# AN ANALYTIC STUDY OF GEOTHERMAL RESERVOIR PRESSURE RESPONSE TO COLD WATER REINJECTION

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## I. Introduction

Various aspects of reinjection of cooled geothermal water into the geothermal reservoir have been studied by many authors. One question of practical relevance is the calculation of reinjection pressures required. These pressures on the one hand determine the pumping requirements which are important inputs to the technical and economical feasibilities of the project. *On* the other hand, they may also be used as baseline data. **As** time goes on, if the pressure measured becomes much in excess of the calculated values, some kind of plugging may be occurring and remedial action would have to be taken.

For isothermal cases where the injected water is at the same temperature as the reservoir water, the pressure change is simply given by the Theis solution in terms of an exponential integral. The solution shows that this pressure change is directly proportional to the viscosity. It turns out that the viscosity is a strong function of temperature. Over a range of temperatures from 100°C to 250°C, the viscosity changes by a factor of 3 (whereas the density of water changes by about *20%).* This is illustrated in Figure 1.

The injection of cold water into a hot reservoir is a moving boundary problem. *On* the inside of a boundary enclosing the injection well, the parameters correspond to that of the injected cold water; and on the outside, the parameters correspond to that of the reservoir hot water. The boundary is, of course, not sharp because of heat conduction between the hot and cold water. The width of the boundary depends on the aquifer heat conductivity and capacity, and it increases with time as the boundary (or cold temperature front) moves outward from the injection well.

There exist numerical models to solve such a problem. In an earlier work, we made a simple study using numerical model "CCC" developed at the Lawrence Berkeley Laboratory. A sample of calculated results is shown in Figure 1. The present work is an analytic calculation of such a problem when several approximations are applied. Solutions are obtained in terms of well-known functions or in terms of one integral. These calculations are checked against numerical model results.

Figure 1 also indicates that the temperature boundary effects show up in the pressure change as a function of time. Thus, the pressure data may be analyzed to obtain reservoir transmissivity, storativity, and other reservoir parameters. For such purposes, the numerical modeling approach is limited in its utility because of the complexity of calculations. The present analytical approach will prove more advantageous for such a well-test analysis.

Derivation of the governing equation, including temperature effects, will be given in the following section where the permeability-viscosity ratio is assumed to be an arbitrary function of $r^2/t$. In Section 111, this function will be represented by a Fermi-Dirac function, whose parameters are determined based upon physical considerations. The solution for the pressure change is analytic except for the final step, where a numerical integration is called for. We discuss the results and implications of our calculations in Section IV. Summary and concluding remarks are contained in Section V.

## II. Derivation

We start with the three equations:

Eq. of continuity $$\frac{\partial}{\partial t}(\rho\phi) = -\nabla \cdot \rho \underset{\sim}{v}, \qquad (1)$$

Darcy's law $$\underset{\sim}{v} = -\frac{k}{\mu}\nabla P = -K\nabla P, \qquad (2)$$

Eq. of state $$\rho = A(T)e^{\beta P}, \qquad (3)$$

where the pertinent variables are $\rho$(density), +(porosity), $\underset{\sim}{v}$(velocity), k(permeability), $\mu$(viscosity), $\beta$(compressibility) and P(pressure). A new variable K is defined in terms of the permeability and viscosity ($K = k/\mu$). The porosity and compressibility of the medium is assumed to be constant in the following derivation. Working in the cylindrical coordinates and combining equations (1), (2), and (3), one obtains:

$$\rho\phi\left(\beta\frac{\partial P}{\partial t} + \frac{\partial \ell nA}{\partial t}\right) = \left(\frac{\rho K}{r} + \rho\frac{\partial K}{\partial r}\right)\frac{\partial P}{\partial r} + \rho\beta K\left(\frac{\partial P}{\partial r}\right)^2 + \rho K\frac{\partial^2 P}{\partial r^2} \cdot \qquad (4)$$

For water, the percentage variation of density with temperature is much smaller than the corresponding viscosity variations over the same temperature range. Under these conditions, we may consider $\partial \ell nA/\partial t$ to be small compared with other terms in the equation. Furthermore, when the variation of pressure is "smooth," it is customary to neglect the $(\partial P/\partial r)^2$ term. Then, Eq. *(4)* reduces to:

$$\beta\phi\frac{\partial P}{\partial t} = \left(\frac{K}{r} + \frac{\partial K}{\partial r}\right)\frac{\partial P}{\partial r} + K\frac{\partial^2 P}{\partial r^2} \quad . \qquad (5)$$

Ideally, for incompressible fluid, the fluid front propagates as $r^2/t$, and it is therefore expedient to apply the Boltzmann transformation and change the variables from $(r,t)$ to $(z = r^2/t, t)$. One obtains from (5) the final equation that governs the pressure as a function of $z = r^2/t$;

$$\frac{d^2P}{dz^2} + \left(\frac{1}{z} + \frac{c}{4K(z)} + \frac{d\ell nK(z)}{dz}\right) \frac{\partial P}{\partial z} = 0 \qquad (6)$$

where the constant $\beta\phi$ is now written as c. The boundary conditions for Eq. (6) are:

(i) $P = P_o$ at $r = \infty$ and $t = 0$,

i.e., $$\lim_{z\to\infty} P(z) = P_o \quad . \qquad (7)$$

(ii) Incompressible fluid flow through a cylindrical surface around a line source implies

$$\lim_{r\to 0}(2\pi rh)v = -(2\pi rh)\, K \frac{\partial P}{\partial r} = Q \qquad (8)$$

where Q is the pumping rate and h is the aquifer thickness. Applying the Boltzmann transformation on Eq. (8), we obtain, in the variable $z = r^2/t$:

$$\lim_{z\to 0} K(z)\, z \frac{\partial P}{\partial z} = - \frac{Q}{4\pi h} \quad . \qquad (9)$$

We have now reduced the physical problem to the solving of a first-order differential equation for $dP/dz$, provided that the relevant permeability-viscosity function of the system is known. Grouping equations (6), (7), and (9) together, we have:

$$d^2P/dz^2 + \left(\frac{1}{z} + \frac{c}{4K(z)} + \frac{d\ell nK(z)}{dz}\right)\frac{dP}{dz} = 0, \qquad (10a)$$

$$\lim_{z\to 0} K(z)z \frac{\partial P}{\partial z} = - \frac{Q}{4\pi h}, \qquad (10b)$$

$$\lim_{z\to\infty} P(z) = P_o . \qquad (10c)$$

## III. Solution

We assume that $K(z) = k/\mu$ may be represented by a Fermi-Dirac function given by:

$$K(z) = \frac{(K_R - K_I)}{1+e^{-(z-d)/a}} + K_I. \qquad (11)$$

The function takes on the values of $K_I$, $K_R$, and $(K_I + K_R)/2$ respectively at $z = 0$, $z = \infty$, and $z = d$ (see Figure 2a). The function varies appreciably only in the neighborhood of $z = d$; the parameter a characterizes the range of z over which $K(z)$ is appreciably different from $K_I$ and $K_R$. The derivative $dK/dz$ is a symmetrical function about $z = d$. The full-width half-maximum of $dK/dz$ about this point may be shown to be 3.52 a (see Figure 2b). This behavior of $K(z)$ may be understood as follows: Near the wellbore the function takes on the value of $K = k/\mu = K_I$, equal to that of injected water; and it takes on the value $K_R$, equal to that of the reservoir at large radial distances from the well. If we assume a sharp temperature front between the cold injected water and the hot reservoir water, then the location of this transition from $K_I$ to $K_R$ is given by:

$$Qt = \pi r^2 h \frac{\rho_a C_a}{\rho_w C_w} \qquad (12)$$

where $\rho_w$, $C_w$, $\rho_a$, $C_a$ are the densities and heat capacities of water and aquifer, respectively. Eq. (12) implies a constant value for $r^2/t = d$, given by:

$$d = \frac{Q\rho_w C_w}{\pi h \rho_a C_a} \qquad (13)$$

which is the same parameter d used in the Fermi-Dirac function. The fact that the temperature front is not sharp is accounted for by the width of variation from $K_I$ to $K_R$, given earlier by the parameter a.

Avdonin has solved the problem of the propagation of temperature front with the injection of hot water into cold water in an aquifer. In the limit, where there is no vertical heat loss, Avdonin's solution is given by:

$$\frac{T-T_R}{T_I-T_R} = \frac{\mathrm{erfc}(o)}{\Gamma(\nu)} \left(\frac{C_a\rho_a}{4\kappa_a}\right)^{\nu} \left(\frac{r^2}{t}\right)^{\nu} \int_0^1 \exp\left[-\frac{1}{4S}\frac{C_a\rho_a}{\kappa_a}\left(\frac{r^2}{t}\right)\right] \frac{dS}{S^{\nu+1}} \qquad (14)$$

$$\nu = Q\, C_w\rho_w/4\pi h\kappa_a$$

where $T_R$ = initial aquifer temperature

$T_I$ = temperature of injection fluid

$\kappa_a$ = aquifer conductivity.

We note that in Eq. (14) the temperature again varies as $r^2/t$, as in the case of $K(z)$. Since the viscosity is a function of temperature, one expects the variation of $K(z) = k(z)/\mu(z)$ in Eq. (11) to be intimately related to $(T-T_R)/(T_I-T_R)$ here. In particular, we may relate the widths of $dK(z)/dz$ and $(d/dz)(T-T_R)/(T_I-T_R)$. The full-width half-maximum of the curve $d/dz(T-T_R)/(T_I-T_R)$ is governed by a transcendental equation. However, in the limit of "narrow width," the full-width half-maximum can be shown to be $4\sqrt{2}\kappa_a/(C_a\rho_a)$. Equating the two full-width half-maxima, we arrive at the simple expression:

$$a = 1.605 \frac{\kappa_a}{C_a \rho_a} \tag{15}$$

which relates the parameter a, in the theoretical model for $K(z)$, to the physical property of the aquifer.

Integrating Eq. (10a) and applying the boundary conditions (10b), one gets

$$\frac{dP}{dz} = - \frac{Q}{4\pi h} \frac{1}{K(z)z} \propto \left( - \frac{c}{4} \int_0^z \frac{1}{K(z')} dz' \right) . \tag{16}$$

Integrating (16) and applying the boundary condition (10c), we get

$$P(z) = P_o - \frac{Q}{4\pi h} \int_z^\infty \frac{1}{K(z)z} \exp\left(- \frac{c}{4} \int_0^z \frac{1}{K(z')} dz'\right) dz . \tag{17}$$

Given the Fermi-Dirac function for $K(z)$ (Eq. 11), we have

$$\int_0^z \frac{c}{4K(z')} dz' = \frac{c}{4} \frac{z}{K_R} + \frac{ca}{4}\left(\frac{1}{K_R} - \frac{1}{K_I}\right) \ell n \frac{K_I \exp(-(z-d)/a) + K_R}{K_I \exp(d/a) + K_R} . \tag{18}$$

Then, Eq. (17) reduces to

$$P(z) = P_o - \frac{Q}{4\pi h} \exp\left(\frac{-cd}{4K_I}\right) \int_z^\infty \frac{1 + \exp(-(z'-d)/a)}{K_R + K_I \exp(-(z'-d)/a)} \frac{1}{z'} \cdot \tag{19}$$

$$\cdot \frac{[K_R + K_I \exp(-(z'-d)/a)]^{(1/K_I - 1/K_R) ca/4}}{\exp\left(\frac{c}{4K_R} z'\right)} dz' .$$

Eq. (19) is integrated numerically. Figures 3 and 4 show the variation of $P(z) - P_o$ with $z$ for various values of d and a. Table 1 summarizes the parameters used.

## IV. Results and Discussion

The most interesting feature of these plots is that the curve follows, for small values of $t/r^2$, a Theis line with parameters corresponding to those of the native hot water, and for large $t/r^2$ it approaches a line parallel to a Theis line with parameters corresponding to those of the injected water. The transition occurs at $z \approx d$, or $t/r^2 \approx 1/d$, where d may be expressed in terms of the flowrate, heat capacities, and reservoir thickness (see Eq. 13). Thus, injection well test data can yield the transition point d and the separation A. These two additional parameters, when coupled with the two which are normally obtained (transmissivity kh and storativity $\phi\beta h$) affords the possibility of determining the parameters h, $\phi$, and k separately, provided the heat capacities are known.

To make the solution more transparent, we break up the K function into three sections as shown in Figure 2. Thus:

$$\begin{aligned} K &= K_I & z &< d-w \\ K &= K_{FD} & d-w &< z < d+w \\ K &= K_R & d+w &< z. \end{aligned} \qquad (20)$$

Here, $(d - w, d + w)$ defines the interval in z where K changes from $K_I$ to $K_R$, and $K_{FD}$ represents the Fermi-Dirac function given in Eq. (11).

Now Equation (17) gives the general pressure solution:

$$P = P_o - \frac{Q}{4\pi h} \int_z^{\infty} \frac{1}{Kz} \exp\left(- \int_o^z \frac{c}{4K(z')} dz'\right) dz \quad .$$

With K given by Eq. (20), we have

$$\begin{aligned} \int_o^z \frac{c}{4K} dz' &= \frac{c}{4K_I} z & &\text{for } z < d-w \\ &= \frac{c}{4K_I}(d-w) + \int_o^z \frac{c}{4K_{FD}} dz' & &d-w < z < d+w \qquad (21) \\ &= \frac{c}{4K_I}(d-w) + \int_o^{d+w} \frac{c}{4K_{FD}} dz' + \frac{c}{4K_R}(z-d-w) & &z > d+w \ . \end{aligned}$$

Then, the solution is given for

(a) $z > d+w$

$$P_1(z) = P_o + \frac{Q}{4\pi h K_R} C_1 \, Ei\left(-\frac{c}{4K_R} z\right) \tag{22}$$

where

$$C_1 = \exp\left(\frac{cw}{4K}\right) \left(K_R + K_I \exp\left(-\frac{w}{a}\right)\right)^{\frac{ca}{4}\left(\frac{1}{K_I} - \frac{1}{K_R}\right)}$$

(b) $d+w > z > d-w$

$$P_2(z) = P_1(d+w) - \frac{Q}{4\pi h} \exp\left(-\frac{c}{4K_I}(d-w)\right) \int_z^{d+w} \frac{1}{K_{FD} z} \exp\left(-\int_0^z \frac{c}{4K_{FD}} dz'\right) dz \tag{23}$$

(c) $z < d-w$

$$P_3(z) = P_2(d-w) + \frac{Q}{4\pi K_I h} Ei\left(-\frac{c}{4K_I} z\right) \tag{24}$$

For large $z$ or small $t/r^2$, the solution behaves as a constant times the Theis solution using reservoir water parameters (Eq. 22). This constant is approximately one, since w and a are "small" quantities describing the transition width from $K_I$ to $K_R$.

On the other hand, for small $z$ or large $t/r^2$, pressure behaves as the Theis solution with injected water parameters, but with a constant shift, $\Delta$, given by the first term in Eq. (24), i.e.,

$$\Delta = P_2(d-w) - P_o$$

$$= \frac{Q}{4\pi h} \left\{ \frac{C_1}{K} Ei\left(-\frac{c}{4K_R}(d+w)\right) - \exp\left(-\frac{c}{4K_I}(d-w)\right) \int_{d-w}^{d+w} \frac{1}{K_{FD} z} \cdot \exp\left(-\int_0^z \frac{c}{4K_{FD}} dz'\right) dz \right\}$$

where the second term may be obtained by numerical integration (see Eq. 19) or by assuming the integrand to be approximately constant over the interval $(d - w, d + w)$.

In well-test analysis, Figures 3 and 4 may be used directly. The early data may first be compared with the curve for $t/r^2 = 1/d$ yielding **kh** and $\phi\beta h$ values. Matching of later data will give parameter d from which h may be estimated (see Eq. 13). Thus, **k,** $6, and h are evaluated. Of course, in actual field-data analysis other possible effects (e.g., boundaries) may enter and great care has to be exercised.

## V. Summary

A governing equation is obtained assuming temperature-dependent viscosity. The solution is obtained by assuming the parameter $k/\mu$ to be a Fermi-Dirac function of $r^2/t$. The constants in the function are related to the cold water injection problem by a comparison with Avdonin's solution. The result displays an interesting transient pressure curve which initially (small $t/r^2$ values) follows the Theis solution with parameters corresponding to reservoir hot water and at large $t/r^2$ values, turns and becomes parallel to the Theis solution with cold water parameters. Use of these results for cold water injection well-test analysis is briefly discussed.

Work performed under the auspices of the U.S. Department of Energy.

Table 1

Parameters Used in Calculations

$$Q = 80 \text{ kg/sec}$$

$$h = 150 \text{ m}$$

$$c = \phi\beta = 1.2168 \times 10^{-8} \text{ ms}^2/\text{kg}$$

$$K(300^\circ C) = \left(\frac{k}{\mu}\right)_{300^\circ C} = 5.4705 \times 10^{-10} \text{ m}^3\text{s/kg}$$

$$K(100^\circ C) = \left(\frac{k}{\mu}\right)_{100^\circ C} = 1.7857 \times 10^{-10} \text{ m}^3\text{s/kg}$$

$$d = \begin{Bmatrix} 3.2568 \\ 4.2568 \\ 5.2566 \end{Bmatrix} \times 10^{-4} \text{ m}^2/\text{s}$$

$$a = \begin{Bmatrix} 0.1 \\ 1.0 \\ 2.0 \end{Bmatrix} \times 10^{-4} \text{ m}^2/\text{s}$$

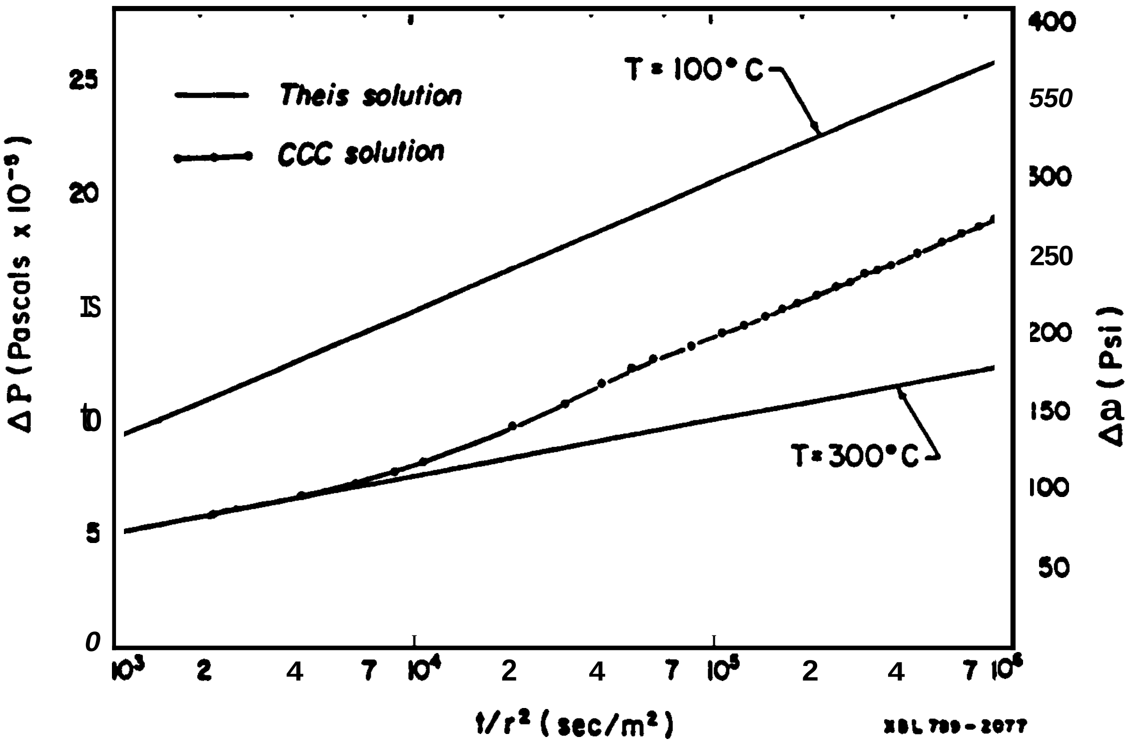


FIGURE 1

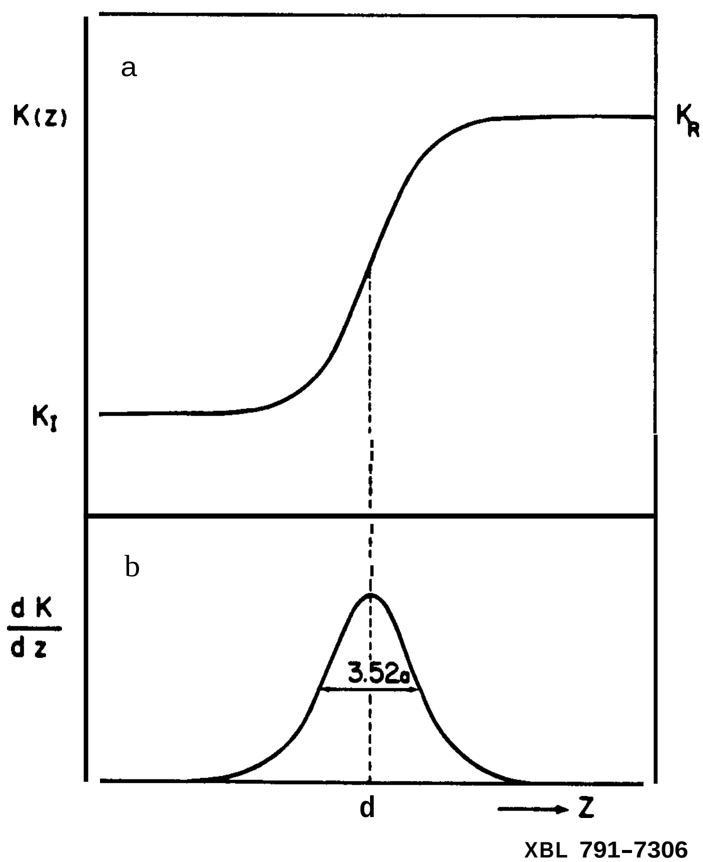


FIGURE 2

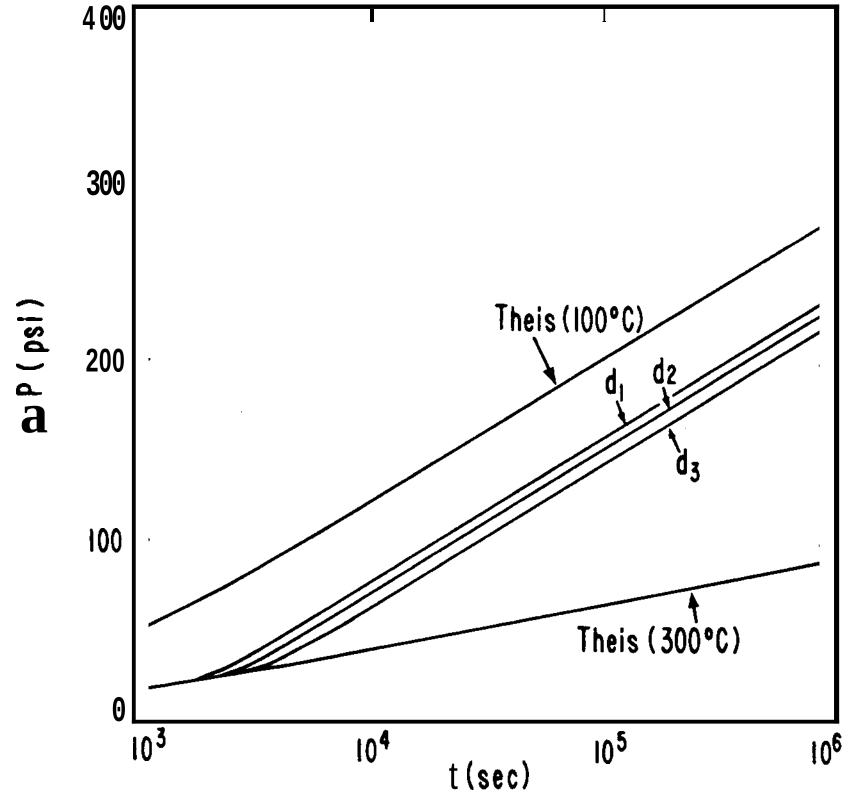


FIGURE 3

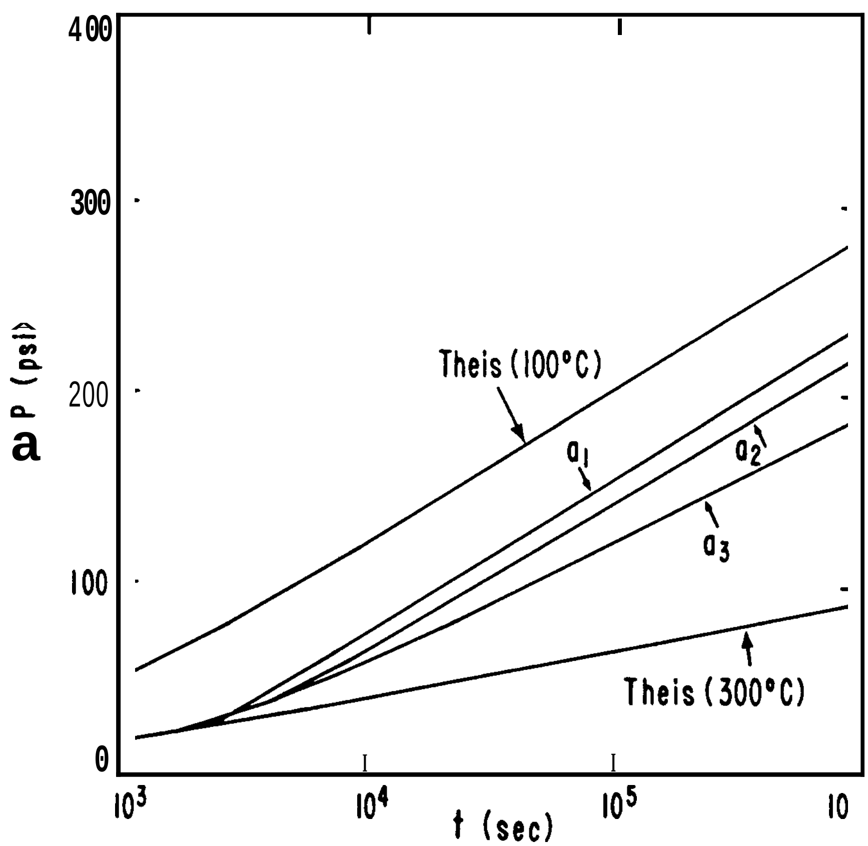


FIGURE 4