METHODS OF SOLUTION OF THE EQUATIONS FOR CONVECTION IN POROUS MEDIA, WITH GEOTHERMAL APPLICATIONS

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Various approaches to the solution of the equations of thermal convection in fluids may be classified,, for convenience, under such headings as: (1) the Stuart-Watson method, which deals with the behavior of finite-amplitude instabilities, for which $R/R_c \simeq 1$, where R is the Rayleigh number and R_c is its critical value for neutral stability; (2) the Galerkin method, a well-known numerical technique utilizing truncated expansions in orthogonal functions, which has been applied up to $R/R_c \simeq 0(10)$; (3) the variational method, which seeks to establish bounds on the heat transport, for given Rayleigh number, up to large R/R_c; (4) direct numerical solution of the convection equations, usually in finite-difference form, up to $R/R_c \approx 0(10)$.

These techniques are considered in relation to the equations of convection of variable-viscosity fluid in a porous medium.

Since the particular application is intended to be geothermal convection, many simplifications may have to be accepted. First, it is assumed that the flow can be treated as flow through porous media. This is not necessarily true, although the approximation becomes more satisfactory if only large-scale motions are considered. Secondly, the medium may not be isotropic. This is not a serious difficulty, but isotropy will be assumed for convenience. Thirdly, salt may be transported as well as heat, and can exert an influence upon fluid buoyancy. Evidently, the transport of salt would involve a straightforward generalization of the treatment for heat transport (although some new phenomena are encountered), and is not considered here. Fourthly, chemical interaction of the fluid with the medium, which would introduce great complications, is assumed to be negligible.

Very large temperature differences are encountered in geothermal applications, so that the dependence upon temperature of fluid properties needs to be taken into account. The most important of these is the variation of viscosity, which may involve an order-of-magnitude change. The fluid and porous medium are assumed to be incompressible, but dilation and contraction of the fluid with temperature changes may lead to a 20% change in density, which is of some significance. This is most readily taken into account, while retaining a convenient form of the equations, by introducing a vector $\underset{\sim}{q}_m$ proportional to the mass flow rate, and relating this to the volume flow rate $\underset{\sim}{q}$ by

$$\rho_o \underset{\sim}{q}_m = \rho \underset{\sim}{q} \tag{1}$$

Here ρ is the fluid density at temperature T, and ρ_o is a density

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corresponding to a reference temperature T_o. Then the equations of continuity, motion and heat transport take a convenient form (c.f. Wooding, 1957, 1960):

$$\frac{\varepsilon}{\rho_o}\frac{\partial \rho}{\partial t} + \nabla \cdot \underset{\sim}{q}_m = 0 \tag{2}$$

$$\frac{1}{\rho_o} \nabla P + \underset{\sim}{k} g \frac{\rho}{} + \frac{\nu}{k} \underset{\sim}{q}_m = 0 \tag{3}$$

$$E \frac{\partial T}{\partial t} + \underset{\sim}{q}_m \cdot \nabla T = \kappa \nabla^2 T \tag{4}$$

Here t is time, P is pressure, ε is the porosity of the medium,

$$E = \{(1 - \varepsilon)c_s\rho_s + \varepsilon c\rho\}/c\rho_o \tag{5}$$

is the ratio of the heat capacity per unit volume of saturated medium (at temperature T) to that of the fluid (at temperature T_o), c signifies specific heat and suffix s denotes the solid medium. Also, k is the intrinsic permeability, ν is the kinematic viscosity of the fluid, κ is the thermal diffusivity of the saturated medium, here taken constant, g is the magnitude of gravity and $\underset{\sim}{k}$ is a unit vector, directed vertically upwards. While Equations (2) and (4) are straightforward conservation relations, Equation (3) (Darcy's Law) is a force-flux relation which involves some assumptions--notably the existence of the permeability k.

In addition to the foregoing, there must exist an equation of state for each temperature-dependent quantity. Here the relation for density often is taken as

$$\frac{\rho - \rho_o}{\rho_o} = - \frac{T - T_o}{\alpha T_o} \tag{6}$$

i.e., the Boussinesq approximation, α being the temperature coefficient of linear expansion of the fluid. Thermal expansion of the medium is neglected. A more satisfactory representation of the thermal expansion law requires a polynomial.

The variation of viscosity with temperature is quite strongly non-linear. For geothermal applications a simple relation for water is

$$\text{—} = \quad + a\,(T - T_o)\}^{-1} \tag{7}$$

where the coefficient a may be $O(10)$ (Wooding, 1957). This constitutes one of the main obstacles to the direct use of some of the standard methods of solution of the convection equations.

A further important source of nonlinearity is the term $\underset{\sim}{q}_m \cdot \nabla T$ in (4). Since most convection studies in the literature correspond to the case of constant viscosity, the latter nonlinearity has received considerable attention, whereas the particular situation of variable viscosity has received relatively little.

In viscous-fluid convection, the important case of a small variation of viscosity has been treated by perturbation methods by Palm (1960), Segel and Stuart (1962), Palm and Øiann (1964), Segel (1965a, b) and others, with considerable success, since the reason for the existence of hexagonal convection cells over a finite range of Reynolds number has been satisfactorily explained.

An equivalent analysis for flow in porous media has not been carried out, as far as is known. However, most cases of interest in porous-media flow, particularly with geothermal applications, involve very large changes of viscosity, for which a perturbation analysis on the above lines would not be satisfactory. Generally, it is considered necessary to resort to numerical techniques., as in papers by Wooding (1957, 1963) and recent studies by Horne (1975) and Kassoy and Zebib (1975), or by the use of variational techniques (e.g., Wooding, 1960, 1975).

Techniques of Solution of the Convection Equations

A convenient classification is the following:

1. The Stuart-Watson Method (Stuart, 1958; Watson, 1960; Stuart, 1964; etc.) is used for treating finite-amplitude instability problems, i.e., to find answers to the question: What happens to an infinitesmal disturbance as it grows to finite amplitude in a situation which is linearly unstable? Clearly, a single disturbance mode will, through nonlinearities, generate a "normal-mode cascade" (Segel, 1965c) and these in turn will interact *to* modify the fundamental disturbance amplitude. Generally the effect is to introduce nonlinear damping of the fundamental, so that the disturbance grows to a finite amplitude and a new stable equilibrium results. However, special cases of reinforcement (e.g., resonance) may be encountered.

Since the Stuart-Watson method involves expansion in normal modes about the neutral disturbance, and the expansion is truncated after the third-order terms (c.f. Segel, 1965a), it is limited to flow situations where the amplitude remains small (although finite) throughout all time. This generally restricts its use in convection problems to cases where the Rayleigh number $R \simeq R_c$ --the critical value for neutral stability. In spite of this limitation, the method yields great physical insight into mechanisms of fluid instability.

2. The Galerkin Method (Veronis, 1966; Straus, 1974; Clever and Busse, 1974; etc.) is one of the oldest and best known. Briefly, expansions of the dependent variables in the convection equations are sought in terms of orthonormal functions which satisfy the boundary conditions term by term. The method of truncation of these series, first described by Veronis, is to

choose a "maximum total wavenumber," i.e., to retain only those terms for which the sum of the wavenumbers in the various spatial directions does not exceed a given upper bound. When the differential equation is linear, it is possible to obtain relationships between the coefficients by term-by-term comparison. Otherwise each partial differential equation may be reduced to a set of ordinary differential equations (say in time) or an algebraic equation, by multiplying by successive terms of the orthonormal set, and integrating over space. The resultant set of ordinary differential equations, or of algebraic equations, can then be solved by conventional numerical methods.

For the case of two-dimensional convection in a porous medium with constant viscosity, Straus (1974) has calculated the dependence of Nusselt number (Nu) upon Rayleigh number up to $R \approx 380$, above which point (from linearized stability analysis), two-dimensional solutions are unstable. For $R > 100$, the (R, Nu)-curve shows a significant change in slope. The results are in good agreement with experimental measurements.

3. The variational method of Howard (1963) and Busse (1969) has been used by Busse and Joseph (1972) and Gupta and Joseph (1973) to calculate upper bounds to the Nusselt number, as a function of Rayleigh number, for three-dimensional convection in a porous medium at constant viscosity. In this approach, the equations of motion and heat transport are recast as a variational problem, involving averages over the entire porous layer. Then the dependent variables appearing in the variational problem are replaced by a "class of admissible functions" which includes all statistically stationary solutions, and which satisfies the boundary conditions and any supplementary conditions which may be specified. The Euler equations of the variational problem embrace a wide class of solutions, corresponding to extreme values of the system, and these may be represented by expansions in orthonormal functions based upon horizontal wavenumbers α_n. A single wavenumber is adequate up to $R = 221.5$ (Gupta and Joseph, 1973), at which point the solution bifurcates and two α-values are needed. These calculations have been carried up to about $R = 500$ with very good agreement with experiment. At higher R, an asymptotic (boundary-layer) analysis based on that of Chan (1971) predicts appropriate qualitative behavior, but these results are not in good quantitative agreement with the numerical studies.

4. Methods of numerical solution of the convection equations are now the subject of a very large literature, and extensive reviews such as those by Orszag and Israeli (1974), or of Horne (1975) for porous media, are necessary to ensure adequate treatments. Because of limitations in computer capacity and speed, most convection studies have been limited to two-dimensional flows. Convection in viscous fluids with large variations in viscosity has been considered, for two-dimensional flows, by Torrance and Turcotte (1971) and by Houston and De Bremaecker (1974).

Horne (1975) has carried out some calculations with variable viscosity for two-dimensional convection in porous media. in discussing his results, Horne comments that equally-vigorous convection occurs with variable viscosity at lower apparent Rayleigh number than in the constant-viscosity case, since R is defined for $T = T_o$, where viscosity is high. He also observes that the representation of a variable-viscosity convection system with a constant-viscosity model is "inexact, but not entirely without use." This suggests that an intermediate value of Rayleigh number might be found which corresponds to the constant-viscosity value at the same Nusselt number. However, in studying the onset of convection in porous media with variable viscosity, Kassoy and Zebib (1975) conclude that the viscosity variations have substantial effects upon the flow pattern and that a mean value of viscosity cannot be taken to estimate a suitable intermediate value of R.

Numerical studies of convection in three dimensions in a viscous fluid, based on the early work of Chorin (1966), have been performed by Veltishchev and Zelnin (1975), taking viscosity constant. In this approach the equations are represented in finite-difference form, using ''primitive" variables, i.e., velocity components (u, v, w), temperature and pressure. Calculations were carried out in a rectilinear domain with horizontal dimensions 2.34 and 4.03 times the depth. (In common with other numerical models, the domain is limited to a finite box.) Interesting stable convective flow patterns are obtained, notably two-dimensional rolls for low to intermediate values of the Rayleigh number, three-dimensional flows in a higher, somewhat narrower, range, and unsteady motions above that. These flow transitions are accompanied by changes in slope of the Rayleigh number-heat flux curve.

For three-dimensional Convection in a porous medium, relatively few references can be found. Holst and Aziz (1972) used a combination of successive over-relaxation for the solution of the equation of motion (reduced to Poisson's equation) with centered differencing for the first derivations of the heat equation. However, the more advanced techniques of direct solution utilized by Horne and O'Sullivan (1974) and Horne (1975) are faster and more accurate. These employ an Arakawa (1966) finite-difference scheme to evaluate the terms arising from $\tilde{q}_m \cdot \nabla T$ in (4), and an extension of the Buneman algorithm (Buzbee, Golub and Nielson, 1970) to evaluate the Poisson equation. Horne (1975) has used these techniques to calculate solutions for three-dimensional convection in a cubical box, taking a 17 x 17 x 17 mesh, at R = 500. For a uniformly heated lower boundary, convection is found to take the form of two-dimensional rolls, even when the initial perturbation is three-dimensional.

It is planned to publish a more detailed treatment at a later date.

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