SIMULATION OF SATURATED-UNSATURATED DEFORMABLE POROUS MEDIA

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A multiphase consolidation theory is presented which considers a three-dimensional deformation field coupled with a three-dimensional hydrologic flow field. The governing system of equations describes the components of displacement, the fluid pressures and the saturations. The system of equations governing saturated-unsaturated consolidation is obtained as a subset of the above equations. A mixed stress-displacement formulation of the governing equations is introduced, and it facilitates handling of load type boundary conditions while solutions in terms of displacements are still possible. Finite element Galerkin theory is used for spatial approximations, and a weighted implicit finite difference time-stepping scheme is employed to approximate the time derivative terms. Due to the nonlinear nature of the problem, an iterative solution scheme is necessary within each time step.

The model predicts the commonly ignored horizontal displacements in a variably saturated system undergoing simultaneous desaturation and deformation, while using a completely interconnected coupling of the stress and pressure fields within the medium. The model is applied to obtain vertical and horizontal displacements, pressure (head) and saturation values due to pumpage in a phreatic aquifer.

Introduction

Vertical and horizontal ground motions due to changes in pore pressure have been observed in several areas around the world (e.g., at Wairakei, New Zealand, 4.5m vertical and 0.8m horizontal and at Long Beach, California, 8.8m vertical and 3.6m horizontal). In analyzing soil consolidation, research efforts have traditionally focused on the saturated zone. However, unsaturated flow plays an important role in a large number of engineering problems and is the first step in analyzing the multiphase geothermal system.

Narasimhan and Witherspoon, 1977[10], have considered the problem. Their model uses Terzaghi's theory for determining consolidation. Thus, it ignores the lateral soil movement.

In the present work, an iterative Galerkin type finite element method is used to solve the equations of transient flow in saturated-unsaturated deformable porous media in regions having complex geometry. Flow and deformation can take place in both the vertical and horizontal planes, or in a three-dimensional system displaying radial symmetry.

Governing Equations

The system of nonlinear partial differential equations governing saturated-unsaturated flow in a deforming porous medium are[10]:

$$\mu \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_i}\right) + (\lambda+\mu)\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_i}\right) - \left(S_w P_w\right)\delta_{ij} = 0 \tag{1a}$$

$$\frac{\partial}{\partial x_i}\left[\frac{K_{ij}K_{rw}}{\gamma_w}\left(\frac{\partial P_w}{\partial x_j} + \rho_w F\right)\right] = \frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_i}\right) + nS_w\beta_w\frac{\partial P_w}{\partial t} \tag{1b}$$

$$(i,j = 1,2,3)$$

To solve Equations (1) additional information on the relationship between relative hydraulic conductivity and pressure head and degree of saturation and pressure head is required. These functions are usually determined experimentally for each soil type encountered. Two typical curves, one for a coarse soil and one for a fine soil, are presented in Figures 1 and 2. The following mathematical functions are used to characterize the hydraulic properties of the two soil materials.

$$K_{rw} = \left\{1 + \left(a'|h_w|\right)^{b'}\right\}^{-r'} \tag{2a}$$

$$S_w = \frac{\theta'_r}{\theta'_s} + \left(1 - \frac{\theta'_r}{\theta'_s}\right)\left\{1 + \left(\beta'|h_w|\right)^{\gamma'}\right\}^{-r'} \tag{2b}$$

where values for the various coefficients entering Equations (2) are given in Table 1.

The system of equations (1) in terms of the stress tensor and displacement vector becomes [12, chapters 5 and 7]:

$$\frac{\partial t'_{ij}}{\partial x_i} - \frac{\partial}{\partial x_i}\left(S_w P_w\right) = 0 \tag{3a}$$

$$\frac{\partial}{\partial x_i}\left[\frac{K_{ij}K_{rw}}{\gamma_w}\left(\frac{\partial P_w}{\partial x_j} + \rho_w F_j\right)\right] = \frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial x_i}\right) + n\beta_w S_w\frac{\partial P_w}{\partial t} \tag{3b}$$

$$(i,j = 1,2,3)$$

This system of equations is employed in our analysis.

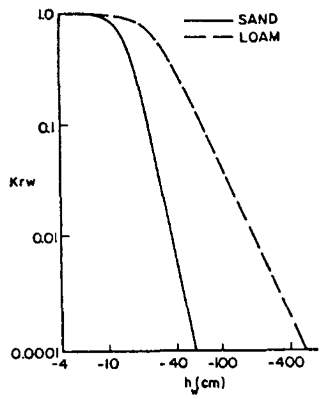

FIGURE 1: RELATIVE HYDRAULIC CONDUCTIVITY VERSUS THE HEAD FOR TWO SOIL MATERIALS. AFTER VAN GENUCHTEN ET AL., 1976 [8].

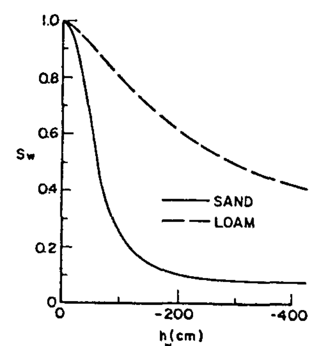

FIGURE 2: DEGREE OF FLUID SATURATION VERSUS THE PRESSURE HEAD FOR TWO SOIL MATERIALS. AFTER VAN GENUCHTEN ET AL., 1976 [8].

VARIABLE	SAND	LOAM	UNITS
θ_r'	0.031	0.10	dimensionless
θ_s'	0.45	0.50	dimensionless
β'	0.0174	0.00481	cm^{-1}
γ'	2.5	1.5	dimensionless
a'	0.0667	0.04	cm^{-1}
b'	5.0	3.5	dimensionless
r'	1.0	0.64	dimensionless

TABLE 1: PHYSICAL DATA OF SAND AND LOAM.

Subsidence Due to Pumpage from a Phreatic Aquifer

The saturated-unsaturated consolidation equations have been applied to the system whose r-z cross section is shown in Figure 3. The finite element solution predicts the vertical and horizontal displacements, excess pore pressure and change in water saturation arising during the pumping process. The bottom impervious layer is assumed fixed and no vertical movement can take place. The wall of the well is also restrained from any lateral movement, and there is seepage into the well. At sufficiently large distance away from the pumping area, the pressure head is not disturbed. Here and at the top surface the soil is free to move both vertically and laterally.

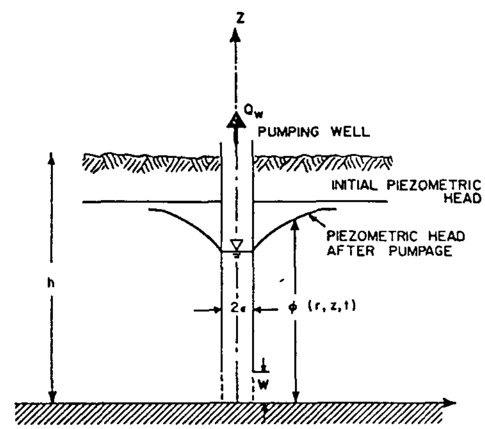

FIGURE 3: A FULLY PENETRATING SCREENED WELL IN AN UNCONFINED AQUIFER. THE SECTION OF THE WELL WHERE SEEPAGE INTO THE WELL CAN TAKE PLACE IS DESIGNATED BY W.

In the numerical solution the spatial domain is divided into isoparametric quadrilateral elements. The time domain is discretized using unequal time intervals and a weighted implicit finite difference iterative scheme [12, chapter 4]. The time step size is calculated according to:

$$\Delta t^{(k)} = K \times \Delta t^{(k-1)}$$

and the elapsed time is given by:

$$t^{(k)} = t^{(k-1)} + \Delta t^{(k)}$$

where K takes values of 0.8, 1.0 and 1.25 depending on the number of iterations required for convergence at the preceding time step. If the required number of iterations is greater than nine, k = -1, and we reset our calculations at an earlier time, i.e.:

$$t^{(k)} = t^{(k-1)} - \Delta t^{(k)}$$

and choose a smaller size time step.

A few of the selected results are presented in Figures 4 - 10.

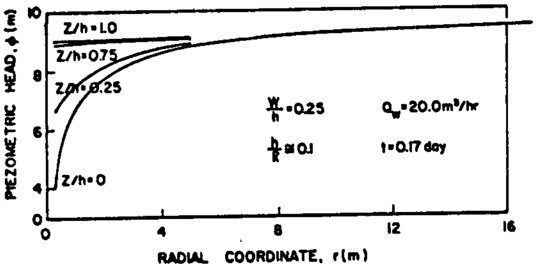

FIGURE 4: HORIZONTAL DISTRIBUTION OF HEAD AT DIFFERENT VERTICAL POSITIONS.

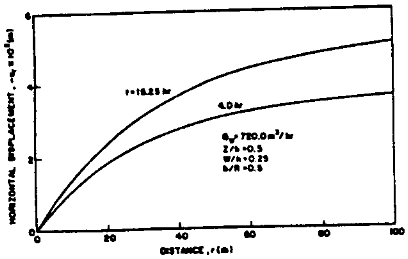

FIGURE 5: HORIZONTAL VARIATIONS OF HORIZONTAL DISPLACEMENT AT THE GIVEN TIME.

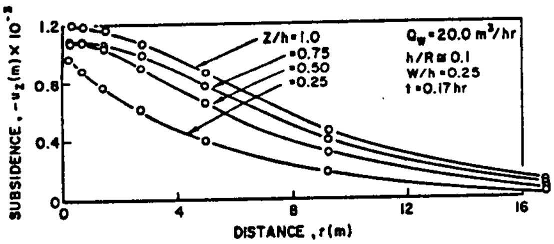

FIGURE 6: HORIZONTAL VARIATION OF SUBSIDENCE FOR DIFFERENT VERTICAL LOCATIONS.

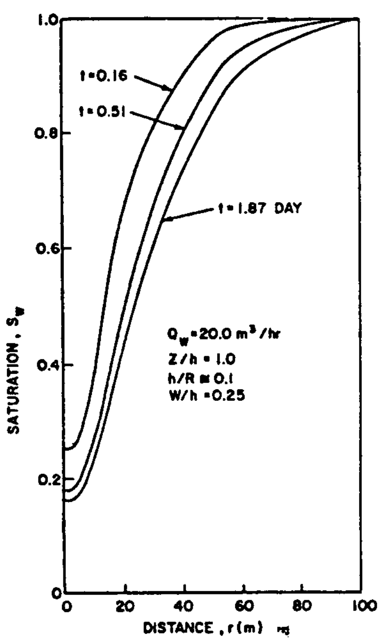

FIGURE 7: HORIZONTAL DISTRIBUTION OF SATURATION FOR THE GIVEN TIMES.

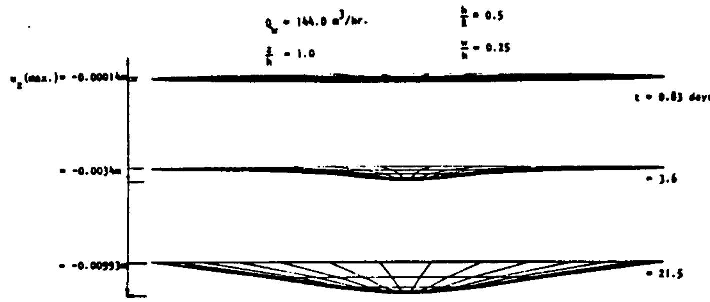

FIGURE 8: THE LAND SURFACE LOWERING AS VIEWED BY AN OBSERVER POSITIONED ON THE SURFACE (ZERO ELEVATION).

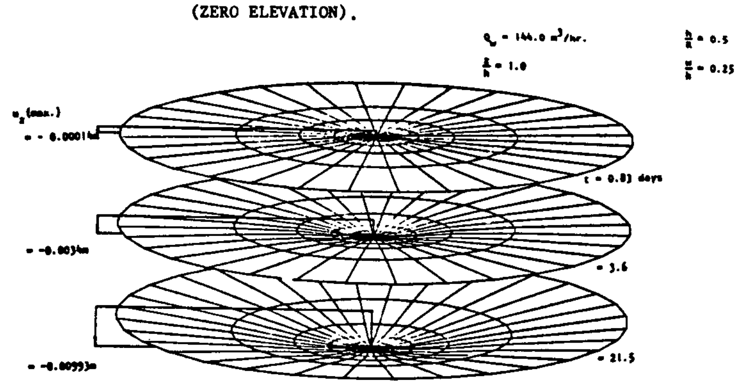

FIGURE 9: THE CONE OF DEPRESSION AS VIEWED FROM ABOVE THE SURFACE FOR SELECTED DIFFERENT TIMES.

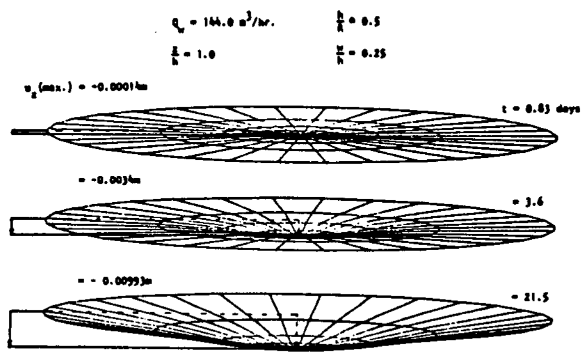

FIGURE 10: THE CONE OF DEPRESSION AS VIEWED BY AN OBSERVER LOCATED BELOW THE SURFACE LEVEL AT DIFFERENT TIMES.

Summary and Results

A mathematical model was developed and used to simulate the deformation field in a desaturating porous medium in which the air phase is assumed to be continuous in the unsaturated zone and to remain at atmospheric pressure. The mathematical model is not applicable to liquid which contains dissolved gas (air bubbles) at different pressures. When the soil is extremely dry and relative hydraulic conductivity, as well as saturation, becomes highly non-linear the approximating equations may become very difficult to solve. Therefore, the model is best suited to soils of moderate to high saturations.

Acknowledgements

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Nomenclature

u_i	displacement vector
P_w	pressure of water
S_w	water saturation
K_{rw}	relative hydraulic conductivity of water
K_{ij}	hydraulic conductivity tensor
β_w	compressibility of water
γ_w	specific weight of water
ρ_w	density of water
λ	Lamé constant
μ	Lamé constant
δ_{ij}	Kronecker delta
x_i	coordinate variables
t	time variable
n	porosity
F_i	body force
h_w	pressure head
θ'_r	residual moisture content
θ'_s	moisture content at saturation
t'_{ij}	effective stress tensor
h	vertical depth
Q_w	volumetric rate of pumping
ϵ	radius of the pumping well
$\phi(r,z,t)$	piezometric head
W	z-coordinate of the seepage zone
R	lateral extent of the aquifer

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