Auxiliary Material for Near-Source Ground Motion from Steady State Dynamic Rupture Pulses

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1. Derivation of the Steady State Solution

Consider two identical isotropic linear elastic half-spaces, having a shear modulus μ and P- and S-wave speeds c_p and c_s, joined along the interface $y = 0$. Along this fault a steady state two-dimensional rupture pulse propagates in the positive x direction at a constant velocity V. The fields are a function of $x - Vt$ and y only, and for convenience we shall take $t = 0$ such that $x - Vt \to x$ in the following expressions, equivalent to transforming into a frame of reference moving with the rupture. We shall take the origin of the coordinate system to coincide with the rupture front. On the fault, we specify the length of the slip zone L and the shear traction within the slip zone $\tau(x)$. The mixed boundary value problem is defined formally as

$$\begin{aligned} v_x(\infty < x < -L, y = 0) &= 0 \\ \sigma_{xy}(-L < x < 0, y = 0) &= \tau(x) - \tau_0 \qquad (1) \\ v_x(0 < x < \infty, y = 0) &= 0 \\ \sigma_{yy}(-\infty < x < \infty, y = 0) &= 0, \qquad (2) \end{aligned}$$

where v_i are velocities, σ_{ij} are stresses, and τ_0 is the initial shear stress on the fault far ahead of and behind the slip zone. The healing process at the trailing edge of the slip zone ($x = -L$) is assumed to be energy neutral (i.e., it is accompanied by neither dissipation nor release of energy).

We first consider the general case of an arbitrary $\tau(x)$, but later specialize to the linear distance-weakening model of *Palmer and Rice* [1973]:

$$\tau(x) = \begin{cases} \tau_p + (\tau_p - \tau_r)x/R, & -R < x < 0 \\ \tau_r, & -L < x < -R, \end{cases} \qquad (3)$$

where τ_p is the peak strength, τ_r is the residual strength, and R is the length of the breakdown zone.

Expressions for the shear traction and slip velocity on the fault for both subshear ($V < c_s$) and supershear ($V > c_s$) ruptures having an arbitrary $\tau(x)$ have been given by *Broberg* [1978, 1989], who also examined the off-fault fields only in the limiting case of a vanishingly small breakdown zone. We present the complete derivation for all fields, both on and off of the fault, at supershear speeds. A recent derivation of the subshear expressions is given by *Rice et al.* [2004], although they do not present results for the velocity fields. First, we decompose the displacement field u_i into dilatational and shear components:

$$u_i(x, y) = u_i^p(x, y) + u_i^s(x, y), \qquad (4)$$

governed by

$$\left(\alpha_p^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u_i^p(x, y) = 0 \qquad (5)$$

and

$$\left(-\beta_s^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u_i^s(x, y) = 0, \qquad (6)$$

with $\alpha_p = \sqrt{1 - V^2/c_p^2}$ and $\beta_s = \sqrt{V^2/c_s^2 - 1}$. The P-wave equation is elliptic with evanescent (i.e., inhomogeneous) wave eigenfunctions. In contrast to the subshear expressions, the S-wave equation at supershear velocities is now hyperbolic and its eigenfunctions are radiating S waves, which form a planar Mach front extending from the rupture front.

A solution to these equations, restricting ourselves to $y \geq 0$ for now and taking into account the conditions $\nabla \times \mathbf{u}^p = 0$ and $\nabla \cdot \mathbf{u}^s = 0$, is

$$\begin{aligned} u_x^p(x, y) &= \Re P(z_p) \\ u_y^p(x, y) &= -\alpha_p \Im P(z_p) \\ u_x^s(x, y) &= -\beta_s S(z_s) \\ u_y^s(x, y) &= S(z_s), \qquad (7) \end{aligned}$$

where $z_p = x + i\alpha_p y$ and $z_s = x + \beta_s y$. Here, $P(z)$ is a function that is analytic except across the slip zone, and $S(z)$ is a real function. In terms of these functions,

$$\sigma_{yy} = (\beta_s^2 - 1)\Re P'(z_p) + 2\beta_s S'(z_s), \qquad (8)$$

where the prime indicates differentiation with respect to the argument. Setting this expression equal to zero on the fault, as required by the boundary condition (2), yields

$$S'(x) = -\frac{\beta_s^2 - 1}{2\beta_s} \Re P'(x). \qquad (9)$$

This expression remains valid away from the fault under the substitution $x \to z_s$. We use (9) to eliminate $S'(z)$ and rewrite expressions for the fields in terms of a new analytic function

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$$N(z) = 2i\mu\alpha_p P'(z), \tag{10}$$

yielding

$$\begin{aligned}
v_x &= -\frac{V}{\mu}\left[\frac{1}{2\alpha_p}\Im N(z_p) + \frac{\beta_s^2 - 1}{4\alpha_p}\Im N(z_s)\right] \\
v_y &= -\frac{V}{\mu}\left[\frac{1}{2}\Re N(z_p) - \frac{\beta_s^2 - 1}{4\alpha_p\beta_s}\Im N(z_s)\right] \\
\sigma_{xx} &= \frac{1 + \beta_s^2 + 2\alpha_p^2}{2\alpha_p}\Im N(z_p) + \frac{\beta_s^2 - 1}{2\alpha_p}\Im N(z_s) \\
\sigma_{xy} &= \Re N(z_p) + \frac{(\beta_s^2 - 1)^2}{4\alpha_p\beta_s}\Im N(z_s) \\
\sigma_{yy} &= \frac{\beta_s^2 - 1}{2\alpha_p}\Im\left[N(z_p) - N(z_s)\right].
\end{aligned} \tag{11}$$

At this point, we apply the mode II symmetry properties:

$$\begin{aligned}
\Re N(z) &= \Re N(\bar{z}) \\
\Im N(z) &= -\Im N(\bar{z}),
\end{aligned} \tag{12}$$

where the overbar denotes complex conjugation, an operation that replaces y by $-y$ (effectively reflecting the observation point across the fault). This implies that $N(\bar{z}) = \bar{N}(z)$; hence,

$$\begin{aligned}
\Re N(x) &= \frac{N_+(x) + N_-(x)}{2} \\
\Im N(x) &= \frac{N_+(x) - N_-(x)}{2i},
\end{aligned} \tag{13}$$

where $N_\pm(x) = \lim_{y\to 0^\pm} N(z)$. Inserting (13) into the traction boundary condition (1) yields

$$N_+(x) = \frac{B - iA}{B + iA}N_-(x) + 2\frac{\tau(x) - \tau_0}{1 - iB/A} \tag{14}$$

for $-L < x < 0$, where $A = 4\alpha_p\beta_s$ and $B = (\beta_s^2 - 1)^2$. Writing

$$e^{-2\pi iq} = \frac{B - iA}{B + iA}, \tag{15}$$

yields

$$q = \frac{1}{\pi}\arctan\frac{A}{B} \tag{16}$$

with $0 < q \leq 1/2$. This allows us to recast (14) as

$$\begin{aligned}
N_+(x) &= e^{-2\pi iq}N_-(x) + \\
&\quad 2i\sin(\pi q)e^{-i\pi q}\left[\tau(x) - \tau_0\right].
\end{aligned} \tag{17}$$

This defines a Hilbert problem [*Muskhelishvili*, 1953], which is solved by first introducing a solution $X(z)$ to the homogeneous problem

$$X_+(x) = e^{-2\pi iq}X_-(x). \tag{18}$$

Using this result to eliminate the factor $e^{-2\pi iq}$ yields

$$\frac{N_+(x)}{X_+(x)} = \frac{N_-(x)}{X_-(x)} + 2i\sin(\pi q)e^{-i\pi q}\frac{\tau(x) - \tau_0}{X_+(x)}. \tag{19}$$

An appropriate selection of $X(z)$ for a semi-infinite crack is $X(z) = z^{1-q}$, and for a pulse of length L is $X(z) = z^{1-q}(z + L)^q$. The branch cut is taken along $-L < x < 0$, $y = 0$. Defining

$$F(z) = \frac{N(z)}{X(z)} \tag{20}$$

and

$$f(x) = 2i\sin(\pi q)e^{-i\pi q}\frac{\tau(x) - \tau_0}{X_+(x)} \tag{21}$$

allows us to rewrite (19) as

$$F_+(x) - F_-(x) = f(x). \tag{22}$$

Using the Cauchy integral theorem to relate the value of F at some point z away from the fault to the jump in F across the slip zone (see *Rice et al.* [2004] for a complete description) solves the equation as

$$F(z) = \frac{1}{2\pi i}\int_{-L}^{0}\frac{f(w)}{w - z}dw + E(z), \tag{23}$$

where $E(z)$ is an entire function (i.e., a polynomial in z). Since the fields vanish at infinity (except within the region of S-wave radiation), $E(z) = 0$. Thus,

$$\begin{aligned}
N(z) &= -\frac{\sin(\pi q)}{\pi}z^{1-q}(z + L)^q \times \\
&\quad \int_{-L}^{0}\frac{\tau(w) - \tau_0}{(-w)^{1-q}(w + L)^q(w - z)}dw.
\end{aligned} \tag{24}$$

Note that $N(z)$ is of the form $(\tau_p - \tau_r)h(R/L, V/c_s)$, which is quite different from the scaling for subshear ruptures, in which the dependence on V/c_s, here arising through q, is absent. The physical reason for this is related to the velocity-dependent S-wave radiation at supershear speeds. These waves alter the energy balance within the breakdown zone and the concentration of the stress fields around the rupture front.

To complete the solution, we must place a constraint on τ_0 by using the fact that the fields vanish at infinity (outside of the radiating S-wave region). Inserting (24) into (11) and taking the limit that $z \to \infty$ ahead of the Mach front yields the constraint

$$\tau_0 = \frac{\sin(\pi q)}{\pi}\int_{-L}^{0}\frac{\tau(x)dx}{(-x)^{1-q}(x + L)^q}. \tag{25}$$

For subshear ruptures, an analogous derivation yields

$$v_x = -\frac{V}{\mu}\left[\frac{1}{2\alpha_p}\Im M(z_p) + \frac{\beta_s^2 - 1}{4\alpha_p}\Im M(z_s)\right]$$

$$
\begin{aligned}
v_y &= -\frac{V}{\mu}\left[\frac{1}{2}\Re M(z_p) - \frac{\beta_s^2-1}{4\alpha_p\beta_s}\Im M(z_s)\right] \\
\sigma_{xx} &= \frac{1+\beta_s^2+2\alpha_p^2}{2\alpha_p}\Im M(z_p) + \frac{\beta_s^2-1}{2\alpha_p}\Im M(z_s) \\
\sigma_{xy} &= \Re M(z_p) + \frac{(\beta_s^2-1)^2}{4\alpha_p\beta_s}\Im M(z_s) \\
\sigma_{yy} &= \frac{\beta_s^2-1}{2\alpha_p}\Im\left[M(z_p)-M(z_s)\right], \qquad (26)
\end{aligned}
$$

where

$$
\begin{aligned}
M(z) &= -\frac{1}{\pi}z^{1/2}(z+L)^{1/2}\times \\
&\int_{-L}^{0}\frac{\tau(w)-\tau_0}{(-w)^{1/2}(w+L)^{1/2}(w-z)}dw. \qquad (27)
\end{aligned}
$$

The expressions for the stress components have been given by *Rice et al.* [2004]. Note that $M(z)$ takes the form $(\tau_p-\tau_r)h(R/L)$, and is independent on rupture velocity (since for any subshear V, the fields in the singular model diverge with an inverse square-root singularity). This results in the ability to factor the speed dependent function out of the expressions for the fields, which cannot be done for supershear ruptures.

2. Shear-Wave Radiation from Supershear Ruptures

In this section, we examine properties of the S-wave radiation from supershear ruptures. First, note that the shear component of the fields given in (11) involves only $\Im N(z)$; consequently, when $N(z)$ is purely real, no S-wave radiation will be present. It is thus desirable to study the properties of $N(z)$ in more detail.

Let us define

$$
I(z) = \int_{-L}^{0}\frac{\tau(w)-\tau_0}{(-w)^{1-q}(w+L)^q(w-z)}dw, \qquad (28)
$$

which is the singular integral in (24). This has real and imaginary parts given by

$$
\Re I(z) = \int_{-L}^{0}\frac{\left[\tau(w)-\tau_0\right](w-z_R)}{(-w)^{1-q}(w+L)^q\left[(w-z_R)^2+z_I^2\right]}dw \qquad (29)
$$

and

$$
\Im I(z) = \int_{-L}^{0}\frac{\left[\tau(w)-\tau_0\right]z_I}{(-w)^{1-q}(w+L)^q\left[(w-z_R)^2+z_I^2\right]}dw, \qquad (30)
$$

where $z_R = \Re z$ and $z_I = \Im z$. Now let us take the limit that $z_I \to 0$, equivalent to letting the observation point approach the fault. Care must be taken in (30), and we note that

$$
\lim_{y\to 0}\frac{y}{(w-x)^2+y^2} = \pi\delta(w-x). \qquad (31)
$$

Thus,

$$
\Re I_+(x) = \int_{-L}^{0}\frac{\tau(w)-\tau_0}{(-w)^{1-q}(w+L)^q(w-x)}dw, \qquad (32)
$$

which is interpreted in the Cauchy principal value sense when $-L<x<0$, and

$$
\Im I_+(x) = \pi\frac{\tau(x)-\tau_0}{(-x)^{1-q}(x+L)^q} \qquad (33)
$$

for $-L<x<0$; otherwise, $\Im I_+(x)=0$.

Now we shall examine the behavior of $N(x)$ on the fault. Ahead of the slip zone ($x>0$), $\Im I_+(x)=0$ and we are left with the purely real expression

$$
\begin{aligned}
N_+(x) &= -\frac{\sin(\pi q)}{\pi}x^{1-q}(x+L)^q\times \\
&\int_{-L}^{0}\frac{\tau(w)-\tau_0}{(-w)^{1-q}(w+L)^q(w-x)}dw. \qquad (34)
\end{aligned}
$$

This result applies away from the fault as well, upon the substitution $x\to z_s$ since z_s is also real. Since the S-wave terms involve only $\Im N(z_s)$, this proves that the S waves vanish ahead of the S Mach front, defined by $x+\beta_s|y|=0$.

Within the slip zone ($-L<x<0$), (24) reduces to the complex expression

$$
\begin{aligned}
N_+(x) &= \frac{\sin(\pi q)}{\pi}e^{-i\pi q}\Bigg\{(-x)^{1-q}(x+L)^q\times \\
&\int_{-L}^{0}\frac{\left[\tau(w)-\tau_0\right]}{(-w)^{1-q}(w+L)^q(w-x)}dw \\
&+i\pi\left[\tau(x)-\tau_0\right]\Bigg\}, \qquad (35)
\end{aligned}
$$

which can be used in (11) to verify that $\sigma_{xy}(-L<x<0)=\tau(x)-\tau_0$.

Behind the slip zone ($x<-L$), $\Im I_+(x)=0$ and (24) reduces to the purely real expression

$$
\begin{aligned}
N_+(x) &= \frac{\sin(\pi q)}{\pi}(-x)^{1-q}(-x-L)^q\times \\
&\int_{-L}^{0}\frac{\left[\tau(w)-\tau_0\right]}{(-w)^{1-q}(w+L)^q(w-x)}dw. \qquad (36)
\end{aligned}
$$

Equations (34)-(36), when combined with (11), reveal an important result: S waves radiate only from the slip zone, as would be expected if working from the representation theorem.

Consequently, we focus on the behavior of the fields off of the fault between the passage of the Mach front originating from $x=0$ and that from $x=-L$ (i.e., when $-L<z_s<0$). First, we observe that the slip velocity Δv on the fault may be written in terms of $\Im N(x)$ only as

$$
\Delta v(x) = -2\frac{V}{\mu}\frac{\beta_s^2+1}{4\alpha_p}\Im N_+(x). \qquad (37)
$$

Using this in the off-fault expressions for velocity (11) yields

$$
\begin{aligned}
v_x &= -\frac{V}{\mu}\left[\frac{1}{2\alpha_p}\Im N(z_p)\right] + \frac{\beta_s^2-1}{2(\beta_s^2+1)}\Delta v(z_s) \qquad (38) \\
v_y &= -\frac{V}{\mu}\left[\frac{1}{2}\Re N(z_p)\right] - \frac{1}{2\beta_s}\frac{\beta_s^2-1}{\beta_s^2+1}\Delta v(z_s), \qquad (39)
\end{aligned}
$$

which proves that both components of the off-fault velocity fields trace out the exact slip velocity function on the fault during the passage of the S waves. The usefulness of this feature rests on the separation of the shear and dilatational fields, which occurs for points sufficiently removed from the fault.

We also use our results to obtain the fracture energy G and final slip δ. We merely numerically integrate the following expressions

$$G = \frac{1}{V} \int_{-L}^{0} \Delta v(x) \left[\tau(x) - \tau_r\right] dx \quad (40)$$

and

$$\delta = \frac{1}{V} \int_{-L}^{0} \Delta v(x) dx, \quad (41)$$

where $\Delta v(x)$ is also obtained by numerical integration. Note that (40) and (41) are strictly valid only for steady state conditions. We find no problems with this procedure, and validate our numerical results against the analytical solution for subshear ruptures having a specific $\tau(x)$ given by *Rice et al.* [2004].

Dimensional considerations allow G to be written as

$$G = h(R/L, V/c_s)(\tau_p - \tau_r)^2 R/\mu. \quad (42)$$

Only for subshear ruptures can $h(R/L, V/c_s)$ be factored into the product of $f(V/c_s)$ and $g(R/L)$, as follows from the discussion after equations (24) and (27).

At supershear speeds, not all of the work done by the remote stress field in excess of friction goes into fracture energy, as is the case for subshear ruptures [*Rice et al.*, 2004]. Instead, there is a flow of energy out to infinity associated with the radiating S waves. Let us denote this energy flux (per unit area advance of the rupture) as G_s. This quantity may be calculated by integrating the outward energy flux associated with the shear field over the slip zone. The fault-normal component of the energy flux is given by

$$\mathcal{F}_y = -\sigma_{xy} v_x - \sigma_{yy} v_y, \quad (43)$$

which may be simplified by eliminating the second term using $\sigma_{yy} = 0$, and G_s is given by

$$G_s = \frac{2}{V} \int_{-L}^{0} \mathcal{F}_y^s(x) dx \quad (44)$$

under steady state conditions. The factor of two arises from radiation leaving both sides of the fault. Substituting only the shear part of the field expressions in (11) for the stress and velocity components in (43), and using (37) to eliminate $\Im N_+(x)$ in favor of $\Delta v(x)$ yields

$$G_s = 2 g_s(V/c_s) \rho \int_{-L}^{0} \Delta v(x)^2 dx, \quad (45)$$

where ρ is the density of the medium and

$$g_s(V/c_s) = \frac{1}{\beta_s} \left(\frac{\beta_s^2 - 1}{\beta_s^2 + 1} \right)^2. \quad (46)$$

Dimensional considerations show that G_s, like G, takes the form

$$G_s = h_s(R/L, V/c_s)(\tau_p - \tau_r)^2 R/\mu, \quad (47)$$

such that the energetics of supershear ruptures depends strongly on both the size of the breakdown zone and the rupture velocity.

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