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The dynamics of viscous compressible fluid in a fracture

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ABSTRACT

We solve a problem of fluid dynamics in an arbitrarily shaped fracture containing proppant. The fracture is embedded in an absolutely rigid solid. The fluid flow is induced by normal harmonic oscillations of fracture walls that are permeable, allowing the fluid to filtrate into the surrounding formation. The amplitude of oscillations can be described as a function of a spatial coordinate along the fracture. Fluid flow inside the fracture is described by Biot's modification of Darcy's law that includes both inertia and filtration terms. Filtration inside the fracture may drastically change its acoustic characteristics. The acoustic resonance along the length of the fracture filled with proppant can be observed only if the permeability of the proppant is extremely high; otherwise filtration of the fluid in the fracture dampens the resonance. If the resonance is observed, its period differs from that in a fracture without internal filtration. The apparent acoustic velocity in the fracture is lower than sound velocity in the fluid because of coupling between the fluid and the proppant. In a fracture without proppant, fluid filtration into the surrounding rock acts to decrease resonance peaks and to increase resonance period. Resonance period changes significantly in fractures of varying shape, an effect which is important when estimating the size of a tapered fracture.

INTRODUCTION

The acoustic properties of both microfractured rocks and rocks with macrofractures are strongly influenced by the relative motion between solid and fluid. The dependence of velocity and attenuation of acoustic waves on the geometry of microfractures and on fluid properties can, in principle, serve to evaluate material structure and transport characteristics both in situ and in the laboratory. The investigation of dynamic oscillatory fluid motion in large cracks is crucial in detecting and evaluating natural and hydrofractures.

Significant progress has been made in the quantitative modeling of the dynamic interaction of a fracture with surrounding elastic rock. Using the finite-difference method, Chouet (1986) described the transient response of a fluid-driven crack to an impulsive pressure in the fluid. This impulse triggers resonance with periods proportional to the crack stiffness and to the width and length of the crack. The crack wave sustaining the resonance is analogous to a tube wave propagating in a fluid-filled borehole. Its wave speed is always lower than the acoustic velocity of the fluid. This theoretical model has been used successfully to explain the very long period of a volcanic tremor (Chouet, 1988).

Ferrazzini and Aki (1987) analytically studied normal modes trapped in a liquid layer sandwiched between two elastic half-spaces. They also found a slow wave similar to a tube wave. The phase velocity and amplitude of this slow wave are in close correlation with those obtained by numerical models. Using a three-dimensional (3-D) fluid-filled crack model, Ferrazzini et al. (1990) were able to quantitatively estimate the size of a fracture during hydrofracture experiments at Fenton Hill, New Mexico.

An experimental proof to the existence of slow fracture waves as predicted in Chouet (1986) and Ferrazzini and Aki (1987) has been given in Tang and Cheng (1988).

Hornby et al. (1989) used the low-frequency reflected Stoneley-wave mode to locate permeable fractures intersecting a borehole and to estimate their effective apertures. The fracture is idealized as a plane-parallel fluid slab of infinite extent. The authors show that within the considered range of parameters, the results do not change significantly when the condition of finite extent is included.

Mahrer and Mauk (1987) modeled a two-dimensional (2-D) hydraulic fracture surrounded by a low-velocity zone to investigate hydraulic fracture treatment-induced microseismicity. They found that a fixed-aperture fracture acted as a leaky waveguide confining seismic energy and altering propagation characteristics. Another important conclusion of this work was that the presence of the fracture accentuated the

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P-wave energy propagated along its length and introduced a secondary P-wave reflection from the opposite fracture tip.

In spite of significant progress achieved in predicting and interpreting far-field radiation due to the excitation of a fluid-filled crack, the influence of some factors on the fracture dynamics remains unclear. Among those are:

1) the spatial variation in a fracture's aperture;
2) filtration from the fracture into the surrounding formation;
3) filtration inside the fracture.

Dvorkin et al. (1990) gave a general solution to a 2-D problem of oscillatory flow of a viscous compressible fluid in a long thin arbitrarily shaped fracture embedded in an absolutely rigid solid. Fluid motion in the fracture was modeled as a one-dimensional (1-D) laminar oscillatory flow. The solution of the problem was reduced to an ordinary second-order differential equation that can be solved numerically or, in the case of a step-shaped pore, analytically. This model was extended to the case when the walls of the pore are permeable.

In this paper, we solve a new problem of fluid dynamics in an arbitrarily shaped fracture containing proppant. The walls of the fracture are assumed to be absolutely rigid. The fluid flow is induced by normal harmonic oscillations of fracture walls that are permeable and allow the fluid to filtrate into the surrounding formation. The amplitude of oscillations can be described as a function of a spatial coordinate along the fracture. Thus, the solution presented is useful as part of a more general approach where the fracture is surrounded by an elastic medium.

Fluid flow inside the fracture is described by Biot's modification of Darcy's law that includes both inertia and filtration terms. This development is important both methodologically and practically: (1) for the first time Biot's modification of Darcy's law (see Appendix) is used to model the flow of fluid inside the fracture; (2) this innovation may be relevant to hydrofractures filled with proppant, natural fractures filled with sand and clay, or can be used to model fluid flow in a rough-walled thin fracture with fluid filtrating through asperities.

Our model of hydrodynamic relaxation in a fracture is close to the solution given by Murphy et al. (1986). However, we consider the variation of fracture aperture, the effects of filtration through fracture walls, and filtration inside the fracture.

We show that filtration inside the fracture may drastically change its acoustic characteristics. The acoustic resonance along the length of the fracture filled with proppant can be observed only if the permeability of the proppant is extremely high. Otherwise, filtration of the fluid in the fracture dampens the resonance. If the resonance is observed, its period differs from that in a fracture without internal filtration. The apparent acoustic velocity in the fracture is lower than acoustic velocity in the fluid because of coupling between the fluid and the proppant.

In this paper, we also employ our previous solution (Dvorkin et al., 1990) to examine the dynamics of a fracture without proppant. We show that fluid filtration into surrounding rock acts to decrease resonance peaks and to increase resonance period. This effect may provide a possible method of estimating in-situ permeability from resonance peak offset.

Another important result is a significant change of resonance period in fractures of varying shape. This effect has to be taken into account when estimating the dimension of a tapered fracture.

We also investigate the limiting cases of inviscid and very viscous fluid, incompressible fluid, and high-frequency oscillations. The results indicate the importance of the fluid compressibility effect at high frequencies and for filtration in a long thin fracture.

FRACTURE WITH PROPPANT—GOVERNING EQUATION

We examine the dynamics of a compressible viscous fluid inside a long thin arbitrarily shaped fracture embedded in an absolutely rigid solid. The fracture is filled with porous material (proppant). The fluid flow is induced by normal harmonic oscillations of fracture walls that are permeable and allow the fluid to filtrate into the surrounding formation. The movement of the fluid is 2-D, and volumetric flow rate components are u and v in the Eulerian coordinate system (x, y) shown in Figure 1. Without the loss of generality, we can assume that the fracture is symmetrical relative to the x-axis. The thickness of the fracture $2\tilde{a}$ is a function of the spatial coordinate x and time t:

$$\tilde{a}(x, t) = a(x) + a_0(x)\varepsilon \exp(i\omega t),$$

where $a(x)$ is zero-frequency aperture of the fracture, ω is the angular frequency of oscillation of the walls, and the product $a_0(x)\varepsilon$ gives the amplitude of this oscillation. This amplitude can change along the crack in accordance with the function $a_0(x)$. This amplitude is small compared to the thickness of the fracture. Therefore the compressibility of the fluid in the fracture is described by a linear acoustic relation between pressure and density variations.

Fluid flow through the proppant is described in Biot's (1956) equation (see Appendix). This equation is the modification of Darcy's law and includes both inertia and filtration terms. Filtration into the surrounding formation is

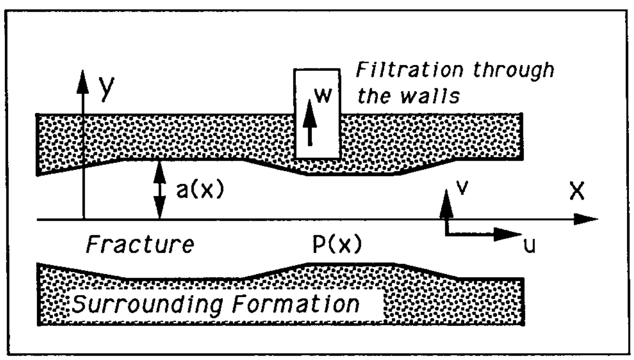

FIG. 1. Viscous compressible fluid in an arbitrarily shaped fracture filled with proppant: $P(x)$ is fluid pressure inside the fracture, u and v are fluid volumetric rate components, w is the volumetric rate component of filtration through the walls.

modeled as a (1-D) process in the y-direction. However, the parameters of filtration may change along the fracture depending on the inside pressure.

We assume that fluid pressure is constant across the fracture, and therefore the density of the fluid does not depend on the y-coordinate. The solution of the problem is obtained for the case where the pressure is a harmonic function of time: $P(x, t) = P_0(x)e^{i\omega t}$.

The following governing equation for function $P_0(x)$ follows from the condition of mass conservation in the fracture (see Appendix):

$$P_0''\left(\frac{\alpha}{\phi} - i\frac{\nu}{k\omega}\right)^{-1} + P_0\left[\phi\frac{\omega^2}{c_0^2} + (1-i)\frac{\omega k_0}{a\nu}\sqrt{\frac{\omega}{2\kappa}}\right] = -\phi\omega^2\rho_0\frac{a_0\varepsilon}{a}, \quad (1)$$

where α is the dynamic tortuosity parameter; ϕ is the porosity of the proppant; k is its permeability; ν is the kinematic viscosity of the fluid; c_0 is fluid acoustic velocity; k_0 is the permeability of the surrounding formation; κ is its hydraulic diffusivity coefficient; ρ_0 is the reference "undisturbed" density of the fluid; P_0'' and P_0' are the derivatives of function $P_0(x)$.

This ordinary differential equation of the second order can be integrated using boundary conditions at $x = 0$ and $x = L$, where L is the length of the fracture. Complete saturation of the fracture leads to the conditions of zero velocity or zero pressure gradient at the edges: $P_0' = 0$ at $x = 0$ and $x = L$. The condition of constant pressure $P_0 = 0$ can be used for a partially saturated fracture (it is understood that pressure is measured from its reference "undisturbed" level in the fracture). Equation (1) can also be solved with more complicated boundary conditions representing, for example, the inclusion of gas in the fracture or its connection with a wellbore.

If the inertia term α/ϕ is dropped, equation (1) readily yields the governing equation for the case when Darcy's law is used instead of Biot's equation:

$$P_0''\frac{k}{\nu} - P_0\left[i\phi\frac{\omega}{c_0^2} + (1+i)\frac{\omega k_0}{a\nu}\sqrt{\frac{\omega}{2\kappa}}\right] = i\phi\omega\rho_0\frac{a_0\varepsilon}{a}.$$

For a fracture without proppant, the following governing equation was derived in Dvorkin et al. (1990):

$$\left[1 - \frac{\tanh\left(a\sqrt{i\omega/\nu}\right)}{a\sqrt{i\omega/\nu}}\right]P_0'' + \frac{\partial a}{a\partial x}\tanh^2\left(a\sqrt{i\omega/\nu}\right)P_0' + \left[\frac{\omega^2}{c_0^2} + (1-i)\frac{\omega k_0}{a\nu}\sqrt{\frac{\omega}{2\kappa}}\right]P_0 = -\omega^2\rho_0\frac{a_0\varepsilon}{a}. \quad (2)$$

FRACTURE WITH PROPPANT-EXAMPLES

Two-point problems for ordinary differential equations (1) and (2) can be easily solved numerically by using the polynomial approximation method (Hamming, 1962). For the case of constant aperture $a(x) = $ const, and conditions $P_0 = 0$ at $x = 0$ and $P_0' = 0$ at $x = L$ (a fracture with one open end) the analytical solution of equation (1) is available:

$$P_0(x) = A\left[1 - \frac{e^{-i\lambda(L-x)} + e^{i\lambda(L-x)}}{e^{i\lambda L} + e^{-i\lambda L}}\right],$$

where

$$\lambda^2 = \left(\frac{\alpha}{\phi} - i\frac{\nu}{k\omega}\right)\left[\phi\frac{\omega^2}{c_0^2} + (1-i)\frac{\omega k_0}{a\nu}\sqrt{\frac{\omega}{2\kappa}}\right];$$

$$A = -\phi\omega^2\rho_0\frac{a_0\varepsilon}{a}\left[\phi\frac{\omega^2}{c_0^2} + (1-i)\frac{\omega k_0}{a\nu}\sqrt{\frac{\omega}{2\kappa}}\right]^{-1}.$$

Pressure averaged over the length of the fracture is:

$$P_{av} = \frac{1}{L}\int_0^L P_0(x)\,dx = A\left[1 - \frac{1}{i\lambda L}\frac{e^{i\lambda L} - e^{-i\lambda L}}{e^{i\lambda L} + e^{-i\lambda L}}\right]. \quad (3)$$

Considering the case of very low viscosity fluid in a fracture with impermeable walls, we find from equation (3) at $\nu \to 0$:

$$P_{av} = -\rho_0 c_0^2\frac{a_0\varepsilon}{a}\left[1 - \frac{\tan(\lambda L)}{\lambda L}\right], \quad (4)$$

where $\lambda = \omega\sqrt{\alpha}/c_0$.

Resonance in the fracture occurs at

$$\lambda L = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$$

when

$$\omega = \frac{\pi}{2}\frac{c_0/\sqrt{\alpha}}{L}, \frac{3\pi}{2}\frac{c_0/\sqrt{\alpha}}{L}, \frac{5\pi}{2}\frac{c_0/\sqrt{\alpha}}{L}, \ldots. \quad (5)$$

In a fracture without proppant, resonance occurs at

$$\omega = \frac{\pi}{2}\frac{c_0}{L}, \frac{3\pi}{2}\frac{c_0}{L}, \frac{5\pi}{2}\frac{c_0}{L}, \ldots,$$

(Dvorkin et al., 1990). Therefore, resonance in a fracture with internal filtration occurs at lower angular frequencies and has a higher period than in a fracture without proppant. This difference is due to coupling between the fluid and the proppant that is characterized by the dynamic tortuosity parameter $\alpha \geq 1$. Hence the apparent acoustic velocity in a fracture with proppant $c_{ap} = c_0/\sqrt{\alpha}$ is lower than acoustic velocity in the fluid: $c_{ap} \leq c_0$.

To investigate the influence of proppant permeability we examined a 10-m long fracture with one open end (a fracture connected with a wellbore) and constant aperture 1 cm. The fracture contains water and is embedded in an impermeable formation. The amplitude of the oscillation of the walls is constant along the fracture. The porosity of the proppant is 35 percent; nondimensional parameter α is 1.5.

Average pressure amplitude $Abs(P_{av})$ was normalized by $\rho c_0^2 a_0\varepsilon/a$ and plotted in Figure 2 versus frequency $f = \omega/2\pi$ for different permeabilities of the proppant: 0.25, 1 and 4 kdarcy.

In the case of high permeability proppant (4 kdarcy), we observe a pronounced resonance peak at frequency 32 Hz whereas formula (5) predicts the first resonance peak at f = 30.6 Hz. In the second case of k = 1 kdarcy, the resonance peak is very small; maximum of $Abs(P_{av})$ can be observed at f = 37.1 Hz. In the third case of k = 250 darcy, the resonance is totally suppressed by the filtration damping. Now the amplitude of average pressure increases monotonously with increasing frequency. This effect is the result of the filtration term $i\nu/k\omega$ dominating the inertial one α/ϕ in equation (1).

This example shows that the domination of the filtration term not only suppresses the resonance but also shifts resonance frequency towards higher values.

High Viscosity and High Frequency Limits

At the high-frequency limit $\omega \to \infty$, we find that $\lambda^2 = \alpha\omega^2/c_0^2 \to \infty$ and $A = -\rho_0 c_0^2 a_0 \varepsilon/a$. Therefore, equation (3) yields:

$$P_{av} = -\rho_0 c_0^2 \frac{a_0 \varepsilon}{a}.$$

If the viscosity of the fluid is very high $\nu \to \infty$, we have the same result:

$$\lambda^2 = -i\phi \frac{\nu\omega}{kc_0^2} \to \infty,\ A = -\rho_0 c_0^2 a_0 \varepsilon/a;\ P_{av} = -\rho_0 c_0^2 \frac{a_0 \varepsilon}{a}.$$

In these two cases, fluid in the fracture is "frozen" because it cannot relax and reacts only at a local compression of fracture walls.

The Effect of Compressibility

Examining the case of a fracture filled with incompressible fluid ($c_0 \to \infty$) and embedded into a nonpermeable formation (k_0 = 0), we find from equation (1):

$$P_0'' = -\phi\omega^2\rho_0 \frac{a_0 \varepsilon}{a}\left(\frac{\alpha}{\phi} - i\frac{\nu}{k\omega}\right).$$

Solving this equation with conditions P_0 = 0 at x = 0 and P_0' = 0 at $x = L$, we arrive at:

$$P_{av} = \phi\omega^2\rho_0 \frac{a_0 \varepsilon}{a}\left(\frac{\alpha}{\phi} - i\frac{\nu}{k\omega}\right)\frac{L^2}{3}.$$

This formula indicates that average pressure in the fracture increases limitlessly with increasing frequency or increasing length of the fracture. Significant errors may occur if the compressibility of the fluid is not taken into account.

FRACTURE WITHOUT PROPPANT—EXAMPLES

To investigate the effects of (1) filtration into a surrounding formation and (2) varying shape of a fracture, a two-point problem for ordinary differential equation (2) was solved numerically by using the polynomial approximation method (Hamming, 1962). We examined a 10-m long fracture with one open end (boundary conditions P_0 = 0 at x = 0 and P_0' = 0 at $x = L$). The fracture was filled with water.

Filtration into the Formation

The plots of average pressure amplitude $Abs(P_{av})$ normalized by $\rho c_0^2 a_0 \varepsilon/a$ versus frequency $f = \omega/2\pi$ are presented in Figure 3 for different permeabilities of the surrounding formation: 0.09, 0.27, 0.81, 2.43, 7.29, and 21.87 mdarcy. The fracture has constant aperture of 1 cm; amplitude $a_0\varepsilon$ is also constant along the fracture.

Acoustic resonance in a fracture surrounded by an impermeable formation is expected in this case at $f = c_0/4L$ = 37.5 Hz. The actual resonance peaks are observed at frequencies 36.45 Hz for 0.09 mdarcy; 35.81 Hz for 0.27 mdarcy; 34.53 Hz for 0.81 mdarcy; 33 Hz for 2.43 mdarcy; 30.4 Hz for 7.29 mdarcy; and 26.9 Hz for 21.87 mdarcy. This example shows that the resonance amplitude strongly decreases with increasing permeability of the surrounding rock. At the same time, the resonance frequency shifts to

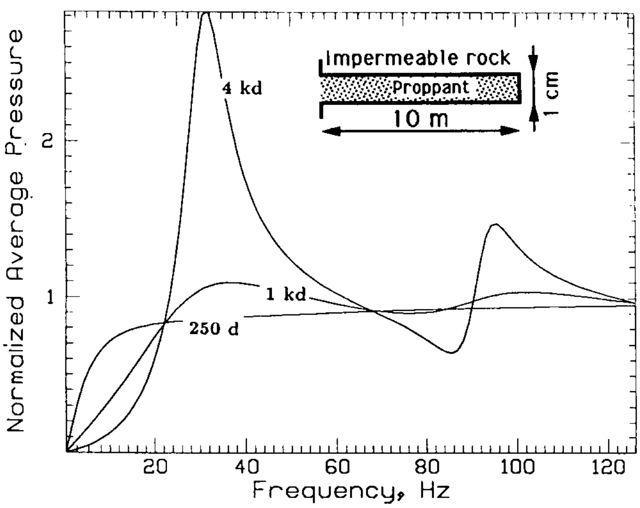

FIG. 2. The effect of the permeability of proppant: Normalized $Abs(P_{av})$ versus frequency in a fracture with proppant for different permeabilities of the proppant; 4, 1, and 0.25 *kd*.

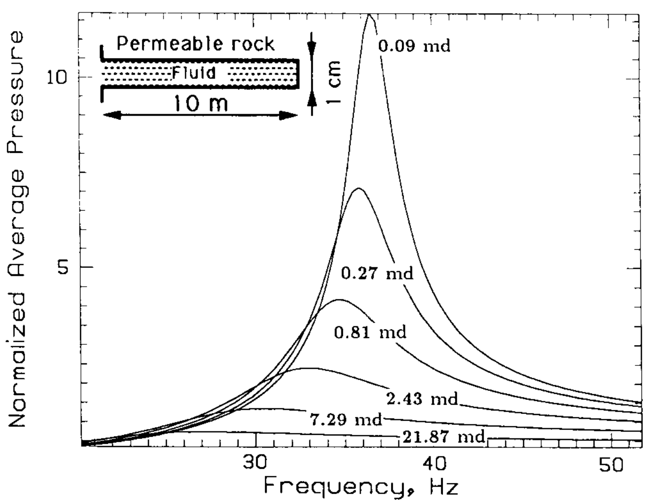

FIG. 3. The effect of filtration into a surrounding formation: Normalized $Abs(P_{av})$ versus frequency in a fracture without proppant for different permeabilities of surrounding formation: 0.09, 0.27, 0.81, 2.43, 7.29, and 21.87 md.

lower values. Therefore, fluid filtration into surrounding rock acts to suppress the resonance and to increase its period.

The Effect of Changing Shape

A fracture with straight walls is embedded in an impermeable formation. The aperture at its left open end is 1 cm. We examined five cases of varying aperture at its right closed end: 0.1, 0.55, 1, 1.45, and 1.9 cm. The amplitude of oscillations was proportional to the aperture.

The plots of average pressure amplitude $Abs(P_{av})$ normalized by $\rho c_0^2 a_0 \varepsilon / a$ versus frequency are presented in Figure 4.

Acoustic resonance in a fracture of constant aperture is expected in this case at $f = c_0/4L = 37.5$ Hz. This is precisely the case when the aperture at the right end is 1 cm. In the cases of tapered fractures with apertures at the right end of 0.1 and 0.55 cm, resonance frequencies are 51.75 and 41.72 Hz. This effect means that in a tapered fracture the period of resonance decreases and apparent acoustic velocity increases relative to the case of a constant aperture fracture.

In widening fractures with apertures at the right end 1.45 and 1.9 cm, resonance frequencies are 34.55 and 32.64 Hz. The period of resonance increases and apparent acoustic velocity decreases relative to the case of constant aperture fracture.

Theoretically, the effects of resonance frequency variation due to fluid filtrating into a surrounding formation and the changing shape of a fracture can be explained by the form of the governing equation (2). Indeed, the coefficients at P_0'' and P_0' depend on the aperture and its first derivative, and the coefficient at P_0 depends on the permeability of the surrounding formation.

Limiting Cases

For a fracture of constant aperture with one open end, the analytical solution of equation (2) yields:

$$P_{av} = A\left[1 - \frac{1}{i\lambda L}\frac{e^{i\lambda L} - e^{-i\lambda L}}{e^{i\lambda L} + e^{-i\lambda L}}\right], \tag{6}$$

where

$$\lambda^2 = \left[\frac{\omega^2}{c_0^2} + (1-i)\frac{\omega k_0}{a\nu}\sqrt{\frac{\omega}{2\kappa}}\right]\left[1 - \frac{\tanh(a\sqrt{i\omega/\nu})}{a\sqrt{i\omega/\nu}}\right]^{-1};$$

$$A = -\omega^2\rho_0\frac{a_0\varepsilon}{a}\left[\frac{\omega^2}{c_0^2} + (1-i)\frac{\omega k_0}{a\nu}\sqrt{\frac{\omega}{2\kappa}}\right]^{-1}.$$

For the case of very low-viscosity fluid in a fracture with impermeable walls, P_{av} is expressed by equation (4) with $\lambda = \omega/c_0$. The first resonance frequency is $c_0/4L$.

Examining high-viscosity and high-frequency limits, we find as in the case of a fracture with proppant,

$$P_{av} = -\rho_0 c_0^2\frac{a_0\varepsilon}{a}.$$

Again, fluid cannot relax and is "frozen" because of high viscosity or high frequency.

For a fracture with incompressible fluid embedded in an impermeable formation we find:

$$P_{av} = \frac{\omega^2\rho_0 a_0\varepsilon/a}{1 - \dfrac{\tanh(a\sqrt{i\omega/\nu})}{a\sqrt{i\omega/\nu}}}\frac{L^2}{2}.$$

As in the case of a fracture with proppant, taking into account the compressibility of the fluid is important for accurate estimates of its reaction.

CONCLUSIONS AND DISCUSSION

We have solved a new problem of fluid dynamics in an arbitrarily shaped long thin fracture containing proppant. The fracture is embedded in an absolutely rigid formation. Fluid flow inside the fracture is described by Biot's modifi-

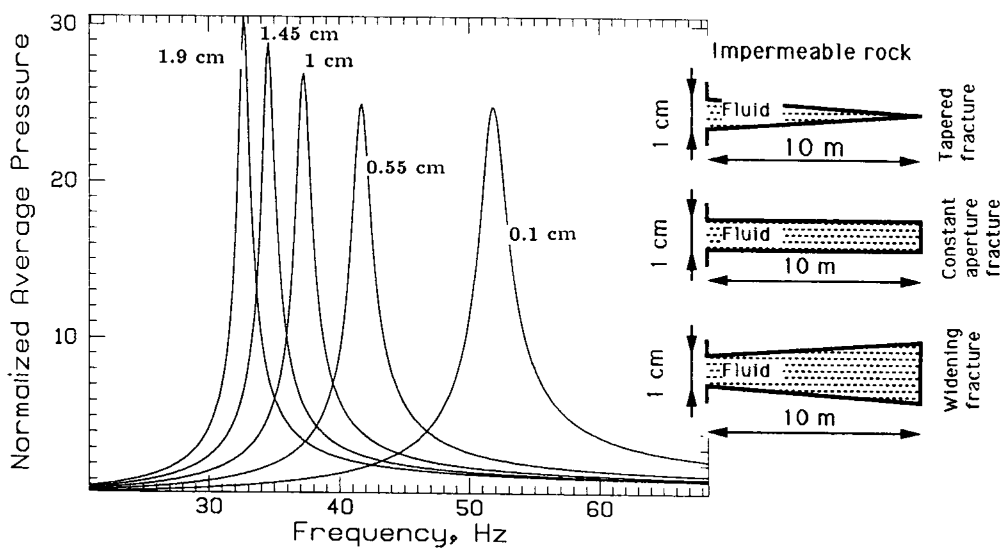

FIG. 4. The effect of the changing shape of a fracture: Normalized $Abs(P_{av})$ versus frequency in a fracture without proppant for constant aperture of 1 cm at the left open end and varying aperture at the right closed end: 0.1, 0.55, 1, 1.45, and 1.9 cm.

cation of Darcy's law with inertia and filtration terms. The acoustic resonance along the length of the fracture filled with proppant is dampened by the internal fluid filtration. The apparent acoustic velocity in the fracture is lower than sound velocity in the fluid because of coupling between the fluid and the proppant. Fluid filtration into the surrounding rock acts to decrease resonance peaks and to increase resonance period. Resonance period changes significantly in fractures of varying shape.

In this paper, we have concentrated on some important issues of fracture fluid dynamics:

1) the effect of proppant on the response of a fracture to external oscillatory excitation,
2) the effect of filtration through the walls of a fracture, and
3) the effect of a fracture changing shape.

We have shown that these factors can significantly affect the acoustic characteristics of a fracture and the magnitude of its response to external excitation. These results were derived under the assumption that the fracture was embedded in an absolutely rigid formation without taking into account the elastic compliance of the walls. Yet, the conclusions are important both methodologically and practically:

1) The new solutions for fracture dynamics problems can be incorporated into more general problems of interaction with an elastic formation.
2) The results show that the estimates of fracture size based on crack wave theory (e.g., Ferrazzini et al., 1990) have to be corrected for tapered cracks, cracks surrounded by a permeable rock, or cracks filled with proppant. Fluid acoustic resonance may be totally suppressed in fluid-filled cracks with proppant or in rough-walled cracks. Therefore such cracks may not react to external excitation with resonance. Fluid filtration through the walls of a crack into the surrounding formation may also be responsible for the damping of the resonance.

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APPENDIX

THE DERIVATION OF THE GOVERNING EQUATION

We examine the dynamics of a compressible viscous fluid inside a long thin arbitrarily shaped fracture filled with porous material (proppant). The movement of the fluid is 2-D and volumetric flow rate components are u and v in the coordinate system (x, y) as shown in Figure 1. Without the loss of generality, we can assume that the fracture is symmetrical relative to the x-axis. The thickness of the fracture $2\tilde{a}$ is a function of the spatial coordinate x and time t:

$$\tilde{a}(x, t) = a(x) + a_0(x)\varepsilon \exp(i\omega t), \quad \text{(A-1)}$$

where $a(x)$ is zero-frequency aperture of the fracture, ω is the angular frequency of oscillation of the walls, and the product $a_0(x)\varepsilon$ gives the amplitude of this oscillation. This amplitude can change along the crack in accordance with the function $a_0(x)$. The amplitude is small compared to the thickness of the fracture:

$$a_0(x)\varepsilon \ll a(x). \quad \text{(A-2)}$$

Therefore the compressibility of the fluid in the fracture can be described by a linear relation:

$$dP = c_0^2 d\rho, \quad \text{(A-3)}$$

where P is the pressure in the fluid, ρ is its density, and c_0 is fluid acoustic velocity. Density variation $d\rho$ in equation (A-3) is much smaller than its reference "undisturbed" value ρ_0:

$$d\rho \ll \rho_0. \quad \text{(A-4)}$$

We assume that the pressure is constant across the fracture and therefore the density of the fluid does not depend on the y-coordinate.

The equation of mass conservation in the fracture is:

$$\frac{\partial(\rho\phi)}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \quad \text{(A-5)}$$

where the porosity of the fracture filling ϕ is assumed to be constant.

The integration of equation (A-5) in the y-direction from 0 to $\tilde{a}$ gives:

$$\tilde{a}\phi \frac{\partial \rho}{\partial t} + \rho v|_0^{\tilde{a}} + \int_0^{\tilde{a}} \frac{\partial(\rho u)}{\partial x}\, dy = 0. \qquad \text{(A-6)}$$

Due to the symmetry of the fluid flow relative to the x-axis,

$$v|_{y=0} = 0. \qquad \text{(A-7)}$$

At the upper wall of the fracture

$$v|_{y=\tilde{a}} = w + \phi \frac{\partial \tilde{a}}{\partial t}, \qquad \text{(A-8)}$$

where w is the volumetric flow rate of fluid filtration into the surrounding formation (Figure 1). This filtration is modeled as a 1-D process in the y-direction. However, the parameters of filtration may change along the fracture depending on the inside pressure. The 1-D filtration equation is solved assuming that the fracture is embedded in an infinite porous space. The boundary conditions are mass conservation and pressure continuity at the fracture walls. The details of this solution are given in Dvorkin et al. (1990). The resulting formula relates w to P:

$$w|_{y=\tilde{a}} = (1+i)\frac{k_0}{\mu}\sqrt{\frac{\omega}{2\kappa}}\, P, \qquad \text{(A-9)}$$

where a hydraulic diffusivity coefficient $\kappa = \rho_0 k_0 c_0^2/(\phi_0 \mu)$; μ is fluid's viscosity; k_0 is the permeability of the surrounding material; and ϕ_0 is its porosity.

Assuming that all functions are harmonically time dependent:

$$P(x, t) = P_0(x)e^{i\omega t}, \quad w(x, t) = w_0(x)e^{i\omega t},$$

$$u(x, y, t) = u_0(x, y)e^{i\omega t}, \quad v(x, y, t) = v_0(x, y)e^{i\omega t}$$

we find from equation (A-8) by using equations (A-1) and (A-9):

$$v|_{y=\tilde{a}} = \left[(1+i)\frac{k_0}{\mu}\sqrt{\frac{\omega}{2\kappa}}\, P_0 + \phi i\omega a_0 \varepsilon\right]e^{i\omega t}. \qquad \text{(A-10)}$$

From equation (A-3) we have:

$$\frac{\partial \rho}{\partial t} = \frac{1}{c_0^2}\frac{\partial P}{\partial t} = \frac{i\omega}{c_0^2} P_0 e^{i\omega t}. \qquad \text{(A-11)}$$

Fluid flow through the proppant is described by Biot's (1956) equations for the propagation of elastic waves in a fluid-saturated porous solid. If the displacements of the proppant are ignored relative to the displacements of the fluid, these equations yield the following relation for 1-D flow in the x-direction:

$$-\frac{\partial P}{\partial x} = \frac{\rho_{22}}{\phi^2}\frac{\partial u}{\partial t} + \frac{\mu}{k} u,$$

where k is the permeability of the proppant; $\rho_{22} = \alpha\rho\phi$; and $\alpha \geq 1$ is the dynamic tortuosity parameter (Bourbie et al., 1987). Therefore we arrive at

$$-\frac{\partial P}{\partial x} = \frac{\alpha\rho}{\phi}\frac{\partial u}{\partial t} + \frac{\mu}{k} u. \qquad \text{(A-12)}$$

This equation includes inertia $\alpha\rho/\phi\ \partial u/\partial t$ and filtration $\mu/k\ u$ terms. Darcy's law of filtration immediately follows from equation (A-12) by dropping the inertia term.

Relating P_0 to u_0 from equation (A-12) we have:

$$-\frac{dP_0}{dx} = \left(i\omega\frac{\alpha\rho}{\phi} + \frac{\mu}{k}\right)u_0. \qquad \text{(A-13)}$$

In the following derivations, we neglect the variations of $\tilde{a}$ and ρ compared to their reference values:

$$\tilde{a}(x, t) \approx a(x); \quad \rho(x, t) \approx \rho_0.$$

The validity of these approximations follows from equations (A-2) and (A-4). Yet, the variations of these functions have to be considered when taking their derivatives.

Relations (A-3) and (A-13) are used to transform the integral in (A-6):

$$\int_0^{\tilde{a}} \frac{\partial(\rho u)}{\partial x}\, dy = \int_0^{\tilde{a}} \left(u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x}\right) dy$$

$$= \int_0^{\tilde{a}} \left[-\left(i\omega\frac{\alpha\rho_0}{\phi} + \frac{\mu}{k}\right)^{-1} P_0' \frac{1}{c_0^2} P_0' e^{i\omega t} - \rho_0 P_0''\left(i\omega\frac{\alpha\rho_0}{\phi} + \frac{\mu}{k}\right)^{-1}\right] e^{i\omega t}\, dy$$

$$\approx -\int_0^{\tilde{a}} P_0''\left(i\omega\frac{\alpha\rho_0}{\phi} + \frac{\nu}{k}\right)^{-1} e^{i\omega t}\, dy, \qquad \text{(A-14)}$$

where $\nu = \mu/\rho_0$ and P_0' and P_0'' are first and second derivatives with respect to x. In equation (A-14), the nonlinear term with P_0' is dropped because pressure variation is assumed to be small.

Substituting equations (A-3), (A-7), (A-10), and (A-14) into equation (A-6), we arrive at:

$$a\phi\frac{i\omega}{c_0^2} P_0 e^{i\omega t} + \rho_0\left[(1+i)\frac{k_0}{\mu}\sqrt{\frac{\omega}{2\kappa}}\, P_0 + \phi i\omega a_0\varepsilon\right]e^{i\omega t} - aP_0''\left(i\omega\frac{\alpha}{\phi} + \frac{\nu}{k}\right)^{-1} e^{i\omega t} = 0.$$

This relation can be easily transformed into the governing equation:

$$P_0''\left(\frac{\alpha}{\phi} - i\frac{\nu}{k\omega}\right)^{-1} + P_0\left[\phi\frac{\omega^2}{c_0^2} + (1-i)\frac{\omega k_0}{a\nu}\sqrt{\frac{\omega}{2\kappa}}\right] = -\phi\omega^2\rho_0\frac{a_0\varepsilon}{a}. \qquad \text{(A-15)}$$

If the inertia term α/ϕ is dropped, equation (A-15) readily yields the governing equation for the case when Darcy's law is used instead of equation (A-12):

$$P_0''\frac{k}{\nu} - P_0\left[i\phi\frac{\omega}{c_0^2} + (1+i)\frac{\omega k_0}{a\nu}\sqrt{\frac{\omega}{2\kappa}}\right] = i\phi\omega\rho_0\frac{a_0\varepsilon}{a}.$$