The Effect of Radially Varying Transmissivity on the Transient Pressure Phenomenon

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1. Introduction

During reinjection of cooled geothermal fluid into a reservoir, chemical precipitation and other processes may occur changing the permeability of the aquifer. In general, the permeability becomes a function both of time and space. This will, of course, affect the injection well. Some attempts† have been made to analytically predict the pressure response. The present paper describes our calculations which yield analytic expressions, in terms of a single integral, for a wide class of physically reasonable permeability functions. Results are presented for a few typical examples.

2. Governing Equations

Consider an aquifer consisting of a horizontal slab of thickness, h, penetrated normally by a line source supplying a flow Q. The aquifer medium is taken to be isotropic. In our simplified model we neglect gravity, consider the system to be isothermal, and consider only a single fluid phase. The governing equation is then given by

$$\beta_o \mu \phi \frac{\partial p}{\partial t} = \underset{\sim}{\nabla} \cdot K \underset{\sim}{\nabla} p \qquad (1)$$

if we assume that $\beta \underset{\sim}{\nabla} p \cdot \underset{\sim}{\nabla} p << \frac{1}{K} \underset{\sim}{\nabla} \cdot K \underset{\sim}{\nabla} p$. Here $\beta_o \equiv$ compressibility, $\mu \equiv$ viscosity. $\phi \equiv$ porosity,are taken constant*, and $K \equiv$ permeability, $p \equiv$ pressure.

Given a permeability function of space and time, (1) yields the pressure distribution that results. The present work solved equation (1) for a large class of physically reasonable permeability function. In particular, we look for a family of constant K surfaces in space-time which may be physically reasonable. Let r_o be the distance from the line source to the fluid front. Since the volume of fluid pumped into aquifer equals the volume of aquifer occupied, we see that the fluid front propagates to

$$r_o(t) = Ct^{1/2}$$

where C is a constant. Thus if r is the distance of any point in the

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* The same analysis can be easily adapted to the case where μ is not a constant, but that K/μ is in the form of the permeability functions described below.

† For example, A. Sklar, Lawrence Livermore Laboratory Annual Report (1977) unpublished.

aquifer to the source, then points with $r^2/t < C^2$ will have permeability K_∞ and if $r^2/t > C^2$ they will have permeability K_o. Points with the same value of the ratio r^2/t will have the same permeability. We shall solve equation *(1)* first for permeabilities of the form,

$$K(\underset{\sim}{r},t) = K_o\ \beta(r^2/t) \qquad (2)$$

where r is the cylindrical radial coordinate, and β is an arbitrary function. We shall then extend the class of solutions to those of the more general permeability function

$$K(\underset{\sim}{r},t) = K_o \propto (t)\ \beta \left[r \Big/ \int_o^t \propto(t')dt' \right] \qquad (3)$$

where $\propto$ is an arbitrary positive function of t.

3. Solution

To make equation *(1)* dimensionless, units are chosen so that $\beta_o\phi = 1$, $\mu = 1$, and $\lim\limits_{r \to o} K(\underset{\sim}{r},t) \equiv K_o = 1$ then dimensionless quantities are:

$$m' = \frac{m(gm)}{c} \quad ; \quad r = \frac{r(cm)}{b} \quad ; \quad t' = \frac{t(sec)}{a}$$

where typically $a = \beta_o\mu\phi \sim 10^{-13}$sec; $b = \sqrt{K_o} \sim 10^{-5}$ cm; $c = \mu ab \sim 10^{-20}$gm.

Thus *(1)* becomes

$$\frac{\partial p}{\partial t} = \underset{\sim}{\nabla} \cdot \beta\ \underset{\sim}{\nabla}\ p \qquad (4)$$

If we look at the solutions where p is a function of r only, $p(\underset{\sim}{x},t) = \bar{p}(r,t)$. Then

$$\frac{\partial \bar{p}}{\partial t} = \beta \frac{\partial^2 \bar{p}}{\partial r^2} + \left[\frac{\beta}{r} + \frac{\partial \beta}{\partial r}\right]\frac{\partial \bar{p}}{\partial r} \qquad (5)$$

Next we change variables $\begin{matrix} r \\ t \end{matrix} \longrightarrow \begin{matrix} z \equiv r^2/t \\ w \equiv t \end{matrix}$, and apply the separation of variables,

$$\bar{p}(r,t) = P(z,w) = \phi(w)\chi(z)$$

On substitution, we find $\phi(w) = 1$ and χ satisfies

$$\frac{\partial^2\chi}{\partial z^2} + \left[\frac{1}{z} + \frac{\partial \ell n\beta}{\partial z} + \frac{1}{4\beta(z)}\right]\frac{\partial \chi}{\partial z} = 0 \qquad (6)$$

which is really a first order differential equation for $\partial\chi/\partial z$.

It remains only to integrate the equation and impose the remaining boundary conditions, which are

$$p(r,o) = p_o \qquad\qquad p(\infty,t) = p_o$$

$$\underset{r \to o}{limit}\ 2\pi r h K(\underset{\sim}{r},t)\ \frac{\partial p}{\partial r} = -Q$$

$$\underset{t \to \infty}{limit}\ 2\pi r h K(\underset{\sim}{r},t)\ \frac{\partial p}{\partial r} = -Q$$

In terms of the variable z, these boundary conditions correspond to

$$\chi(\infty) = p_o \qquad\qquad \underset{z \to o}{limit}\ z\ \frac{\partial \chi}{\partial z} = \frac{-Q}{4\pi h}$$

Thus the solution of (6) after putting in units is,

$$p(\underset{\sim}{x},t) = p_o + \frac{\mu Q}{4\pi h K_o}\ e^{(1/4)I(0)} \int_{\frac{\beta_o \mu \phi}{K_o}\frac{r^2}{t}}^{\infty} \frac{e^{-(1/4)\ I(z')}}{z'\beta(z')}\ dz' \qquad (7)$$

where $$I(z) \equiv \int^{z} \frac{dz'}{\beta(z')}\ .$$

Finally, we obtain the solutions for the more general permeability (3) from those solutions already obtained. The method depends on a property of the differential equation

$$\frac{\partial f}{\partial t} = K(r,t)\ \frac{\partial^2 f}{\partial r^2} + \left[\frac{K}{r} + \frac{\partial K}{\partial r}\right] \frac{\partial f}{\partial r} \qquad (8)$$

and does not depend on the specific form of K other than its being a function of r and t only (e.g., the same method could be used to generate new solutions if [8] is initially solved for other forms of K).

To get the new solutions assume that (8) has been solved for f, with a given K. Then consider the transformed function

$$f_\alpha(r,t) \equiv f\left[r, \int_o^t \alpha(t')\ dt'\right]$$

where α is an arbitrary positive function. f_α does not satisfy (8) since

$$\frac{\partial f_\alpha}{\partial t} = \alpha(t)\ \frac{\partial f}{\partial t}$$

Rather f_α satisfies,

$$\frac{\partial f_\alpha}{\partial t} = K_\alpha(r,t)\ \frac{\partial^2 f_\alpha}{\partial r^2} + \left[\frac{K_\alpha}{r} + \frac{\partial K_\alpha}{\partial r}\right] \frac{\partial f}{\partial r}$$

where $$K_\alpha(r,t) \equiv \alpha(t)\ K\left[r, \int_o^t \alpha(t')dt'\right]$$

Furthermore, the boundary conditions on f_α and f are the same so that if p is the pressure response due to K, then to find the pressure at the point (r,t) when the permeability is K_α, we just evaluate p at the point (r,τ) where

$$\tau = \int_0^t \alpha(t')dt' \ .$$

4. Results

We have calculated the pressure distributions resulting from the following permeability functions

$$K_0 = K_0$$

$$K_1 = K_0\left[1 - \frac{\rho}{z+\varepsilon}\right], \ \rho = 10^{-7} \ \varepsilon = 4\rho/3$$

$$K_2 = K_0\left[1 - \frac{\rho}{z+\varepsilon} + \frac{1}{3}\frac{\rho^2}{(z+\varepsilon)^2}\right], \ \rho = 10^{-6} \ \varepsilon = 2\rho/3$$

$$K_3 = K_0 \exp\left[-\rho/(z+\varepsilon)\right], \ \rho = 10^{-6} \ \varepsilon = 4\rho/3$$

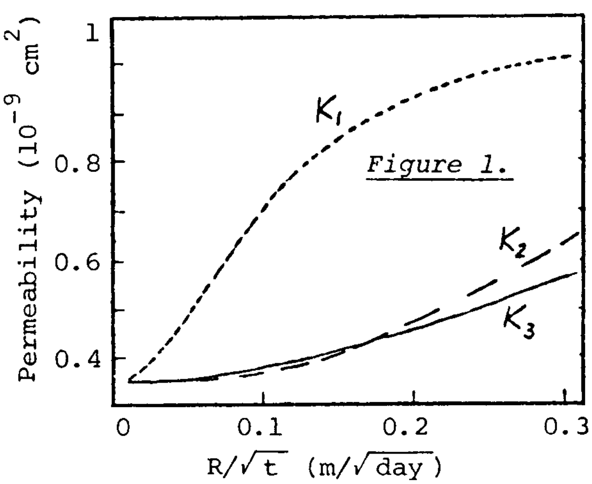

Figure 1.

For comparison permeabilities $K_1 - K_3$ are graphed v s. $r/\sqrt{t}$ in Figure 1*.

The constant permeability K_0 leads to the Theis Solution which is graphed in Figure 2. Figures 3-5 give graphs of $K_1 - K_3$ and the corresponding calculated pressure distributions.

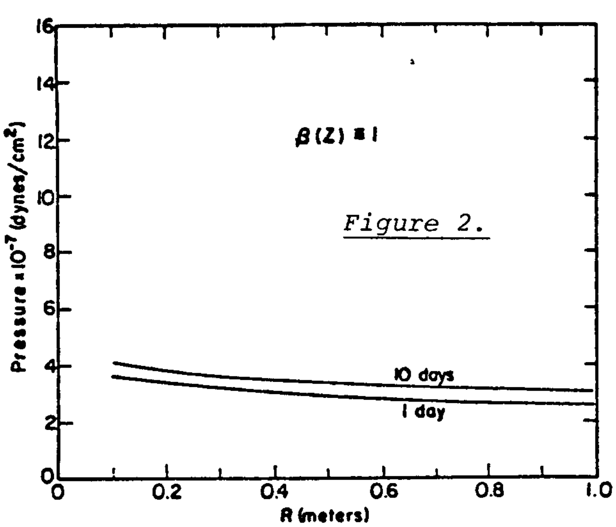

Figure 2.

5. Summary

We have obtained an analytical solution for the pressure response in a reservoir with permeability of the form $K = K(r^2/t)$. It has been found that these solutions may be used to generate additional solutions for

$$K = \alpha(t)K\left[r^2 \Big/ \int_0^t \alpha(t')dt'\right]$$

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* In this Figure, the parameters in K_3 are $\rho = 1.85 \times 10^{-6}$ and $\varepsilon = (4/3)\rho$. Hence, at z=o all the permeability functions K_1 to K_4 have the value $0.25\,K_0$.

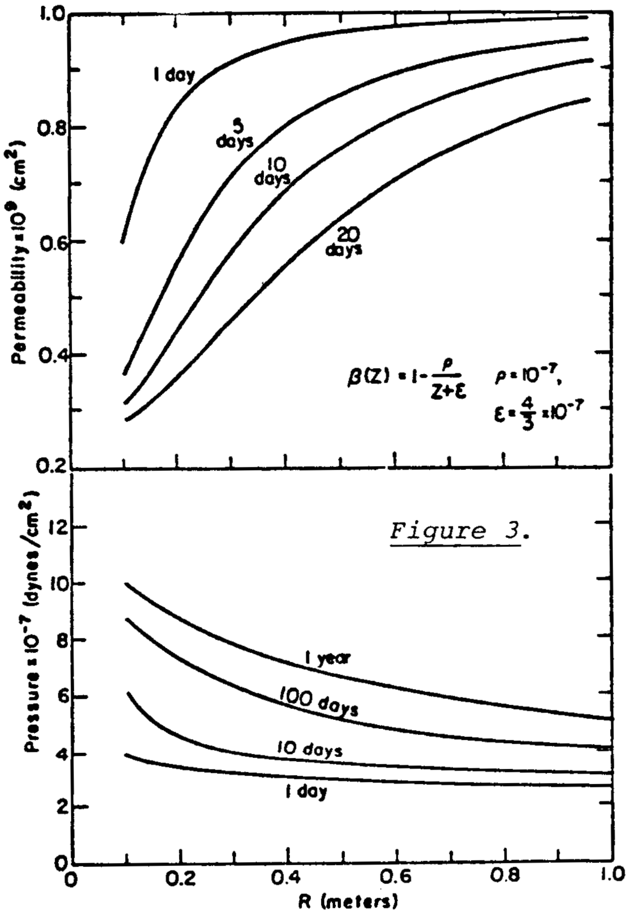


Figure 3.

Several function forms for $K(r^2/t)$ have been studied and the resulting pressure distributions calculated.

A general method for generating the pressure response $\bar{p}$ for a permeability

$$\bar{K}(r,t) = \alpha(t) K \left[r, \int_0^t \alpha(t')dt \right]$$

was developed, once the solution $p(r,t)$ is previously found (analytically or numerically) for a permeability function $K(r,t)$. The solution for $\bar{K}(r,t)$ is

$$\bar{p}(r,t) = p \left[r, \int_0^t \alpha(t')dt \right]$$

The only restriction on α is that it be positive.

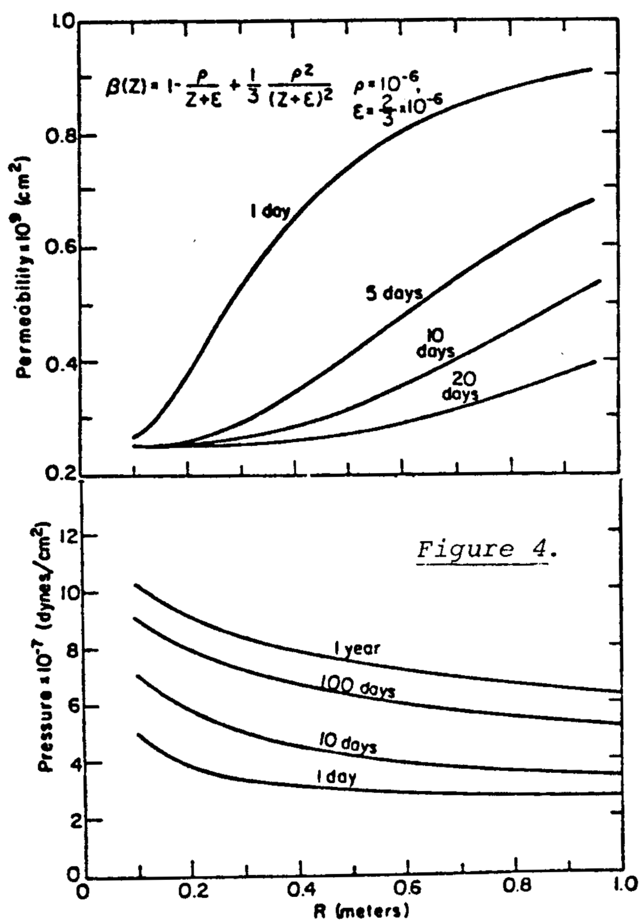


Figure 4.

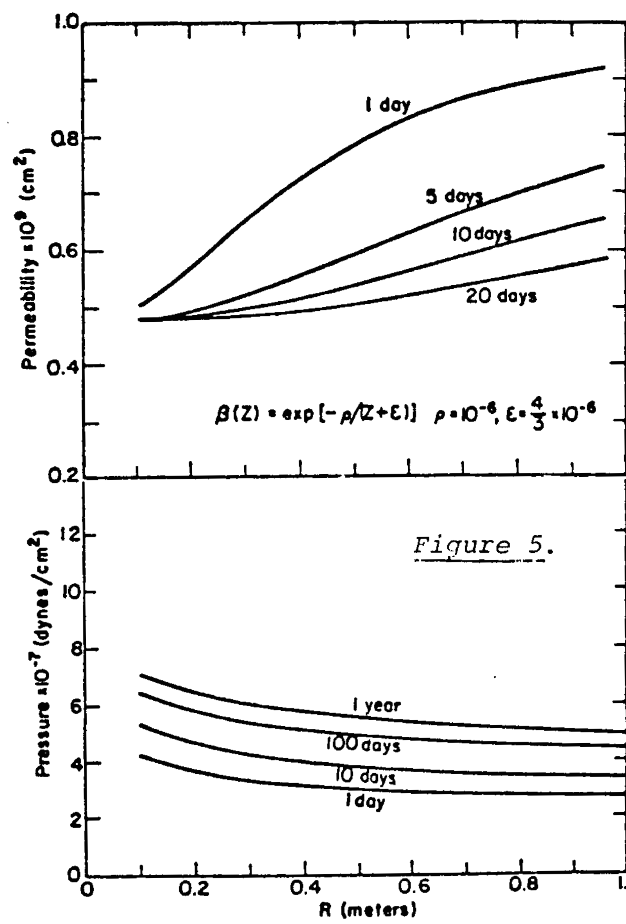
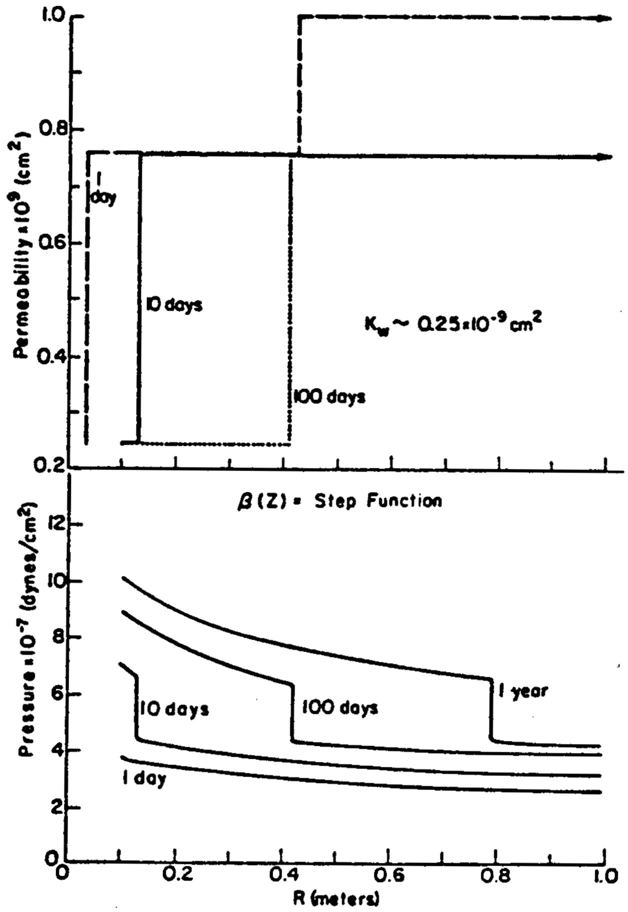


Figure 5.



1.0
0.8
0.6
0.4
0.2
Permeability x10^9 (cm^2)
1 day
10 days
100 days
K_w ~ 0.25x10^-9 cm^2
β(Z) = Step Function
Pressure x10^-7 (dynes/cm^2)
12
10
8
6
4
2
0
10 days
100 days
1 year
1 day
0
0.2
0.4
0.6
0.8
1.0
R (meters)

Figure 6.