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A THREE-DIMENSIONAL SOLUTION FOR HEAT EXTRACTION FROM A FRACTURE IN HOT DRY ROCK USING THE BOUNDARY ELEMENT METHOD

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ABSTRACT

Heat extraction from a fracture in a geothermal reservoir is typically modeled based on the assumption that heat conduction is one-dimensional and perpendicular to the fracture. In this paper an integral equation formulation is used to model the three-dimensional heat flow in the reservoir. This method results in a numerical procedure in which the discretization of the reservoir geometry is entirely eliminated, leading to a much more efficient scheme. In addition to providing the temperature distribution in the reservoir, the three-dimensional boundary integral equation formulation provides an efficient means for calculating the induced thermal stresses on the facture surface and in the reservoir.

INTRODUCTION

The Hot Dry Rock (HDR) concept of geothermal exploitation involves drilling two or more wells to suitable depths to connect permeable fractures of natural or man-made origin, injecting cold water into one well, and recovering hot water from the other. A number of analytical and numerical solutions exist for modeling heat extraction from a fracture. See, for example, Willis-Richards and Wallroth (1995) for a comprehensive review. With few exceptions such as the finite element solution by Kolditz (1995), GOECRACK, and the boundary element model by Cheng et al., 2001; the heat conduction in the reservoir is modeled as one-dimensional and perpendicular to the fracture surface. The primary reason for such simplification is due to the difficulty in modeling an unbounded domain by numerical discretization. In this work, the governing equations for the three-dimensional heat extraction from a single fracture embedded in an infinite geothermal reservoir are transformed into an integral equation. The procedures for solving the resulting integral equation are described and some numerical results are presented.

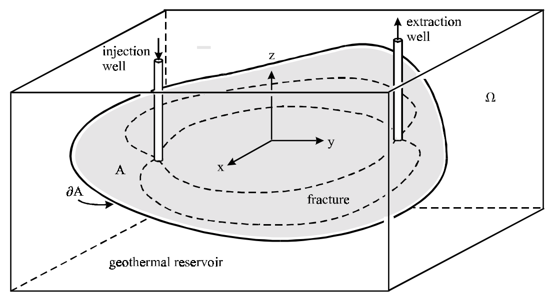

Figure 1: Heat extraction from a planar fracture.

MATHEMATICAL MODEL

Figure 1 illustrates, schematically, a view of heat extraction from a hot dry rock system by circulating water through a naturally existing or man-made fracture. The fracture is assumed to be flat and of finite size. The geothermal reservoir containing the fracture is assumed to be of infinite extent. Other assumptions are similar to these postulated in Cheng et al., 2001. Specifically, it is assumed that (i) all properties such as fracture thickness, permeability, and reservoir heat capacity are constant; (ii) the injection rate of cold water is steady; the reservoir is impermeable to water and the fracture has no storage capacity; hence the production rate of hot water is equal to the injection rate; and (iii) the heat storage and dispersion effects in the fracture fluid flow are negligible.

FRACTURE FLOW

It is assumed that the fracture width is small such that the flow in the fracture is laminar and governed by the lubrication flow equation:

$$\nabla_2\, p(x,y) = -\frac{\pi^2 \mu}{w^3(x,y)}\, q(x,y); \qquad x,y \in A \tag{1}$$

where (∇_2) is the gradient operator in two spatial dimensions, p is the fluid pressure, μ the fluid viscosity, w the fracture width, and A is the fracture surface (see Figure 1). Note that:

$$q = w\bar{v} \tag{2}$$

is the discharge per unit width, and $\bar{v}$ is the average flow velocity given by:

$$\bar{v}(x,y) = \frac{1}{w(x,y)} \int_0^{w(x,y)} v(x,y,z)\, dz \tag{3}$$

where v is the flow velocity. Assuming that (i) the fluid is incompressible, (ii) the fracture wall is impermeable to fluid flow, and (iii) the fracture width does not change with time, the fluid continuity equation can be written as:

$$\nabla_2 \cdot q(x,y) = 0 \tag{4}$$

where $(\nabla_2 \cdot)$ is the two-dimensional divergence operator. Solving equation (1) for $q(x,y)$ and substituting into equation (4) yields a second order partial differential equation:

$$\nabla_2 \cdot \left[w^3(x,y)\, \nabla_2\, p(x,y) \right] = 0 \tag{5}$$

The above equation can be used to solve for the pressure distribution within the fracture using appropriate boundary conditions and the known fracture width, . The discharge and average velocity can then be obtained from (1) and (2).

For the current problem, the boundary condition is:

$$\frac{\partial p}{\partial n} = 0 \qquad \text{on } \partial A \tag{6}$$

In addition, there exist flow singularities in the fracture plane due to the injection and extraction wells. Hence equation (4) must be modified to include these singularities, i.e.,

$$\nabla_2 \cdot q(x,y) = Q\,[\delta(X_e) - \delta(X_i)] \tag{7}$$

where Q is the injection and extraction rate (must be equal), δ is the Dirac delta function, $X_e = [(x - x_e), (y - y_e)]$, $X_i = [(x - x_i), (y - y_i)]$, (x_i, y_i) and (x_e, y_e) are the coordinates of the injection and extraction wells, respectively. Similarly, (5) needs to be modified to:

$$\nabla_2 \cdot \left[w^3(x,y) \nabla_2\, p(x,y) \right] = c \cdot [\delta(X_i) - \delta(X_e)] \tag{8}$$

where $c = \pi^2 \mu\, Q$.

HEAT TRANSPORT IN FRACTURE

The heat transport equation for the fracture can be written as:

$$\nabla_2 \cdot [q(x,y) T(x,y,0,t)] = \frac{2K_r}{\rho_w c_w} \cdot \left.\frac{\partial T(x,y,z,t)}{\partial z}\right|_{z=0^+} ; \text{ for } x,y \in A \tag{9}$$

where the fracture flow field, $q(x,y)$, is known by solving the fluid flow equation; T is the water temperature; ρ_w is the water density; c_w is the specific heat of water; and K_r is the rock thermal conductivity. Note that heat storage and dispersion terms have been neglected based on earlier findings (Cheng et al., 2001).

The heat conduction in the rock can be modeled by a diffusion equation in three dimensions:

$$\nabla_3^2 T(x,y,z,t) = \frac{\rho_r c_r}{K_r}\frac{\partial T(x,y,z,t)}{\partial t}; \quad \text{for } x,y,z \in \Omega \tag{10}$$

where ρ_r is the rock density, c_r is the specific heat of rock, and ∇_3^2 is the three-dimensional Laplacian operator. Note that the same notation, T, is used for the temperature of the reservoir rock and water in the fracture, because temperature must be continuous between the two media.

The governing equations, (9) and (10), are subject to initial and boundary conditions. The initial temperature of the rock and the water in fracture is assumed to be a constant:

$$T(x,y,z,0) = T_{ro} \tag{11}$$

At the injection point $(x_i, y_i, 0)$, the temperature is equal to the injection water temperature:

$$T(x_i,y_i,0,t) = T_{wo} \tag{12}$$

It is clear that there exist singularities at the injection and extraction points; these must be included in the governing equation (9) as source and sink terms:

$$\begin{aligned}&\nabla_2 \cdot [q(x,y)T(x,y,0,t)] \qquad (13)\\ &= \frac{2K_r}{\rho_w c_w}\left.\frac{\partial T(x,y,z,t)}{\partial z}\right|_{z=0^+} + \\ &Q\,[T(x_e,y_e,0,t)\cdot\delta(X_e) - T_{wo}\,\delta(X_i)]\end{aligned}$$

Or, by expanding the quantities under the divergence operator and noting the flow equation (7), the above equation can be written as:

$$q(x,y)\cdot\nabla_2 T(x,y,0,t) - \frac{2K_r}{\rho_w c_w}\left.\frac{\partial T(x,y,z,t)}{\partial z}\right|_{z=0^+} = 0 \tag{14}$$

LAPLACE TRANSFORM

In order to facilitate the use of the Laplace transform technique, it is more desirable to have a zero value for the initial condition. Hence a dimensionless temperature deficit is defined:

$$T_d = \frac{T_{ro} - T}{T_{ro} - T_{wo}} \tag{15}$$

Equations (14) and (10) preserve the same form in the new variable, i.e.,

$$q(x,y)\cdot\nabla_2 T_d(x,y,0,t) = \frac{2K_r}{\rho_w c_w}\left.\frac{\partial T_d(x,y,z,t)}{\partial z}\right|_{z=0^+} \tag{16}$$

$$K_r \nabla_3^2 T_d(x,y,z,t) = \rho_r c_r \frac{\partial T_d(x,y,z,t)}{\partial t} \tag{17}$$

The initial condition then becomes:

$$T_d(x,y,z,0) = 0 \tag{18}$$

At the injection point, the dimensionless temperature deficiency is:

$$T_d(x_i,y_i,0,t) = 1 \tag{19}$$

From the flow singularity in (7), we can write (16) in this form:

$$\begin{aligned}&\rho_w c_w \nabla_2 \cdot [q(x,y)T_d(x,y,0,t)] - \\ &2K_r \left.\frac{\partial T_d(x,y,z,t)}{\partial z}\right|_{z=0^+} = \qquad (20)\\ &\rho_w c_w Q\,[T_d(x_e,y_e,0,t)\,\delta(X_e) - \delta(X_i)]\end{aligned}$$

Performing Laplace transform on (17)-(20), results in:

$$\begin{aligned}&\rho_w c_w \nabla_2 \cdot \left[q(x,y)\bar{T}_d(x,y,0,s)\right] - \\ &2K_r \left.\frac{\partial \bar{T}_d(x,y,z,s)}{\partial z}\right|_{z=0^+} = \qquad (21)\\ &\rho_w c_w Q\left\{\bar{T}_d(x_e,y_e,0,s)\,\delta(X_e) - \frac{1}{s}\,\delta(X_i)\right\}\end{aligned}$$

$$K_r \nabla_3^2 \bar{T}_d(x,y,z,s) = s\rho_r c_r \bar{T}_d(x,y,z,s) \tag{22}$$

$$\bar{T}_d(x_i,y_i,0,s) = \frac{1}{s} \tag{23}$$

where s is the Laplace transform parameter.

INTEGRAL EQUATION

Equations (21) and (22) are defined in three spatial dimensions. Solving transient, three-dimensional problems in an infinite domain still poses a challenge for modern day computers. However, by utilizing Green's function, the system can be converted into a two-dimensional integral equation defined on the fracture surface, thus significantly reducing the computational effort. To model the temperature in the reservoir due to a continuous point heat source with strength $\tilde{Q}$, we introduce the following equation:

$$K_r \nabla_3^2 \tilde{T}_d(x,y,z,s) - s\rho_r c_r \tilde{T}_d(x,y,z,s) = -\tilde{Q}\,\delta\left(\mathbf{x}-\mathbf{x}'\right) \qquad (24)$$

where δ is the Dirac delta function, and $\mathbf{x}'$ is the source location. The solution of (24) is given by the Green's function:

$$\tilde{G} = \frac{\tilde{Q}}{4\pi K_r}\frac{1}{R}\exp\left(-\sqrt{\frac{\rho_r c_r s}{K_r}}\,R\right) \qquad (25)$$

where:

$$R = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \qquad (26)$$

Then it can be shown that the temperature in the reservoir due to a continuous distribution of sources on the fracture surface A is given by:

$$\tilde{T}_d(x,y,z,s) = \qquad (27)$$

$$\frac{-\rho_w c_w}{4\pi K_r}\int_A \nabla_2\cdot\left[\mathbf{q}(x',y')\tilde{T}_d(x',y',0,s)\right]\cdot$$

$$f_1\;dx'dy' - \frac{1}{s}f_i\cdot\frac{\rho_w c_w Q}{4\pi K_r}\tilde{T}_d(x_e,y_e,0,s)\cdot f_e$$

where $\beta = \sqrt{\frac{\rho_r c_r s}{K_r}}$, $f_1 = \frac{1}{R_1}\exp\left(-\beta R_1\right)$, $f_i = \frac{1}{R_i}\exp\left(-\beta R_i\right)$, $f_e = \frac{1}{R_e}\exp\left(-\beta R_e\right)$, and:

$$R_e = \sqrt{(x-x_e)^2+(y-y_e)^2+z^2} \qquad (28)$$

$$R_i = \sqrt{(x-x_i)^2+(y-y_i)^2+z^2} \qquad (29)$$

$$R_1 = \sqrt{(x-x')^2+(y-y')^2+z^2} \qquad (30)$$

Because the flow field $\mathbf{q}$ contains singularities in the fracture domain A due to injection and extraction, equation (27) cannot be readily integrated. To remove the singularity, a regularization technique can be applied to obtain:

$$\tilde{T}_d(x,y,z,s) = \qquad (31)$$

$$-\frac{\rho_w c_w Q}{4\pi K_r}\int_A \nabla_2\cdot\left\{\mathbf{q}(x',y')\cdot\tilde{T}_d^*\right\} f_i\,dx'dy' +$$

$$\frac{\rho_w c_w}{4\pi K_r}\tilde{T}_d^e\cdot f_e - \left(\frac{1}{s}-\tilde{T}_d(x,y,0,s)\right) f_i$$

where $\tilde{T}_d^* = \left[\tilde{T}_d(x',y',0,s)-\tilde{T}_d(x,y,0,s)\right]$ and $\tilde{T}_d^e = \left[\tilde{T}_d(x_e,y_e,0,s)-\tilde{T}_d(x,y,0,s)\right]$. The integrand is no longer singular at the injection and extraction points, $(x_i,y_i,0)$ and $(x_e,y_e,0)$. To avoid dealing with derivatives in the integrand, the integral in (31) can be transformed based on the divergence theorem and noting that the integral performed over the fracture boundary, ∂A, vanishes due to the no flux condition $q_n = 0$ on ∂A (see (6)):

$$\tilde{T}_d(x,y,z,s) = \qquad (32)$$

$$\frac{\rho_w c_w Q}{4\pi K_r}\left\{\tilde{T}_d^e\cdot f_e - \left(\frac{1}{s}-\tilde{T}_d(x,y,0,s)\right)\cdot f_i\right\} -$$

$$\frac{\rho_w c_w}{4\pi K_r}\int_A \tilde{T}_d^*\cdot g_1\exp\left(-\beta R_1\right)\cdot q^*\,dx'dy'$$

where:

$$q^* = q_x(x',y')\frac{\partial R_1}{\partial x'} + q_y(x',y')\frac{\partial R_1}{\partial y'} \qquad (33)$$

$$g_1 = \frac{1}{R_1^2}\left(1+\beta R_1\right) \qquad (34)$$

Applying the above equation on the fracture surface, $(x,y)\in A$ and $z=0$, results in:

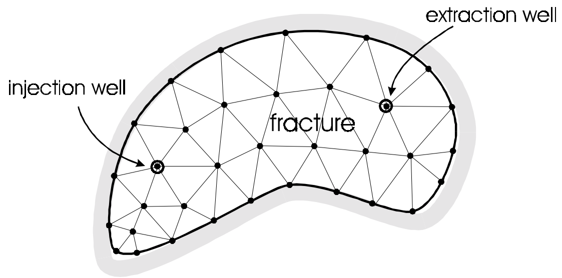

Figure 2: Computational mesh.

$$\tilde{T}_d(x,y,0,s) = \frac{\rho_w c_w Q}{4\pi K_r}\left\{\tilde{T}_d^e \cdot F_e - \left(\frac{1}{s} - \tilde{T}_d(x,y,0,s)\right) \cdot F_i\right\} - \frac{\rho_w c_w}{4\pi K_r}\int_A \tilde{T}_d^* \cdot G \cdot q^* \, dx' dy' \tag{35}$$

where $F_e = \frac{1}{r_e}\exp(-\beta\, r_e)$, $F_i = \frac{1}{r_i}\exp(-\beta\, r_i)$, $G = \frac{1}{r^2}(1+\beta r)\cdot\exp(-\beta\, r)$, and:

$$r = \sqrt{(x-x')^2 + (y-y')^2} \tag{36}$$

$$r_e = \sqrt{(x-x_e)^2 + (y-y_e)^2} \tag{37}$$

$$r_i = \sqrt{(x-x_i)^2 + (y-y_i)^2} \tag{38}$$

Note that equation (35) is now defined in a two-dimensional space, $x, y \in A$ and can be utilized for an integral equation solution.

NUMERICAL IMPLEMENTATION

To solve the system represented by (35), the fracture surface, A, is discritized into a number of elements and is defined by a total of n nodes (see Figure 2) Then, an unknown temperature deficit $\tilde{T}_d^j$ is assigned to each node, except for the node at the injection point where $\tilde{T}_d = 1/s$ is the imposed boundary condition. Hence there exist $n-1$ unknown discrete temperatures. Equation (35) is applied to the $n-1$ nodes (excluding the injection point) by selecting the nodal locations as the base points:

$$\tilde{T}_d^j = \frac{\rho_w c_w Q}{4\pi K_r}\left\{\left[\tilde{T}_d^e - \tilde{T}_d^j\right] F_e - \left(\frac{1}{s} - \tilde{T}_d^j\right) F_i\right\} - \frac{\rho_w c_w}{4\pi K_r}\sum_{n=1}^{N_{el}}\int_{S_{el}}\left[\tilde{T}(\eta',\xi') - \tilde{T}_d^j\right] G \left(q_x(\eta',\xi')\frac{\partial r}{\partial \eta'} + q_y(\eta',\xi')\frac{\partial r}{\partial \xi'}\right) d\eta' d\xi' \tag{39}$$

where $\tilde{T}_d^j$ is the current nodal point, $\tilde{T}_d^e$ is temperature at the extraction point, η' and ξ' are the element local coordinates, and $T(\eta',\xi')$ is the temperature at some point inside element. Note that $\frac{\partial r}{\partial x'} = \frac{\partial r}{\partial \eta'}$ for square elements. After performing numerical integration over the elements, each equation becomes a linear equation involving $\tilde{T}_d^j$, $j = 1,\ldots,n-1$, as unknowns. The linear system can then be solved for the discrete temperatures $\tilde{T}_d^j$.

In the present implementation of the numerical scheme and program development the temperature distribution in a circular crack is found by using square, four-node linear elements. In addition, the dipole solution is used to model fluid flow due to the injection and extraction process in the fracture. The no-flow boundary condition for the circular crack is imposed by using the method of images (Strack, 1989) and superposition to find the appropriate potential function. The potential is then used to calculate the fluid discharge for every fracture element (Figure 3). When the extraction well falls inside an element, it is moved to the nearest nodal point. The temperature inside an element is interpolated using standard shape functions for the element, N_i:

$$N_i = \frac{1}{4}(1 \pm x')(1 \pm y')$$

All other function are calculated exactly at each nodal point.

The double integrals in equation (39) are integrated numerically over each element using the Gaussian integration procedure. For each element both the four-point and nine-point integration schemes are used. If the absolute value of the relative error, $\left|\frac{value\_4 - value\_9}{value\_9}\right|$, is less than 0.01-0.001 then result of the nine-point integration routine is used. Otherwise, the element is further divided into four parts and the nine-point scheme is applied over each sub-element. If the error is still too large, each sub-element is again subdivided into smaller elements until the error is small enough or a specified maximum number of iterations is reached.

A direct solver is used to solve the linear system of equations. An iterative solver can also be used, however, in this case it yields similar results but convergence is very slow. Once the nodal temperatures are obtained, the extraction point temperature can be calculated by (i) using the nodal values (ii) or using a contour integration around the original extraction point. In the latter case, the integration path consists of eight nodes nearest to the extraction point, and the temperature is then given by:

$$\tilde{T}_d^e = \frac{1}{Q} \int_{\Gamma \varepsilon e} q_n(x', y') \tilde{T}_d(x', y') \, ds$$

It is necessary to transform the solution back into the time domain. This can be achieved by using an approximate Laplace inversion method, e.g., the Stehfest (1970) method (Cheng et al., 1994).

EXAMPLE

The solution of the three-dimensional heat conduction is examined below through a numerical example using the following data set:

$$\begin{aligned}
R &= 300 \text{ m}; \quad Q = 5\text{x}10^{-3} \frac{\text{m}^3}{\text{sec}} \\
\rho_w &= 1.0 \text{ g/cm}^3; \quad \rho_r = 2.65 \text{ g/cm}^3; \\
K_r &= 2.58 \text{ W m}^{-1}\text{K}^{-1}; \\
c_w &= 4.05\text{x}10^3 \text{ J kg}^{-1}\text{K}^{-1}; \\
c_r &= 1.1\text{x}10^3 \text{ J kg}^{-1}\text{K}^{-1}
\end{aligned}$$

Figure 4 presents the normalized temperature deficit as a function of time. It can be observed that a finer grid yields a higher value of temperature. However, the curves do converge as a finer grid is used. Also, the values of T_d obtained from the nodal values is higher than the values obtained using the averaging scheme described above. Figure 5 illustrates the normalized extraction temperature for the same problem. Figure 6 shows that a larger fracture results in a higher extraction temperature, as expected.

The 3D model prediction is compared with the one-dimensional heat conduction model of Gringarten and Sauty (1975), as reported by Kolditz (1995), in Figure 7. It can be observed that our solution predicts a higher extraction temperature, this agrees with the observation of Kolditz (1995). The same numerical approach was used to model the one-dimensional heat extraction from a circular fracture using the data of Rodemann (1982). The result is shown in Figure 8. The black contour lines represent the one-dimensional analytic solution of Rodemann (1986). Therefore, the Numerical results presented herein are in qualitative agreement with other published results.

SUMMARY AND CONCLUSION

The three-dimensional boundary integral equation for heat conduction in a geothermal reservoir has been solved numerically. The integral equation solution eliminates the need for discretization of the geothermal reservoir. In addition to rigorous testing of the model, future activities include modeling of reservoir elasticity, thermo-elasticity and poroelasticity. These can cause the fracture width to depend on the fracture fluid pressure, temperature, and reservoir compliance. The present formulation lends nicely itself to these developments. Efforts in these directions are underway.

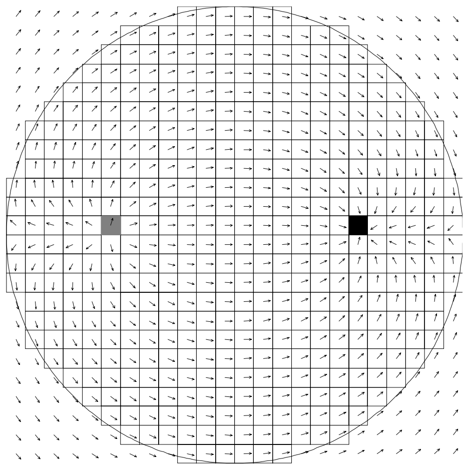

Figure 3: Fluid flow in a circular fracture from an injection/extraction pair (dipole solution).

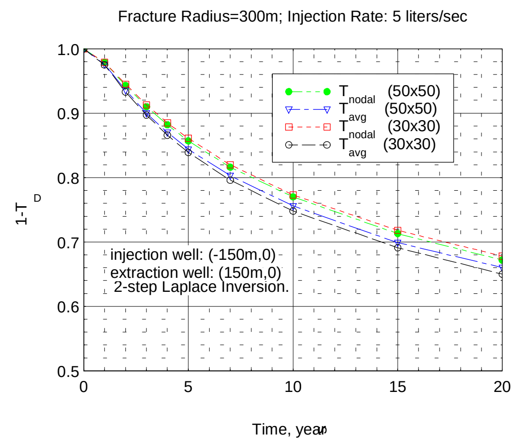

Figure 5: Normalized extraction temperature for example one.

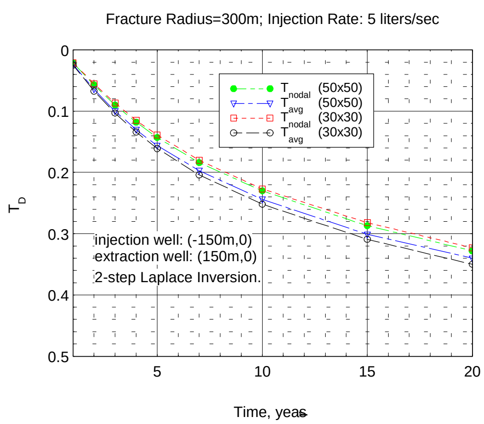

Figure 4: Temperature deficit as a function of time for a 300 m fracture.

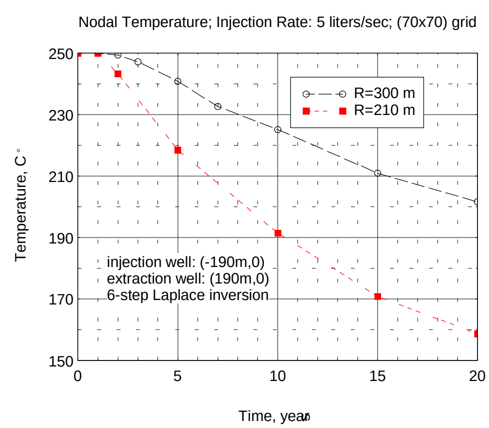

Figure 6: Extraction temperature for two different fracture radii.

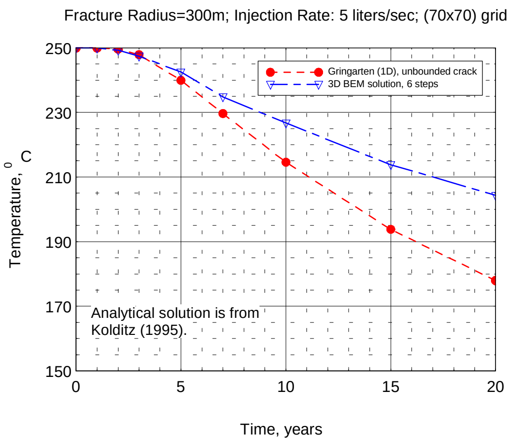

Figure 7: Comparison of 3D numerical and analytical 1D solutions.

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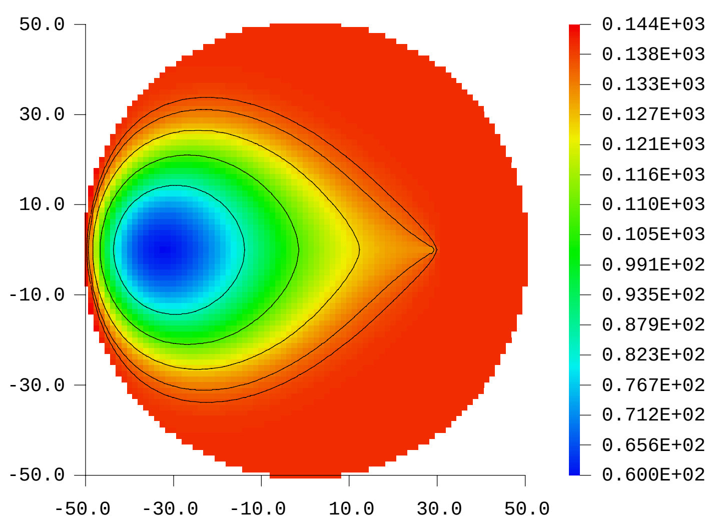

Figure 8. Fluid temperature for the problem of Rodemann (1982).

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