{zf(z)} \right) + C_D z \left[K_0 \left(\sqrt{zf(z)} \right) + S \sqrt{zf(z)} K_1 \left(\sqrt{zf(z)} \right) \right] \right]} \quad (\text{A.152})$$

where

$$f(z) = \frac{\omega (1 - \omega) z + \lambda}{(1 - \omega) z + \lambda} \quad (\text{A.153})$$

Defining the expressions by:

$$u = \frac{r_w^2}{0.000264 \eta} z \quad (\text{A.154})$$

$$v = u \cdot \frac{\omega (1 - \omega) u + \lambda}{(1 - \omega) u + \lambda} \quad (\text{A.155})$$

The wellbore pressure for the double porosity model in field units is given in Laplace space by:

$$\overline{\Delta p} = \frac{qB}{z} \cdot \frac{A_1}{A_2} \quad (\text{A.156})$$

where

$$A_1 = K_0(\sqrt{v}) + S\sqrt{v}K_1(\sqrt{v}) \quad (\text{A.157})$$

$$A_2 = c_1 k \sqrt{v} K_1(\sqrt{v}) + 24Cz \cdot A_1 \quad (\text{A.158})$$

The derivative of the wellbore pressure with respect to permeability is given by:

$$\frac{\partial \overline{\Delta p}}{\partial k} = \frac{qB}{z} \cdot \frac{A_3 \cdot A_2 - A_1 \cdot A_4}{A_2 \cdot A_2} \quad (\text{A.159})$$

where

$$\begin{aligned} A_3 &= \frac{\partial(A_1)}{\partial k} \\ &= -\frac{1}{2\sqrt{v}} \cdot [K_1(\sqrt{v}) + S\sqrt{v}K_0(\sqrt{v})] \cdot A_{31} \end{aligned} \quad (\text{A.160})$$

$$\begin{aligned} A_{31} &= \frac{\partial v}{\partial k} \\ &= -\frac{u}{k} \cdot \left[\frac{\omega(1-\omega)u + \lambda}{(1-\omega)u + \lambda} - \frac{(1-\omega)^2 \lambda u}{((1-\omega)u + \lambda)^2} \right] \end{aligned} \quad (\text{A.161})$$

$$\begin{aligned} A_4 &= \frac{\partial(A_2)}{\partial k} \\ &= c_1 \sqrt{v} K_1(\sqrt{v}) - c_1 k \frac{K_0(\sqrt{v})}{2} \cdot A_{31} + 24Cz \cdot A_3 \end{aligned} \quad (\text{A.162})$$

The following relationships are useful:

$$\frac{\partial u}{\partial k} = -\frac{u}{k} \quad (\text{A.163})$$

$$\frac{\partial v}{\partial u} = \frac{\omega(1-\omega)u + \lambda}{(1-\omega)u + \lambda} - \frac{(1-\omega)^2 \lambda u}{((1-\omega)u + \lambda)^2} \quad (\text{A.164})$$

$$\frac{\partial v}{\partial k} = -\frac{u}{k} \cdot \left[\frac{\omega(1-\omega)u + \lambda}{(1-\omega)u + \lambda} - \frac{(1-\omega)^2 \lambda u}{((1-\omega)u + \lambda)^2} \right] \quad (\text{A.165})$$

$$\frac{\partial \sqrt{v}}{\partial k} = \frac{1}{2\sqrt{v}} \cdot \frac{\partial v}{\partial k} \quad (\text{A.166})$$

$$\frac{\partial}{\partial k} (K_0(\sqrt{v})) = -\frac{K_1(\sqrt{v})}{2\sqrt{v}} \cdot \frac{\partial v}{\partial k} \quad (\text{A.167})$$

$$\frac{\partial}{\partial k} (K_1(\sqrt{v})) = -\left[\frac{K_1(\sqrt{v})}{2v} + \frac{K_0(\sqrt{v})}{2\sqrt{v}} \right] \cdot \frac{\partial v}{\partial k} \quad (\text{A.168})$$

$$\frac{\partial}{\partial k} (\sqrt{v} K_1(\sqrt{v})) = -\frac{K_0(\sqrt{v})}{2} \cdot \frac{\partial v}{\partial k} \quad (\text{A.169})$$

The derivative of the wellbore pressure with respect to skin is given by:

$$\frac{\partial \overline{\Delta p}}{\partial S} = \frac{qB}{z} \cdot \frac{A_5 \cdot A_2 - A_1 \cdot A_6}{A_2 \cdot A_2} \quad (\text{A.170})$$

where

$$\begin{aligned} A_5 &= \frac{\partial (A_1)}{\partial S} \\ &= \sqrt{v} K_1(\sqrt{v}) \end{aligned} \quad (\text{A.171})$$

$$\begin{aligned} A_6 &= \frac{\partial (A_2)}{\partial S} \\ &= 24Cz \cdot A_5 \end{aligned} \quad (\text{A.172})$$

The derivative of the wellbore pressure with respect to wellbore storage constant is given by:

$$\frac{\partial \overline{\Delta p}}{\partial C} = \frac{qB}{z} \cdot \frac{-A_1 \cdot A_7}{A_2 \cdot A_2} \quad (\text{A.173})$$

where

$$\begin{aligned} A_7 &= \frac{\partial (A_2)}{\partial C} \\ &= 24z \cdot A_1 \end{aligned} \quad (\text{A.174})$$

The derivative of the wellbore pressure with respect to storativity (w)s given by:

$$\frac{\partial \overline{\Delta p}}{\partial \omega} = \frac{qB}{z} \cdot \frac{A_8 \cdot A_2 - A_1 \cdot A_9}{A_2 \cdot A_2} \quad (\text{A.175})$$

where

$$\begin{aligned} A_8 &= \frac{\partial(A_1)}{\partial\omega} \\ &= -\frac{1}{2\sqrt{v}} \cdot [K_1(\sqrt{v}) + S\sqrt{v}K_0(\sqrt{v})] \cdot A_{81} \end{aligned} \quad (\text{A.176})$$

$$\begin{aligned} A_{81} &= \frac{\partial v}{\partial\omega} \\ &= u \cdot \left[\frac{(1-\omega)u}{((1-\omega)u + \lambda)} + \frac{(1-\omega)\lambda u}{((1-\omega)u + \lambda)^2} \right] \end{aligned} \quad (\text{A.177})$$

$$\begin{aligned} A_9 &= \frac{\partial(A_2)}{\partial\omega} \\ &= -c_1 k \frac{K_0(\sqrt{v})}{2} \cdot A_{81} + 24Cz \cdot A_8 \end{aligned} \quad (\text{A.178})$$

The following relationships are useful:

$$\frac{\partial v}{\partial\omega} = u \cdot \left[\frac{(1-\omega)u}{((1-\omega)u + \lambda)} + \frac{(1-\omega)\lambda u}{((1-\omega)u + \lambda)^2} \right] \quad (\text{A.179})$$

$$\frac{\partial\sqrt{v}}{\partial\omega} = \frac{1}{2\sqrt{v}} \cdot \frac{dv}{d\omega} \quad (\text{A.180})$$

$$\frac{\partial}{\partial\omega} (K_0(\sqrt{v})) = -\frac{K_1(\sqrt{v})}{2\sqrt{v}} \cdot \frac{\partial v}{\partial\omega} \quad (\text{A.181})$$

$$\frac{\partial}{\partial\omega} (K_1(\sqrt{v})) = -\left[\frac{K_1(\sqrt{v})}{2v} + \frac{K_0(\sqrt{v})}{2\sqrt{v}} \right] \cdot \frac{\partial v}{\partial\omega} \quad (\text{A.182})$$

$$\frac{\partial}{\partial\omega} (\sqrt{v}K_1(\sqrt{v})) = -\frac{K_0(\sqrt{v})}{2} \cdot \frac{\partial v}{\partial\omega} \quad (\text{A.183})$$

The derivative of the wellbore pressure with respect to transmissivity (λ) is given by:

$$\frac{\partial\overline{\Delta p}}{\partial\lambda} = \frac{qB}{z} \cdot \frac{A_{10} \cdot A_2 - A_1 \cdot A_{11}}{A_2 \cdot A_2} \quad (\text{A.184})$$

where

$$\begin{aligned} A_{10} &= \frac{\partial(A_1)}{\partial\lambda} \\ &= -\frac{1}{2\sqrt{v}} \cdot [K_1(\sqrt{v}) + S\sqrt{v}K_0(\sqrt{v})] \cdot A_{12} \end{aligned} \quad (\text{A.185})$$

$$\begin{aligned}
 A_{12} &= \frac{\partial v}{\partial \lambda} \\
 &= u \cdot \left[\frac{(1-2\omega)^2 u}{((1-\omega)u + \lambda)^2} \right]
 \end{aligned} \tag{A.186}$$

$$\begin{aligned}
 A_{11} &= \frac{\partial(A_2)}{\partial \lambda} \\
 &= -c_1 k \frac{K_0(\sqrt{v})}{2} \cdot A_{12} + 24Cz \cdot A_{10}
 \end{aligned} \tag{A.187}$$

The following relationships are useful:

$$\frac{\partial v}{\partial \lambda} = u \cdot \left[\frac{(1-2\omega)^2 u}{(1-\omega)u + \lambda} \right] \tag{A.188}$$

$$\frac{\partial \sqrt{v}}{\partial \lambda} = \frac{1}{2\sqrt{v}} \cdot \frac{dv}{\partial \lambda} \tag{A.189}$$

$$\frac{\partial}{\partial \lambda} (K_0(\sqrt{v})) = -\frac{K_1(\sqrt{v})}{\sqrt{v}} \cdot \frac{\partial v}{\partial \lambda} \tag{A.190}$$

$$\frac{\partial}{\partial \lambda} (K_1(\sqrt{v})) = -\left[\frac{K_1(\sqrt{v})}{2v} + \frac{K_0(\sqrt{v})}{2\sqrt{v}} \right] \cdot \frac{\partial v}{\partial \lambda} \tag{A.191}$$

$$\frac{\partial}{\partial \lambda} (\sqrt{v} K_1(\sqrt{v})) = -\frac{K_0(\sqrt{v})}{2} \cdot \frac{\partial v}{\partial \lambda} \tag{A.192}$$

In cases where skin is negative, r_w is replaced by $r_w e^{-2S}$ and λ is replaced by λe^{-2S} , after which:

$$\frac{\partial v}{\partial S} = -2v \tag{A.193}$$

$$\frac{\partial \sqrt{v}}{\partial S} = -\sqrt{v} \tag{A.194}$$

$$\frac{\partial}{\partial S} (K_0(\sqrt{v})) = \sqrt{v} K_1(\sqrt{v}) \tag{A.195}$$

$$\frac{\partial}{\partial S} (K_1(\sqrt{v})) = K_1(\sqrt{v}) + \sqrt{v} K_0(\sqrt{v}) \tag{A.196}$$

$$\frac{\partial}{\partial S} (\sqrt{v} K_1(\sqrt{v})) = v K_0(\sqrt{v}) \tag{A.197}$$

$$\frac{\partial v}{\partial \lambda} = u \cdot \left[\frac{(1-2\omega)^2 u e^{-2S}}{((1-\omega)u + \lambda)^2} \right] \tag{A.198}$$

$$\frac{\partial \sqrt{v}}{\partial \lambda} = \frac{1}{2\sqrt{v}} \cdot \frac{\partial v}{\partial \lambda} \quad (\text{A.199})$$

$$\frac{\partial}{\partial \lambda} (K_0(\sqrt{v})) = -\frac{K_1(\sqrt{v})}{2\sqrt{v}} \cdot \frac{\partial v}{\partial \lambda} \quad (\text{A.200})$$

$$\frac{\partial}{\partial \lambda} (K_1(\sqrt{v})) = -\left[\frac{K_1(\sqrt{v})}{2v} + \frac{(\sqrt{v})}{2\sqrt{v}} \right] \cdot \frac{\partial v}{\partial \lambda} \quad (\text{A.201})$$

$$\frac{\partial}{\partial \lambda} (\sqrt{v} K_1(\sqrt{v})) = -\frac{K_0(\sqrt{v})}{2} \cdot \frac{\partial v}{\partial \lambda} \quad (\text{A.202})$$

The following expressions need to be modified:

$$A_1 = K_0(\sqrt{v}) \quad (\text{A.203})$$

$$A_3 = -\frac{K_1(\sqrt{v})}{2\sqrt{v}} \cdot A_{31} \quad (\text{A.204})$$

$$A_6 = c_1 k v K_0(\sqrt{v}) + 24Cz \cdot A_5 \quad (\text{A.205})$$

$$A_8 = -\frac{K_1(\sqrt{v})}{2\sqrt{v}} \cdot A_{81} \quad (\text{A.206})$$

$$A_{10} = -\frac{K_1(\sqrt{v})}{2\sqrt{v}} \cdot A_{12} \quad (\text{A.207})$$

$$A_{12} = u \cdot \left[\frac{(1 - 2\omega)^2 u e^{-2S}}{((1 - \omega)u + \lambda)^2} \right] \quad (\text{A.208})$$

The double porosity and sealing fault model has six parameters, which are k , S , C , w , λ , and r_e .

The dimensionless wellbore pressure for the double porosity and sealing fault model is given in Laplace space by:

$$\overline{p_D} = \frac{K_0(\sqrt{zf(z)}) + S\sqrt{zf(z)}K_1(\sqrt{zf(z)}) + K_0(2r_{eD}\sqrt{zf(z)})}{z \left[\sqrt{zf(z)}K_1(\sqrt{zf(z)}) + C_D z \left[K_0(\sqrt{zf(z)}) + S\sqrt{zf(z)}K_1(\sqrt{zf(z)}) \right] \right]} \quad (\text{A.209})$$

$$f(z) = \frac{\omega(1-w)z + \lambda}{(1-w)z + \lambda} \quad (\text{A.210})$$

Defining the expressions by:

$$u = \frac{r_w^2}{0.000264\eta} z \quad (\text{A.211})$$

$$v = u \cdot \frac{\omega(1-\omega)u + \lambda}{(1-\omega)u + \lambda} \quad (\text{A.212})$$

$$w = \frac{4r_e^2}{r_w^2} \cdot v \quad (\text{A.213})$$

The wellbore pressure for the double porosity and sealing fault model in field units is given in Laplace space by:

$$\overline{\Delta p} = \frac{qB}{z} \cdot \frac{A_1}{A_2} \quad (\text{A.214})$$

where

$$A_1 = B_1 + B_2 \quad (\text{A.215})$$

$$B_1 = K_0(\sqrt{v}) + S\sqrt{v}K_1(\sqrt{v}) \quad (\text{A.216})$$

$$B_2 = K_0(\sqrt{w}) \quad (\text{A.217})$$

$$A_2 = c_1 k \sqrt{v} K_1(\sqrt{v}) + 24Cz \cdot A_{11} \quad (\text{A.218})$$

The derivative of the wellbore pressure with respect to permeability is given by:

$$\frac{\partial \overline{\Delta p}}{\partial k} = \frac{qB}{z} \cdot \frac{A_3 \cdot A_2 - A_1 \cdot A_4}{A_2 \cdot A_2} \quad (\text{A.219})$$

where

$$\begin{aligned} A_3 &= \frac{\partial(A_1)}{\partial k} \\ &= A_{32} + A_{33} \\ A_{32} &= \frac{\partial(B_1)}{\partial k} \end{aligned} \quad (\text{A.220})$$

$$= -\frac{1}{2\sqrt{v}} \cdot [K_1(\sqrt{v}) + S\sqrt{v}K_0(\sqrt{v})] \cdot A_{31} \quad (\text{A.221})$$

$$\begin{aligned} A_{33} &= \frac{\partial(B_2)}{\partial k} \\ &= -\frac{2r_e}{r_w} \frac{K_1(\sqrt{w})}{2\sqrt{v}} \cdot A_{31} \end{aligned} \quad (\text{A.2212})$$

$$\begin{aligned} A_{31} &= \frac{\partial v}{\partial k} \\ &= -\frac{u}{k} \cdot \left[\frac{\omega(1-\omega)u + \lambda}{(1-\omega)u + \lambda} - \frac{(1-\omega)^2 \lambda u}{((1-\omega)u + \lambda)^2} \right] \end{aligned} \quad (\text{A.223})$$

$$\begin{aligned} A_4 &= \frac{\partial(A_2)}{\partial k} \\ &= c_1 \sqrt{v} K_1(\sqrt{v}) - c_1 k \frac{K_0(\sqrt{v})}{2} \cdot A_{31} + 24Cz \cdot A_{32} \end{aligned} \quad (\text{A.224})$$

The derivative of the wellbore pressure with respect to skin is given by:

$$\frac{\partial \overline{\Delta p}}{\partial S} = \frac{qB}{z} \cdot \frac{A_5 \cdot A_2 - A_1 \cdot A_6}{A_2 \cdot A_2} \quad (\text{A.225})$$

where

$$\begin{aligned} A_5 &= \frac{\partial(A_1)}{\partial S} \\ &= \sqrt{v} K_1(\sqrt{v}) \end{aligned} \quad (\text{A.226})$$

$$\begin{aligned} A_6 &= \frac{\partial(A_2)}{\partial S} \\ &= 24Cz \cdot A_5 \end{aligned} \quad (\text{A.227})$$

The derivative of the wellbore pressure with respect to wellbore storage constant is given by:

$$\frac{\partial \overline{\Delta p}}{\partial C} = \frac{qB}{z} \cdot \frac{-A_1 \cdot A_7}{A_2 \cdot A_2} \quad (\text{A.228})$$

where

$$\begin{aligned} A_7 &= \frac{\partial(A_2)}{\partial C} \\ &= 24z \cdot A_1 \end{aligned} \quad (\text{A.229'})$$

The derivative of the wellbore pressure with respect to storativity (w) is given by:

$$\frac{\partial \bar{\Delta p}}{\partial \omega} = \frac{qB}{z} \cdot \frac{A_8 \cdot A_2 - A_1 \cdot A_9}{A_2 \cdot A_2} \quad (\text{A.230})$$

where

$$\begin{aligned} A_8 &= \frac{\partial(A_1)}{\partial \omega} \\ &= A_{82} + A_{g3} \end{aligned} \quad (\text{A.231})$$

$$\begin{aligned} A_{82} &= \frac{\partial(B_1)}{\partial \omega} \\ &= -\frac{1}{2\sqrt{v}} \cdot [K_1(\sqrt{v}) + S\sqrt{v}K_0(\sqrt{v})] \cdot A_{81} \end{aligned} \quad (\text{A.232})$$

$$\begin{aligned} A_{83} &= \frac{\partial(B_2)}{\partial \omega} \\ &= -\frac{2r_e K_1(\sqrt{w})}{r_w 2\sqrt{v}} \cdot A_{81} \end{aligned} \quad (\text{A.233})$$

$$\begin{aligned} A_{81} &= \frac{\partial v}{\partial \omega} \left[\frac{(1-\omega)u}{((1-\omega)u + \lambda)} + \frac{(1-w)xu}{((1-\omega)u + \lambda)^2} \right] \\ &= \end{aligned} \quad (\text{A.234})$$

$$\begin{aligned} A_9 &= \frac{\partial(A_2)}{\partial \omega} \\ &= -c_1 k \frac{K_0(\sqrt{v})}{2} \cdot A_{81} + 24Cz \cdot A_{81} \end{aligned} \quad (\text{A.235})$$

The derivative of the wellbore pressure with respect to transmissivity (λ) is given by:

$$\frac{\partial \bar{\Delta p}}{\partial \lambda} = \frac{qB}{z} \cdot \frac{A_{10} \cdot A_2 - A_1 \cdot A_{11}}{A_2 \cdot A_2} \quad (\text{A.236})$$

where

$$\begin{aligned} A_{10} &= \frac{\partial(A_1)}{\partial \lambda} \\ &= A_{10B} + A_{10C} \end{aligned} \quad (\text{A.23'7})$$

$$\begin{aligned} A_{10B} &= \frac{\partial(B_1)}{\partial \lambda} \\ &= -\frac{1}{2\sqrt{v}} \cdot [K_1(\sqrt{v}) + S\sqrt{v}K_0(\sqrt{v})] \cdot A_{10C} \end{aligned} \quad (\text{A.238})$$

$$\begin{aligned}
A_{10C} &= \frac{\partial(B_2)}{\partial\lambda} \\
&= -\frac{2r_e K_1(\sqrt{w})}{r_w 2\sqrt{v}} \cdot A_{10A}
\end{aligned} \tag{A.239}$$

$$\begin{aligned}
A_{10A} &= \frac{\partial v}{\partial\lambda} \\
&= u \cdot \left[\frac{(1-2\omega)^2 u}{((1-\omega)u + \lambda)^2} \right]
\end{aligned} \tag{A.240}$$

$$\begin{aligned}
A_{11} &= \frac{\partial(A_2)}{\partial\lambda} \\
&= -c_1 k \frac{K_0(\sqrt{v})}{2} \cdot A_{10A} + 24C \cdot A_{10B}
\end{aligned} \tag{A.241}$$

The derivative of the wellbore pressure with respect to the distance to the boundary is given by:

$$\frac{\partial \overline{\Delta p}}{\partial r_e} = \frac{qB}{z} \cdot \frac{A_{12}}{A_2} \tag{A.242}$$

where

$$\begin{aligned}
A_{12} &= \frac{\partial(A_1)}{\partial r_e} \\
&= -\frac{\sqrt{w}}{r_e} K_1(\sqrt{w})
\end{aligned} \tag{A.243}$$

In cases where skin is negative, the following expressions need to be modified:

$$A_{11} = K_0(\sqrt{v}) \tag{A.244}$$

$$A_{32} = -\frac{K_1(\sqrt{v})}{2\sqrt{v}} \cdot A_{31} \tag{A.245}$$

$$A_6 = c_1 k v K_0(\sqrt{v}) + 24Cz \cdot A_5 \tag{A.246}$$

$$A_{82} = -\frac{K_1(\sqrt{v})}{2\sqrt{v}} \cdot A_{81} \tag{A.247}$$

$$A_{10B} = -\frac{K_1(\sqrt{v})}{2\sqrt{v}} \cdot A_{10A} \tag{A.248}$$

$$A_{10A} = u \cdot \left[\frac{(1-2\omega)^2 u e^{-2S}}{((1-\omega)u + \lambda)^2} \right] \tag{A.249}$$

A.2.7 Double Porosity and No Flow Outer Boundary Model

The double porosity and no flow outer boundary model has six parameters, which are k , S , C , ω , λ , and r_e .

The dimensionless wellbore pressure for the double porosity and no flow outer boundary model is given in Laplace space by:

$$\overline{p_D} = \frac{A_{01} + S\sqrt{zf(z)} \cdot A_{02}}{z \left[\sqrt{zf(z)} \cdot A_{02} + C_D z \left[A_{01} + S\sqrt{zf(z)} \cdot A_{02} \right] \right]} \quad (\text{A.250})$$

where

$$f(z) = \frac{\omega(1-\omega)z + \lambda}{(1-\omega)z + \lambda} \quad (\text{A.251})$$

$$A_{01} = I_1(r_{eD}\sqrt{zf(z)}) K_0(\sqrt{zf(z)}) + K_1(r_{eD}\sqrt{zf(z)}) I_0(\sqrt{zf(z)}) \quad (\text{A.252})$$

$$A_{02} = I_1(r_{eD}\sqrt{zf(z)}) K_1(\sqrt{zf(z)}) - K_1(r_{eD}\sqrt{zf(z)}) I_1(\sqrt{zf(z)}) \quad (\text{A.253})$$

Defining the expressions by:

$$u = \frac{r_w^2}{0.000264\eta} z \quad (\text{A.254})$$

$$v = u \cdot \frac{\omega(1-\omega)u + \lambda}{(1-\omega)u + \lambda} \quad (\text{A.255})$$

$$w = \frac{r_e^2}{r_w^2} \cdot v \quad (\text{A.256})$$

The wellbore pressure for the double porosity and no flow outer boundary model in field units is given in Laplace space by:

$$\overline{\Delta p} = \frac{qB}{z} \cdot \frac{A_1}{A_2} \quad (\text{A.257})$$

where

$$A_1 = B_1 + S\sqrt{v} \cdot B_2 \quad (\text{A.258})$$

$$B_1 = I_1 \left(\sqrt{w} \right) K_0 \left(\sqrt{v} \right) + K_1 \left(\sqrt{w} \right) I_0 \left(\sqrt{v} \right) \quad (\text{A.259})$$

$$A_{32} = I_1 \left(\sqrt{w} \right) K_1 \left(\sqrt{v} \right) - K_1 \left(\sqrt{w} \right) I_1 \left(\sqrt{v} \right) \quad (\text{A.260})$$

$$B_3 = I_0 \left(\sqrt{w} \right) K_0 \left(\sqrt{v} \right) - K_0 \left(\sqrt{w} \right) I_0 \left(\sqrt{v} \right) \quad (\text{A.261})$$

$$B_4 = I_0 \left(\sqrt{w} \right) K_1 \left(\sqrt{v} \right) + K_0 \left(\sqrt{w} \right) I_1 \left(\sqrt{v} \right) \quad (\text{A.262})$$

$$A_2 = c_1 k \sqrt{v} \cdot B_2 + 24Cz \cdot A_1 \quad (\text{A.263})$$

The derivative of the wellbore pressure with respect to permeability is given by:

$$\frac{\partial \bar{\Delta p}}{\partial k} = \frac{qB}{z} \cdot \frac{A_3 \cdot A_2 - A_1 \cdot A_4}{A_2 \cdot A_2} \quad (\text{A.264})$$

where

$$\begin{aligned} A_3 &= \frac{\partial (A_1)}{\partial k} \\ &= A_{32} + S \cdot A_{33} \end{aligned} \quad (\text{A.265})$$

$$\begin{aligned} A_{32} &= \frac{\partial (B_1)}{\partial k} \\ &= -\frac{1}{2\sqrt{v}} \left[\frac{1}{\sqrt{v}} \cdot B_1 + B_2 - \frac{\sqrt{w}}{\sqrt{v}} \cdot B_3 \right] \cdot A_{31} \end{aligned} \quad (\text{A.266})$$

$$\begin{aligned} A_{33} &= \frac{\partial (\sqrt{v} \cdot B_2)}{\partial k} \\ &= -\frac{1}{2\sqrt{v}} \left[B_2 + \sqrt{v} \cdot B_1 - \sqrt{w} \cdot B_4 \right] \cdot A_{31} \end{aligned} \quad (\text{A.267})$$

$$\begin{aligned} A_{31} &= \frac{\partial v}{\partial k} \\ &= -\frac{u}{k} \cdot \left[\frac{\omega(1-\omega)u + \lambda}{(1-\omega)u + \lambda} - \frac{(1-\omega)^2 \lambda u}{((1-\omega)u + \lambda)^2} \right] \end{aligned} \quad (\text{A.268})$$

$$\begin{aligned} A_4 &= \frac{\partial (A_2)}{\partial k} \\ &= c_1 \sqrt{v} \cdot B_2 + c_1 k \cdot A_{33} + 24Cz \cdot A_3 \end{aligned} \quad (\text{A.269})$$

The derivative of the wellbore pressure with respect to skin is given by:

$$\frac{\partial \bar{\Delta p}}{\partial S} = \frac{qB}{z} \cdot \frac{A_5 \cdot A_2 - A_1 \cdot A_6}{A_2 \cdot A_2} \quad (\text{A.270})$$

where

$$\begin{aligned} A_5 &= \frac{\partial(A_1)}{\partial S} \\ &= \sqrt{u} \cdot B_2 \end{aligned} \quad (\text{A.271})$$

$$\begin{aligned} A_6 &= \frac{\partial(A_2)}{\partial S} \\ &= 24Cz \cdot A_5 \end{aligned} \quad (\text{A.272})$$

The derivative of the wellbore pressure with respect to wellbore storage constant is given by:

$$\frac{\partial \overline{\Delta p}}{\partial C} = \frac{qB}{z} \cdot \frac{-A_1 \cdot A_7}{A_2 \cdot A_2} \quad (\text{A.273})$$

where

$$\begin{aligned} A_7 &= \frac{\partial(A_2)}{\partial C} \\ &= 24z \cdot A_1 \end{aligned} \quad (\text{A.274})$$

The derivative of the wellbore pressure with respect to storativity (w) is given by:

$$\frac{\partial \overline{\Delta p}}{\partial w} = \frac{qB}{z} \cdot \frac{A_8 \cdot A_2 - A_1 \cdot A_9}{A_2 \cdot A_2} \quad (\text{A.275})$$

where

$$\begin{aligned} A_8 &= \frac{\partial(A_1)}{\partial w} \\ &= A_{82} + S \cdot A_{83} \end{aligned} \quad (\text{A.276})$$

$$\begin{aligned} A_{82} &= \frac{\partial(B_1)}{\partial w} \\ &= -\frac{1}{2\sqrt{v}} \left[\frac{1}{\sqrt{v}} \cdot B_1 + B_2 - \frac{\sqrt{w}}{\sqrt{v}} \cdot B_3 \right] \cdot A_8 \end{aligned} \quad (\text{A.277})$$

$$\begin{aligned} A_{83} &= \frac{\partial(\sqrt{v} \cdot B_2)}{\partial w} \\ &= -\frac{1}{2\sqrt{v}} \left[B_2 + \sqrt{v} \cdot B_1 - \sqrt{w} \cdot B_4 \right] \cdot A_8 \end{aligned} \quad (\text{A.278})$$

$$A_{81} = \frac{\partial v}{\partial w}$$

$$- u \cdot \left[\frac{(1-\omega)u}{((1-\omega)u + \lambda)} + \frac{(1-w)xu}{((1-\omega)u + \lambda)^2} \right] \quad (\text{A.279})$$

$$\begin{aligned} A_9 &= \frac{\partial(A_2)}{\partial\omega} \\ &= c_1 k \cdot A_{83} + 24Cz \cdot A_8 \end{aligned} \quad (\text{A.280})$$

The derivative of the wellbore pressure with respect to transmissivity (A) is given by:

$$\frac{\partial \overline{\Delta p}}{\partial \lambda} = \frac{qB}{z} \cdot \frac{A_{10} \cdot A_2 - A_1 \cdot A_{11}}{A_2 \cdot A_2} \quad (\text{A.281})$$

where

$$\begin{aligned} A_{10} &= \frac{\partial(A_1)}{\partial \lambda} \\ &= A_{10B} + S \cdot A_{10C} \end{aligned} \quad (\text{A.282})$$

$$\begin{aligned} A_{10B} &= \frac{\partial(B_1)}{\partial \lambda} \\ &= -\frac{1}{2\sqrt{v}} \left[\frac{1}{\sqrt{v}} B_1 + B_2 - \frac{\sqrt{w}}{\sqrt{v}} \cdot B_3 \right] \cdot A_{10A} \end{aligned} \quad (\text{A.283})$$

$$\begin{aligned} A_{10C} &= \frac{\partial(\sqrt{v} \cdot B_2)}{\partial \lambda} \\ &= -\frac{1}{2\sqrt{v}} [B_2 + \sqrt{v} \cdot B_1 - \sqrt{w} \cdot B_4] \cdot A_{10A} \end{aligned} \quad (\text{A.284})$$

$$\begin{aligned} A_{10A} &= \frac{\partial v}{\partial \lambda} \\ &= u \cdot \left[\frac{(1-2\omega)^2 u}{((1-\omega)u + \lambda)^2} \right] \end{aligned} \quad (\text{A.285})$$

$$\begin{aligned} A_{11} &= \frac{\partial(A_2)}{\partial \lambda} \\ &= c_1 k \cdot A_{10C} + 24Cz \cdot A_{10} \end{aligned} \quad (\text{A.286})$$

The derivative of the wellbore pressure with respect to the distance to the boundary is given by:

$$\frac{\partial \overline{\Delta p}}{\partial r_e} = \frac{qB}{z} \cdot \frac{A_{12} \cdot A_2 - A_1 \cdot A_{13}}{A_2 \cdot A_2} \quad (\text{A.287})$$

where

$$\begin{aligned} A_{12} &= \frac{\partial(A_1)}{\partial r_e} \\ &= A_{12B} + S \cdot A_{12C} \end{aligned} \quad (\text{A.288})$$

$$\begin{aligned} A_{12B} &= \frac{\partial(B_1)}{\partial r_e} \\ &= -\frac{\sqrt{w}}{r_e} \left[\frac{1}{\sqrt{w}} \cdot B_1 - B_3 \right] \end{aligned} \quad (\text{A.289})$$

$$\begin{aligned} A_{12C} &= \frac{\partial(\sqrt{v} \cdot B_2)}{\partial r_e} \\ &= -\frac{\sqrt{w}}{r_e} \left[\frac{\sqrt{v}}{\sqrt{w}} \cdot B_2 - \sqrt{v} \cdot B_4 \right] \end{aligned} \quad (\text{A.290})$$

$$\begin{aligned} A_{13} &= \frac{\partial(A_2)}{\partial r_e} \\ &= c_1 k \cdot A_{12C} + 24Cz \cdot A_{12} \end{aligned} \quad (\text{A.291})$$

In cases where skin is negative, the following expressions need to be modified:

$$A_1 = B_1 \quad (\text{A.292})$$

$$A_3 = A_{32} \quad (\text{A.293})$$

$$A_6 = c_1 k v \cdot B_1 + 24Cz \cdot A_5 \quad (\text{A.294})$$

$$A_8 = A_{82} \quad (\text{A.295})$$

$$A_{10} = A_{10B} \quad (\text{A.296})$$

$$A_{10A} = u \cdot \left[\frac{(1-2w)^2 u e^{-2S}}{((1-w)u + 1)} \right] \quad (\text{A.297})$$

$$A_{12} = A_{12B} \quad (\text{A.298})$$

A.2.8 Double Porosity and Constant Pressure Outer Boundary Model

The double porosity and constant pressure outer boundary model has six parameters, which are k , S , C , w , λ , and r_e .

The dimensionless wellbore pressure for the double porosity and constant pressure boundary model is given in Laplace space by:

$$\overline{p_D} = \frac{A_{03} + S\sqrt{zf(z)} \cdot A_{04}}{z \left[\sqrt{zf(z)} \cdot A_{04} + C_D z \left[A_{03} + S\sqrt{zf(z)} \cdot A_{04} \right] \right]} \quad (\text{A.2919})$$

where

$$f(z) = \frac{\omega(1-\omega)z + \lambda}{(1-\omega)z + \lambda} \quad (\text{A.300})$$

$$A_{03} = I_0(r_{eD}\sqrt{zf(z)}) K_0(\sqrt{zf(z)}) - K_0(r_{eD}\sqrt{zf(z)}) I_0(\sqrt{zf(z)}) \quad (\text{A.301})$$

$$(\text{A.302})$$

|

|

$$A_{04} = K_0(r_{eD}\sqrt{zf(z)}) I_1(\sqrt{zf(z)}) + I_0(r_{eD}\sqrt{zf(z)}) K_1(\sqrt{zf(z)}) \quad (\text{A.303})$$

Defining the expressions by:

$$u = \frac{r_w^2}{0.0002647} z \quad (\text{A.304})$$

$$v = u \cdot \frac{\omega(1-\omega)u + \lambda}{(1-\omega)u + \lambda} \quad (\text{A.305})$$

$$w = \frac{r_e^2}{r_w^2} \cdot v \quad (\text{A.306})$$

The wellbore pressure for the double porosity and constant pressure outer boundary model in field units is given in Laplace space by:

$$\overline{\Delta p} = \frac{qB}{z} \cdot \frac{A_1}{A_2} \quad (\text{A.307})$$

where

$$A_1 = B_3 + S\sqrt{v} \cdot B_4 \quad (\text{A.308})$$

$$B_1 = I_1 \left(K_0(\sqrt{v}) + K_1(\sqrt{w}) I_0(A) \right) \quad (\text{A.309})$$

$$B_2 = I_1 \left(K_1(\sqrt{v}) - K_1(\sqrt{w}) I_1(A) \right) \quad (\text{A.310})$$

$$B_3 = I_0 \left(K_0(\sqrt{v}) - K_0(\sqrt{w}) I_0(A) \right) \quad (\text{A.311})$$

$$B_4 = I_0(\sqrt{w}) K_1(A) + K_0(\sqrt{w}) I_1(\sqrt{v}) \quad (\text{A.312})$$

$$A_2 = c_1 k \sqrt{v} \cdot B_4 + 24Cz \cdot A_1 \quad (\text{A.313})$$

The derivative of the wellbore pressure with respect to permeability is given by:

$$\frac{\partial \Delta p}{\partial k} = \frac{qB}{z} \cdot \frac{A_3 \cdot A_2 - A_1 \cdot A_4}{A_2 \cdot A_2} \quad (\text{A.314})$$

where

$$\begin{aligned} A_3 &= \frac{\partial(A_1)}{\partial k} \\ &= A_{32} + S \cdot A_{33} \end{aligned} \quad (\text{A.315})$$

$$\begin{aligned} A_{32} &= \frac{\partial(B_3)}{\partial k} \\ &= -\frac{1}{2\sqrt{v}} \left[B_4 - \frac{\sqrt{w}}{\sqrt{v}} \cdot B_1 \right] \cdot A_{31} \end{aligned} \quad (\text{A.316})$$

$$\begin{aligned} A_{33} &= \frac{\partial(\sqrt{v} \cdot B_4)}{\partial k} \\ &= -\frac{1}{2\sqrt{v}} [\sqrt{v} \cdot B_3 + \sqrt{w} \cdot B_2] \cdot A_{31} \end{aligned} \quad (\text{A.317})$$

$$\begin{aligned} A_{31} &= \frac{\partial v}{\partial k} \\ &= -\frac{u}{k} \cdot \left[\frac{w(1-\omega)u + \lambda}{(1-\omega)u + \lambda} - \frac{(1-\omega)^2 \lambda u}{((1-\omega)u + \lambda)^2} \right] \end{aligned} \quad (\text{A.318})$$

$$\begin{aligned} A_4 &= \frac{\partial(A_2)}{\partial k} \\ &= c_1 \sqrt{u} \cdot B_2 + c_1 k \cdot A_3 + 24Cz \cdot A_3 \end{aligned} \quad (\text{A.319})$$

The derivative of the wellbore pressure with respect to skin is given by:

$$\frac{\partial \Delta p}{\partial S} = \frac{qB}{z} \cdot \frac{A_5 \cdot A_2 - A_1 \cdot A_6}{A_2 \cdot A_2} \quad (\text{A.320})$$

where

$$\begin{aligned} A_5 &= \frac{\partial(A_1)}{\partial S} \\ &= \sqrt{u} \cdot B_2 \end{aligned} \quad (\text{A.321})$$

$$\begin{aligned} A_6 &= \frac{\partial(A_2)}{\partial S} \\ &= 24Cz \cdot A_5 \end{aligned} \quad (\text{A.322})$$

The derivative of the wellbore pressure with respect to wellbore storage constant is given by:

$$\frac{\partial \overline{\Delta p}}{\partial C} = \frac{qB}{z} \cdot \frac{-A_1 \cdot A_7}{A_2 \cdot A_2} \quad (\text{A.323})$$

where

$$\begin{aligned} A_7 &= \frac{\partial(A_2)}{\partial C} \\ &= 24z \cdot A_1 \end{aligned} \quad (\text{A.324})$$

The derivative of the wellbore pressure with respect to storativity (w) is given by:

$$\frac{\partial \overline{\Delta p}}{\partial k} = \frac{qB}{z} \cdot \frac{A_8 \cdot A_2 - A_1 \cdot A_9}{A_2 \cdot A_2} \quad (\text{A.325})$$

where

$$\begin{aligned} A_8 &= \frac{\partial(A_1)}{\partial w} \\ &= A_{82} + S \cdot A_{83} \end{aligned} \quad (\text{A.326})$$

$$\begin{aligned} A_{82} &= \frac{\partial(B_3)}{\partial w} \\ &= -\frac{1}{2\sqrt{v}} \left[B_4 - \frac{\sqrt{w}}{\sqrt{v}} \cdot B_1 \right] \cdot A_{81} \end{aligned} \quad (\text{A.327})$$

$$\begin{aligned} A_{83} &= \frac{\partial(\sqrt{v} \cdot B_4)}{\partial w} \\ &= -\frac{1}{2\sqrt{v}} \left[\sqrt{v} \cdot B_3 + \sqrt{w} \cdot B_2 \right] \cdot A_{81} \end{aligned} \quad (\text{A.328})$$

$$A_{81} = \frac{\partial v}{\partial w}$$

$$= u \cdot \left[\frac{(1-\omega)u}{(1-\omega)u + \lambda} - \frac{(1-w)u}{((1-w)u + \lambda)^2} \right] \quad (\text{A.329})$$

$$\begin{aligned} A_9 &= \frac{\partial(A_2)}{\partial\omega} \\ &= c_1 k \cdot A_{83} + 24Cz \cdot A_8 \end{aligned} \quad (\text{A.330})$$

The derivative of the wellbore pressure with respect to transmissivity (λ) is given by:

$$\frac{\partial \overline{\Delta p}}{\partial \lambda} = \frac{qB}{z} \cdot \frac{A_{10} \cdot A_2 - A_1 \cdot A_{11}}{A_2 \cdot A_2} \quad (\text{A.331})$$

where

$$\begin{aligned} A_{10} &= \frac{\partial(A_1)}{\partial \lambda} \\ &= A_{10B} + S \cdot A_{10C} \end{aligned} \quad (\text{A.332})$$

$$\begin{aligned} A_{10B} &= \frac{\partial(B_3)}{\partial \lambda} \\ &= -\frac{1}{2\sqrt{v}} \left[B_4 - \frac{\sqrt{w}}{\sqrt{v}} \cdot B_1 \right] \cdot A_{10A} \end{aligned} \quad (\text{A.333})$$

$$\begin{aligned} A_{10C} &= \frac{\partial(\sqrt{v} \cdot B_4)}{\partial \lambda} \\ &= -\frac{1}{2\sqrt{v}} [\sqrt{v} \cdot B_3 + \sqrt{w} \cdot B_2] \cdot A_{10A} \end{aligned} \quad (\text{A.334})$$

$$\begin{aligned} A_{10A} &= \frac{\partial v}{\partial \lambda} \\ &= u \cdot \left[\frac{(1-2\omega)^2 u}{((1-\omega)u + \lambda)^2} \right] \end{aligned} \quad (\text{A.335})$$

$$\begin{aligned} A_{11} &= \frac{\partial(A_2)}{\partial \lambda} \\ &= c_1 k \cdot A_{10C} + 24Cz \cdot A_{10} \end{aligned} \quad (\text{A.336})$$

The derivative of the wellbore pressure with respect to the distance to the boundary is given by:

$$\frac{\partial \overline{\Delta p}}{\partial r_e} = \frac{qB}{z} \cdot \frac{A_{12} \cdot A_2 - A_1 \cdot A_{13}}{A_2 \cdot A_2} \quad (\text{A.337})$$

where

$$\begin{aligned} A_{12} &= \frac{\partial(A_1)}{\partial r_e} \\ &= A_{1213} + S \cdot A_{12C} \end{aligned} \quad (\text{A.338})$$

$$\begin{aligned} A_{12B} &= \frac{\partial(B_3)}{\partial r_e} \\ &= \frac{\sqrt{w}}{r_e} \cdot B_1 \end{aligned} \quad (\text{A.339})$$

$$\begin{aligned} A_{12C} &= \frac{\partial(\sqrt{v} \cdot B_4)}{\partial r_e} \\ &= -\frac{\sqrt{w}}{r_e} \sqrt{v} \cdot B_2 \end{aligned} \quad (\text{A.340})$$

$$\begin{aligned} A_{13} &= \frac{\partial(A_2)}{\partial r_e} \\ &= c_1 k \cdot A_{12C} + 24Cz \cdot A_{12} \end{aligned} \quad (\text{A.341})$$

In cases where skin is negative, the following expressions need to be modified:

$$A_1 = B_1 \quad (\text{A.342})$$

$$A_3 = A_{32} \quad (\text{A.343})$$

$$A_6 = c_1 k v \cdot B_1 + 24Cz \cdot A_5 \quad (\text{A.344})$$

$$A_8 = A_{82} \quad (\text{A.345})$$

$$A_{10} = A_{10B} \quad (\text{A.346})$$

$$A_{10A} = u \cdot \left[\frac{(1 - 2\omega)^2 u e^{-2S}}{((1 - \omega)u + \lambda)^2} \right] \quad (\text{A.347})$$

$$A_{12} = A_{12B} \quad (\text{A.348})$$